Serre's Reduction of Linear Functional Systems: Theory, Implementation and Applications

Thomas Cluzeau

University of Limoges ; CNRS ; XLIM ; UMR 6172 & ENSIL

Joint work with **Alban Quadrat** (INRIA Sophia Antipolis)

ACA 2009 - AADIOS SESSION

É.T.S., Montréal, Québec, Canada

06/25/2009

Motivations

- Objective of this work:

Use computer algebra (algebraic manipulations) to simplify systems coming from mathematical physics, applied mathematics, engineering sciences or control theory

- Interest:
  - Simplify the equations of the system
    $\Rightarrow$ simplify the study of its structural properties
  - Pre-conditioner to numerical analysis methods

Outline of the talk

1. Review of my 2008 ACA/AADIOS talk
2. Serre's Reduction
   Boudellioua (Sultan Qaboos University, Oman), Quadrat
3. Implementation: the SERRE package (in progress)
4. Going further: the holonomic case (in progress)

I

Review of my 2008 ACA/AADIOS talk

Decomposition problem

When can we decouple the equations of a linear functional system?

- Let D be a noncommutative polynomial ring of operators.
- Let $R \in D^{q \times p}$ be a matrix defining a linear functional system.

$$\exists\ W \in \mathrm{GL}_p(D),\ V \in \mathrm{GL}_q(D) : V\,R\,W = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}?$$

Tools used

- Let D be an Ore algebra, $R \in D^{q \times p}$ and a left D-module $\mathcal{F}$.

- Consider $\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}$.

- As in number theory or algebraic geometry, to $\ker_{\mathcal{F}}(R.)$ we associate the finitely presented left D-module:

$$M = D^{1 \times p}/(D^{1 \times q}\,R).$$

- Method:

1. Compute $\mathrm{end}_D(M)$, *i.e.*, solve the equation $R\,P = Q\,R$;
2. Compute idempotents in $\mathrm{end}_D(M)$: $P^2 = P$;
3. Check freeness of certain D-modules and compute bases (QuilenSuslin, Stafford).

Example: tank model (Petit-Rouchon, IEEE TAC 02)

- Consider $D = \mathbb{Q}(\alpha)\,[\partial, \delta]$ with $\partial = \frac{d}{dt}$, the matrix of the system

$$R = \begin{pmatrix} \partial & -\partial\,\delta^2 & \alpha\,\partial^2\,\delta \\ \partial\,\delta^2 & -\partial & \alpha\,\partial^2\,\delta \end{pmatrix} \in D^{2\times 3}.$$

- The matrices $P = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ et $Q = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ satisfy:

$$R\,P = Q\,R, \quad P^2 = P, \quad Q^2 = Q.$$

- Using linear algebra, we get:

$$U = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_3(D), \quad V = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}_2(D),$$

$$\Rightarrow \overline{R} = V\,R\,U^{-1} = \begin{pmatrix} \partial\,(1-\delta)\,(1+\delta) & 0 & 0 \\ 0 & \partial\,(\delta^2+1) & 2\,\alpha\,\partial^2\,\delta \end{pmatrix}.$$

The OreMorphisms package

- Algorithms are implemented in a Maple package called OreMorphisms based on the library OreModules developed by Q. et Robertz:

http://wwwb.math.rwth-aachen.de/OreModules

- List of functions:
  - Morphisms, MorphismsConstCoeff, MorphismsRat,
  - Idempotents, IdempotentsConstCoeff, IdempotentsRat
  - IdempotentsMat, IdempotentsMatConstCoeffs, IdempotentsMatRat
  - KerMorphism(Rat), ImMorphism(Rat), CokerMorphism(Rat), CoimMorphism(Rat),
  - TestSurj(Rat), TestInj(Rat), TestIso(Rat).

- It is freely available with a library of examples at:

http://www.ensil.unilim.fr/~cluzeau/OreMorphisms

II

Serre's Reduction
(Boudellioua, Quadrat)

Serre's reduction problem

When can we reduce the number of equations and unknowns of a linear functional system?

- Let D be a noncommutative polynomial ring of operators.
- Let $R \in D^{q \times p}$ be a matrix defining a linear functional linear.

$$\exists\, W \in \mathrm{GL}_p(D),\ V \in \mathrm{GL}_q(D) : V\,R\,W = \begin{pmatrix} I_r & 0 \\ 0 & S \end{pmatrix}?$$

- When $r = q - 1$, the system is equivalent to a sole equation!

String with an interior mass (Fliess et al, COCV 98)

$$\begin{cases} \phi_1(t)+\psi_1(t)-\phi_2(t)-\psi_2(t)=0, \\ \dot{\phi}_1(t)+\dot{\psi}_1(t)+a\,\phi_1(t)-a\,\psi_1(t)-b\,\phi_2(t)+b\,\psi_2(t)=0, \\ \phi_1(t-2\,h_1)+\psi_1(t)-u(t-h_1)=0, \\ \phi_2(t)+\psi_2(t-2\,h_2)-v(t-h_2)=0. \end{cases} \qquad (\star)$$

$$\partial f(t)=\dot{f}(t), \quad \sigma_1 f(t)=f(t-h_1), \quad \sigma_2 f(t)=f(t-h_2).$$

$$V\begin{pmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ \partial+a & \partial-a & -b & b & 0 & 0 \\ \sigma_1^2 & 1 & 0 & 0 & -\sigma_1 & 0 \\ 0 & 0 & 1 & \sigma_2^2 & 0 & -\sigma_2 \end{pmatrix} W$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial_1+a+b & a\,\sigma_1 & b\,\sigma_2 \end{pmatrix}.$$

$$(\star) \;\Leftrightarrow\; \dot{z}_1(t)+(a+b)\,z_1(t)+a\,z_2(t-h_1)+b\,z_3(t-h_2)=0.$$

String with an interior mass (Fliess et al, COCV 98)

The unimodular matrices V and W are defined by:

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \sigma_1^2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ (a-b-\partial)\,\sigma_1^2+\partial+a & -1 & \partial-a+b & -2\,b \end{pmatrix} \in \mathrm{GL}_4(D),$$

$$W = \begin{pmatrix} 1 & 0 & 0 & -1 & -\sigma_1 & 0 \\ 0 & -1 & 0 & 0 & \sigma_1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \sigma_2 \\ 0 & -1 & -1 & -1 & 0 & -\sigma_2 \\ 0 & 0 & 0 & -\sigma_1 & -\sigma_1^2+1 & 0 \\ 0 & -\sigma_2 & -\sigma_2 & -\sigma_2 & 0 & -\sigma_2^2+1 \end{pmatrix} \in \mathrm{GL}_6(D).$$

- The computations were obtained using the packages:

SERRE and QUILLENSUSLIN.

Generalization of a Serre's result

• Theorem: Let $R \in D^{q \times p}$ be a full row rank matrix, $\Lambda \in D^q$, $P = (R \quad -\Lambda)$ and the two left D-modules $M = D^{1 \times p}/(D^{1 \times q}\, R)$ and $E = D^{1 \times (p+1)}/(D^{1 \times q}\, P)$ defining an extension of D by M:

$$0 \longrightarrow D \xrightarrow{\alpha} E \xrightarrow{\beta} M \longrightarrow 0.$$

We have the equivalent assertions:

1. E is stably free of rank $p+1-q$: $E \oplus D^{1 \times q} \cong D^{1 \times (p+1)}$.
2. $P = (R \quad -\Lambda)$ admits a right-inverse over D.
3. $\mathrm{ext}^1_D(E, D) = D^q/(P\, D^{p+1}) = 0$.
4. $\mathrm{ext}^1_D(M, D) = D^q/(R\, D^p)$ is the cyclic right D-module generated by $\rho(\Lambda)$, where ρ denotes the projection:

$$\rho : D^q \longrightarrow \mathrm{ext}^1_D(M, D) = D^q/(R\, D^p).$$

The previous equivalences only depend on the residue class $\rho(\Lambda)$.

Theorem

• Theorem: Let $R \in D^{q \times p}$ be a full row rank matrix and $\Lambda \in D^q$ such that there exists $U \in \mathrm{GL}_{p+1}(D)$ satisfying:

$$(R \quad -\Lambda)\, U = (I_q \quad 0).$$

Let us denote by

$$U = \begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix} \in \mathrm{GL}_{p+1}(D),$$

where:

$$S_1 \in D^{p \times q},\; S_2 \in D^{1 \times q},\; Q_1 \in D^{p \times (p+1-q)},\; Q_2 \in D^{1 \times (p+1-q)}.$$

Then, we have:

$$\boxed{M = D^{1 \times p}/(D^{1 \times q}\, R) \;\cong\; L = D^{1 \times (p+1-q)}/(D\, Q_2)}$$

The converse result also holds. These results only depend on:

$$\rho(\Lambda) \in \mathrm{ext}^1_D(M, D) \triangleq D^q/(R\, D^p),\; \rho : D^q \longrightarrow D^q/(R\, D^p).$$

Ring conditions

- Proposition: Let $R \in D^{q \times p}$ be a full row rank matrix and $\Lambda \in D^q$ such that $P = (R \quad -\Lambda) \in D^{q \times (p+1)}$ admits a right-inverse over D. Moreover, if D is either a

  1. principal left ideal domain,
  2. commutative polynomial ring with coefficients in a field,
  3. Weyl algebra $A_n(k)$ or $B_n(k)$, where k is a field of characteristic 0, and $p - q \geq 1$,

  then there exists $U \in \mathrm{GL}_{p+1}(D)$ satisfying that $P\,U = (I_q \quad 0)$.

- The matrix U can be obtained by means of:
  1. a Jacobson form (JACOBSON),
  2. the Quillen-Suslin theorem (QUILLENSUSLIN),
  3. Stafford's theorem (STAFFORD).

Example: Wind tunnel model

- The wind tunnel model (Manitius, IEEE TAC 84):

$$\begin{cases} \dot{x}_1(t) + a\,x_1(t) - k\,a\,x_2(t-h) = 0, \\ \dot{x}_2(t) - x_3(t) = 0, \\ \dot{x}_3(t) + \omega^2\,x_2(t) + 2\,\zeta\,\omega\,x_3(t) - \omega^2\,u(t) = 0. \end{cases}$$

- Let us consider $D = \mathbb{Q}(a, k, \omega, \zeta)[\partial, \delta]$, the system matrix

$$R = \begin{pmatrix} \partial + a & -k\,a\,\delta & 0 & 0 \\ 0 & \partial & -1 & 0 \\ 0 & \omega^2 & \partial + 2\,\zeta\,\omega & -\omega^2 \end{pmatrix} \in D^{3\times 4},$$

and the finitely presented D-module $M = D^{1\times 4}/(D^{1\times 3}\,R)$.

- The D-module $\mathrm{ext}^1_D(M, D) = D^3/(R\,D^4)$ is a $\mathbb{Q}(a, k, \omega, \zeta)$- vector space of dimension 1 and $\rho((1 \quad 0 \quad 0)^T)$ is a basis.

Example: Wind tunnel model

- Let us consider $\Lambda = (1 \quad 0 \quad 0)^T$ and $P = (R \quad -\Lambda)$.
- The matrix P admits the following right-inverse S:

$$S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -\frac{\partial + 2\,\zeta\,\omega}{\omega^2} & -\frac{1}{\omega^2} \\ -1 & 0 & 0 \end{pmatrix} \in D^{5 \times 3}.$$

- According to Quillen-Suslin theorem, $E = D^{1\times 5}/(D^{1\times 3}\, P)$ is free D-module of rank 2.

Example: Wind tunnel model

- Computing a basis of E, we obtain that $U \in \mathrm{GL}_5(D)$,

$$U = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \omega^2 \\ 0 & -1 & 0 & 0 & \omega^2\,\partial \\ 0 & -\frac{\partial+2\,\zeta\,\omega}{\omega^2} & -\frac{1}{\omega^2} & 0 & \partial^2+2\,\zeta\,\omega\,\partial+\omega^2 \\ -1 & 0 & 0 & -(\partial+a) & -\omega^2\,k\,a\,\delta \end{pmatrix},$$

satisfies that $P\,U = (I_3 \quad 0)$ (OreModules, QuillenSuslin).

- The wind tunnel model is equivalent to the sole equation:

$$(\partial+a)\,\zeta_1+\omega^2\,k\,a\,\delta\,\zeta_2=0$$

$$\Leftrightarrow \quad \dot{\zeta}_1(t)+a\,\zeta_1(t)+\omega^2\,k\,a\,\zeta_2(t-h)=0.$$

Equivalence

- Theorem: If $\Lambda \in D^q$ admits a left-inverse $\Gamma \in D^{1\times q}$, i.e., $\Gamma \, \Lambda = 1$, then Q_1 admits the left-inverse $T_1 + T_2 \, \Gamma \, R \in D^{(p+1-q)\times p}$ and the left D-module $\ker_D(.Q_1)$ is stably free of rank $q-1$:

$$\ker_D(.Q_1) \oplus D^{1\times(p+1-q)} \cong D^{1\times p}.$$

If the left D-module $\ker_D(.Q_1)$ is free, then $\exists \, Q_3 \in D^{p\times(q-1)}$ s.t.:

$$V = (Q_3 \quad Q_1) \in \mathrm{GL}_p(D).$$

Then, we have $W = (R\, Q_3 \quad \Lambda) \in \mathrm{GL}_q(D)$,

$$W^{-1} = \begin{pmatrix} Y_3 \, S_1 \\ -S_2 + Q_2 \, Y_1 \, S_1 \end{pmatrix},$$

with $V^{-1} = (Y_3^T \quad Y_1^T)^T$, $Y_3 \in D^{(q-1)\times p}$, $Y_1 \in D^{(p-q+1)\times p}$ and:

$$W^{-1} \, R \, V = \begin{pmatrix} I_{q-1} & 0 \\ 0 & Q_2 \end{pmatrix}.$$

Example: Wind tunnel model

- The vector $\Lambda = (1 \quad 0 \quad 0)^T$ admits the left-inverse $\Gamma = \Lambda^T$.
- We compute $Q_3 \in D^{2\times 2}$ such that $V = (Q_3^T \quad Q_1^T) \in \mathrm{GL}_4(D)$:

$$V = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \omega^2 \\ 0 & -1 & 0 & \omega^2\,\partial \\ -\frac{1}{\omega^2} & -\frac{\partial+2\,\zeta\,\omega}{\omega^2} & 0 & \partial^2 + 2\,\zeta\,\omega\,\partial + \omega^2 \end{pmatrix}.$$

- We have $W = (R\,Q_3 \quad \Lambda) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathrm{GL}_3(D)$ and:

$$W^{-1}\,R\,V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -(\partial + a) & -\omega^2\,k\,a\,\delta \end{pmatrix}.$$

III

Implementation

The SERRE package (in progress)

Algorithmic issue

1. Consider an ansatz $\Lambda \in D^q$ of a given order.
2. Compute a Gröbner basis of $D^q/(R\,D^p)$.
3. Compute the normal form $\overline{\Lambda} \in D^q$ of $\rho(\Lambda)$.
4. Compute the obstructions to freeness of the left D-module $\overline{E} = D^{1\times(p+1)}/(D^{1\times q}\,(R \quad -\overline{\Lambda}))$ (π-polynomials).
5. Solve the systems in the arbitrary coefficients obtained by making the obstructions vanish.
6. If a solution $\Lambda_\star$ exists, then compute $U \in \mathrm{GL}_{p+1}(D)$ satisfying that $(R \quad -\Lambda)\,U = (I_q \quad 0)$ and return $Q_2 \in D^{1\times(p+1-q)}$.

The SERRE package

• Algorithms are implemented in a Maple package called SERRE based on the library OREMODULES developed by Q. et Robertz:

http://wwwb.math.rwth-aachen.de/OreModules

• List of functions:

- FreeExtension (distinct variants)
- SerreReduction (distinct variants)
- SerreEquivalence (distinct variants)

• It will be soon freely available with a library of examples at:

http://www.ensil.unilim.fr/~cluzeau/Serre

Demo

Maple demo

IV

Going further

The holonomic case (in progress)

Example: Stress tensor (elasticity)

- Let $D = \mathbb{Q}[\partial_x, \partial_y]$ and $M = D^{1\times 3}/(D^{1\times 2}\, R)$, where:

$$R = \begin{pmatrix} \partial_x & \partial_y & 0 \\ 0 & \partial_x & \partial_y \end{pmatrix} \in D^{2\times 3}.$$

- The D-module $\mathrm{ext}^1_D(M, D) = D^2/(R\, D^3)$ is a $\mathbb{Q}$-vector space of dimension 3 with basis $\{\rho((1 \quad 0)^T),\ \rho((0 \quad 1)^T),\ \rho((0 \quad \partial_x)^T)\}$.
- Let us consider $\Lambda = (a \quad b + c\,\partial_x)^T$, $P = (R \quad -\Lambda)$.
- If we denote by $A = D[a, b, c]$ and $N = A^2/(P\, A^4)$, then we get:

$$\mathrm{ext}^1_A(N, A) = 0, \quad \mathrm{ext}^2_A(N, A) = A/(\partial_x, \partial_y).$$

- Hence, $E = A^{1\times 4}/(A^{1\times 2}\, P)$ is never a projective A-module and

$$\begin{cases} \partial_x\, \sigma^{11} + \partial_y\, \sigma^{12} = 0, \\ \partial_x\, \sigma^{12} + \partial_y\, \sigma^{22} = 0, \end{cases}$$

cannot be defined by a sole equation!

Example: Stress tensor (elasticity)

- Let $D' = A_2(\mathbb{Q})$ be the Weyl algebra
- Remark that the $N' = D'^2/(R\, D'^4)$ is a holonomic D'-module
- It is thus cyclic and a cyclic vector $C \in \mathbb{Q}[x, y]^2$ can be computed using A. Leykin algorithm
- We find that $C = (1 \quad x)^T$ is a cyclic vector which implies that $P' = (R \quad -C)$ admits a right-inverse S:

$$S = \begin{pmatrix} -x & 1 \\ -x^2 & x \\ -x^3 & x^2 \\ -2 - x\,\partial_x - x^2\partial_y & \partial_x + x\,\partial_y \end{pmatrix}.$$

- The D'-module $E' = D'^{1\times 4}/(D'^{1\times 2}\, P')$ is thus free by Stafford theorem.

Example: Stress tensor (elasticity)

- Using SERRE or OREMORPHISMS, we find unimodular matrices U and V such that

$$V\,P'\,U = \begin{pmatrix} \partial_x\,x & -\partial_x + \partial_y\,x & -\partial_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- From these, we can easily deduce that

$$U' = \begin{pmatrix} -x & 1 & -\partial_y & -\partial_x + \partial_y\,x \\ -x^2 & x & -\partial_y\,x & 1 - \partial_x\,x + x^2\partial_y \\ -x^3 & x^2 & 1 - x^2\partial_y & -\partial_x\,x^2 + x^3\partial_y \\ -2 - \partial_x\,x - x^2\partial_y & \partial_x + \partial_y\,x & -\partial_x\,\partial_y - {\partial_y}^2 x & -{\partial_x}^2 + 2\,\partial_y + {\partial_y}^2 x^2 \end{pmatrix}$$

satisfies $P'\,U' = (I_2 \quad 0)$.

Example: Stress tensor (elasticity)

• As in Serre's reduction, we thus conclude that over $D' = A_2(\mathbb{Q})$, the stress tensor system defined by

$$R = \begin{pmatrix} \partial_x & \partial_y & 0 \\ 0 & \partial_x & \partial_y \end{pmatrix}$$

is equivalent to

$$\overline{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \partial_x \, \partial_y + x \partial_y{}^2 & \partial_x{}^2 - 2 \, \partial_y - x^2 \partial_y{}^2 \end{pmatrix}.$$

• The stress tensor system can thus be defined by a sole equation!

General Result

• Theorem: Let D be the Weyl algebra $A_n(k)$ or $B_n(k)$, where k is a field of characteristic 0.
Let $R \in D^{q \times p}$, $M = D^{1 \times p}/(D^{1 \times q}\, R)$ and $N = \mathrm{ext}^1_D(M, D) = D^q/(R\, D^p)$.

If N is a holonomic module and $p - q \geq 1$, then there exists $Q \in D^{1 \times (p+1-q)}$ such that

$$\boxed{M \;\cong\; D^{1 \times (p+1-q)}/(D\, Q).}$$

If furthermore $q \geq 3$, then there exist two unimodular matrices U and V such that

$$V\, R\, U = \begin{pmatrix} I_{q-1} & 0 \\ 0 & Q \end{pmatrix}.$$