Local Reduced Forms of Systems of Linear Functional Equations and Applications

Moulay Barkatou
moulay.barkatou@unilim.fr

XLIM UMR 6172 CNRS-Université de Limoges
FRANCE.

ACA’09
AADIOS Session
Montreal, Canada, 25-28 June, 2009
Moser and super-irreducible forms play a central role in the local analysis (and also in the global study) of linear systems of differential equations


Algorithms for constructing such forms have been developed and implemented in the Maple package ISOLDE http://isolde.sourceforge.net.

This talk is about the generalization of these notions to the class of systems of linear functional equations which comprises common types of systems:

- Difference systems \( Y(x + 1) = M(x)Y(x) \)
- \( q \)-Difference systems \( Y(qx) = M(x)Y(x) \)
- Differential systems \( Y'(x) = M(x)Y(x) \)
Introduce a unifying framework that permits to treat, simultaneously, all types of linear functional systems.

Use the language of pseudo-linear derivations over a field of discrete valuation.

Derive a definition of regularity and develop an algorithm for recognizing regular systems inspired by Moser’s work on differential equations.

Extend the notion of super-irreducible forms to these class of systems and develop an algorithm to compute it.

Application: the approach developed in [Bar 1999, Abramov &Bark 1998] for computing polynomial solution for differential and difference systems is also applicable for $q$—difference systems.

Notation

\( \mathbb{Q} \subset \mathbb{K} \subset \mathbb{C} \)

\( \mathcal{O} = \mathbb{K}[\![x]\!][x^{-1}] \) the ring of formal power series in \( x \) over \( \mathbb{K} \)

\( \mathbb{F} = \mathbb{K}((x)) = \mathbb{K}[\![x]\!][x^{-1}] \) the quotient field of \( \mathcal{O} \)

\( \mathbb{F} \) is equipped with the \( x-\)adic valuation \( \nu : \)

If

\[ f = \sum_{i} f_{i} x^{i} \in \mathbb{F} \setminus \{0\} \]

then

\[ \nu(f) = \inf \{ i \text{ such that } f_{i} \neq 0 \} \]

For a matrix \( A = (a_{ij}) \) with coefficients in \( \mathbb{F} \) put

\[ \nu(A) = \min \nu(a_{ij}) \]

\( \partial = \frac{d}{dx} = ' = \) the standard derivation of \( \mathbb{F} \).
Linear differential systems with a singularity at $x = 0$

Can be written as

$$\partial [A] \quad \partial Y = AY$$

where $Y$ vector of length $n$ and $A \in \mathbb{F}^{n \times n}$ with $v(A) \leq -1$.

**Gauge transformations**

Given $T \in GL(n, \mathbb{F})$, the change of variable

$$Y = TZ$$

does not yield the equivalent system

$$\partial [B] \quad \partial Z = BZ$$

where

$$B = T_\partial [A] := T^{-1}AT - T^{-1}\partial T.$$
Classification of Singularities

Consider a system

\[ \partial[A] \quad \partial Y = A(x) Y \]

\[ A = x^{\nu(A)} \sum_{j=0}^{\infty} A_j x^j, \quad A_j \in K^{n \times n} \quad \text{with} \quad A_0 \neq 0 \]

**Definition** \( \partial[A] \) is

- **regular singular** if \( \exists T \in GL(n, F) \) s.t. \( \nu(T\partial[A]) \geq -1 \).

- **irregular singular** if not regular singular.
How to recognize a regular singular system?

▶ **Problem 1**: Given a system \( \partial [A] \) with \( v(A) < -1 \), to decide whether it is regular singular or not and in case it is regular to compute a gauge transformation \( T \) s. t. \( v(T \partial [A]) \geq -1 \).

▶ **Problem 2**: Given a system \( \partial [A] \) with \( v(A) < -1 \), to decide whether there exists \( T \in \text{GL}(n, F) \) such that \( v(T \partial [A]) > v(A) \).

▶ There is an algorithm due to Moser (1960) which transforms a given system \( \partial [A] \) to an equivalent one with maximal valuation.
Moser Reduced Systems

- **Moser rank:**

\[
m(A) = \begin{cases} 
-1 - \nu(A) + \frac{\text{rank}(A_0)}{n} & \text{if } \nu(A) < -1 \\
0 & \text{if } \nu(A) \geq -1
\end{cases}
\]

- **Moser invariant:**

\[
\mu(A) = \min_{T \in \text{GL}(n, K((x))))} (m(T \partial [A]))
\]

- System \([A]\) is regular singular iff \(\mu(A) = 0\).

- **Definition:** \([A]\) is said to be Moser reducible if \(m(A) > \mu(A)\).
A Reducibility Criterion

**Theorem.** [Moser 1960]

If \( \nu(A) < -1 \) then \([A]\) is Moser reducible iff the polynomial

\[
\Theta_A(\lambda) := x^{\text{rank}(A_0)} \det (\lambda I - A_0/x - A_1)|_{x=0} \equiv 0.
\]

If \([A]\) is Moser reducible then the reduction can be carried out with a transformation of the form

\[
S(x) = C \text{ diag}(x^{d_1}, \ldots, x^{d_n})
\]

where \(C\) is constant matrix and where \(d_i \in \{0, 1\}\).
Moser Algorithm

- Applying the above Theorem several times, if necessary, $\mu(A)$ can be determined.

- Further, a matrix polynomial $T \in GL(n, F)$ such that $m(T[A]) = \mu(A)$ can be computed in this way.

- This leads to an efficient algorithm which transforms a given system into a Moser reduced one [Barkatou&Pfluegel 2007].

- This algorithm and its extensions (super-reduced forms) plays a central role even in the irregular singular case.
Generalization to other linear functional systems

- To adapt existing algorithms and their implementations for use with other common types of linear functional systems.
- Find a unifying framework that allows to treat, simultaneously, all types of linear functional systems.
- Need the language of pseudo-linear derivations over a field of discrete valuation.
- Derive a definition of regularity and develop a method for recognizing regular systems.
- Concepts of Moser- and Super-irreducible forms extended to linear functional systems (Barkatou, Broughton and Pfluegel (ISSAC 2008)).
Discrete valuation fields – Basic Objects

- $F$ a commutative discrete valuation field of characteristic zero and $\nu$ the valuation of $F$.

  $\nu : F \longrightarrow \mathbb{Z} \cup \{+\infty\}$ is a map with the following properties:

  (i) $\nu(f) = +\infty \iff f = 0$,
  (ii) $\nu(fg) = \nu(f) + \nu(g)$,
  (iii) $\nu(f + g) \geq \min (\nu(f), \nu(g))$

- $O = \{f \in F : \nu(f) \geq 0\}$ is the valuation ring of $F$.

- $M = \{f \in F : \nu(f) > 0\}$ is the (unique) maximal ideal of $O$.

- $\overline{F} := O/M$ is the residue field of $F$.

- $\pi$ the canonical homomorphism from $O$ onto $\overline{F}$.

- An element $t \in O$ is a local parameter if $\nu(t) = 1$. 

- An element $t \in O$ is a local parameter if $\nu(t) = 1$. 

The valuation ring $\mathcal{O}$ is a principal ideal ring.

Proper ideals of $\mathcal{O}$ are of the form $t^m\mathcal{O}$, $m \in \mathbb{N}^*$. In particular, $\mathcal{M} = t\mathcal{O}$.

We suppose that $\mathbb{F}$ is complete with respect to the valuation $v$ and fix a local parameter $t$ of $\mathbb{F}$.

Let $R$ be a set of representatives of $\mathcal{O}$ in $\overline{\mathbb{F}}$, that is a subset $R$ of $\mathcal{O}$ such that $0 \in R$ and the map: $a \in R \mapsto \pi(a) \in \mathcal{O}/\mathcal{M} = \overline{\mathbb{F}}$ is bijective.

Every $f \in \mathbb{F}$ can be uniquely expanded as

$$f = \sum_{i=\nu}^{+\infty} f_i t^i,$$

where $\nu = \nu(f)$ and $f_i \in R$ with $f_\nu \neq 0$. 


Example

\( F = \mathbb{C}((x^{-1})) \) equipped with the \( x^{-1} \)-adic valuation:

\[
f = \sum_{i=i_0}^{+\infty} f_i x^{-i}
\]

with \( f_{i_0} \neq 0 \) then \( v(f) = i_0 \).

\begin{itemize}
  \item The valuation ring of \( F \) is \( O = \mathbb{C}[[x^{-1}]] \).
  \item The residue field of \( F \) is \( \mathbb{C} \).
  \item \( \pi(f) = f(\infty) \) for \( f \in O \).
  \item \( t = x^{-1} \) is a local parameter of \( F \).
\end{itemize}
Pseudo-derivations on discrete valuation fields

- Let $\phi$ be an isometry: automorphism of $F$ s. t. $\nu(\phi f) = \nu(f)$ for all $f \in F$.

- A Pseudo-derivation or $\phi$–derivation is a map $\delta : F \to F$ satisfying, for all $a, b \in F$

  $$\delta(a + b) = \delta a + \delta b \quad \text{and} \quad \delta(ab) = \phi a \delta b + \delta a b \quad \text{(Leibniz rule)}$$

- The field of constants of $F$ is

  $$C = \{ f \in F : \phi f = f \ \text{and} \ \delta f = 0 \}.$$

- When $\phi = 1_F$ the identity map of $F$ then $\delta$ is simply a standard derivation.

- When $\phi \neq 1_F$ then $\delta$ is of the form $\delta = \gamma(1_F - \phi)$ for some $\gamma \in F$. 
Examples of Pseudo-derivations

1. Differential case:
   - $F = \mathbb{C}((x - x_0))$, $x_0 \in \mathbb{C}$, or $F = \mathbb{C}((x^{-1}))$
   - $\phi = 1_F$ and $\delta = \partial = \frac{d}{dx}$

2. Difference case:
   - $F = \mathbb{C}((x^{-1}))$
   - $\phi$ is the $\mathbb{C}$–automorphism defined by $\phi x = x - 1$
   - $\delta = 1 - \phi$

3. $q$–Difference case:
   - $F = \mathbb{C}((x))$ or $F = \mathbb{C}((x^{-1}))$,
   - $\phi$ the $\mathbb{C}$–automorphism defined by $\phi x = qx$, $q \in \mathbb{C} \setminus \{0, 1\}$
   - $\delta = 1 - \phi$. 
Systems of Pseudo-Linear Equations

A system of **pseudo-linear** equations over \((F, \phi, \delta)\) is a system of the form

\[
\delta,\phi[A] \quad \delta(Y) = A\phi(Y)
\]

where \(A\) is an \(n \times n\) matrix in \(F\) and where \(Y\) is an unknown \(n\)–dimensional column vector.

**Gauge Equivalent Systems**

The **gauge transformation** \(Y = TZ\), \(T \in \text{GL}(n,F)\) leads to the **equivalent** system

\[
\delta(Z) = B\phi(Z)
\]

where

\[
B = T^{-1} A\phi(T) - T^{-1} \delta(T) =: T_{\delta,\phi}[A].
\]
Degree of a derivation

Given \( \delta \neq 0 \) fixed, we define its \textit{degree} as

\[
\omega(\delta) = \inf_{f \in \mathbb{F}, f \neq 0} v(f^{-1}\delta(f)).
\]

**Proposition** If \( \delta \neq 0 \) is continuous then

\[
\omega(\delta) = v(t^{-1}\delta(t))
\]

for some local parameter \( t \).
Examples

1. $F = \mathbb{C}((x))$, $t = x$, $\phi = 1_F$ and $\delta = \frac{d}{dx}$.

One has $\omega(\delta) = -1 = \nu(x^{-1}\delta x)$

2. $F = \mathbb{C}((x^{-1}))$, $t = x^{-1}$, $\phi = 1_F$ and $\delta = \frac{d}{dx}$.

One has $\omega(\delta) = +1 = \nu(t^{-1}\delta(t)) = \nu(x\frac{1}{x^2})$.

3. $F = \mathbb{C}((x^{-1}))$, $\phi$ the $\mathbb{C}$–automorphism defined by $\phi x = x - 1$ and $\delta = 1 - \phi$.

One has $\omega(\delta) = 1 = \nu(t^{-1}\delta(t)) = \nu(x(\frac{1}{x} - \frac{1}{x-1}))$.

4. $F = \mathbb{C}((x^{-1}))$, $\phi$ the $\mathbb{C}$–automorphism defined by $\phi x = qx$, $q \in \mathbb{C} \setminus \{0, 1\}$ and $\delta = 1 - \phi$.

In this case $\omega(\delta) = 0 = \nu(t^{-1}\delta(t)) = \nu(x(\frac{1}{x} - \frac{1}{qx}))$. 
Regular Systems

Consider a pseudo-linear system $\delta, \phi[A]$ with

$$A = t^{\nu(A)} \sum_{i=0}^{+\infty} A_i t^i$$

where the $A_i$ are matrices with entries in $R$ with $A_0 \neq 0$.

Definition

A pseudo-linear system $\delta, \phi[A]$ is called regular if there exists a gauge transformation $T \in \text{GL}(n, F)$ such that $\nu(T_{\delta, \phi}[A]) \geq \omega(\delta)$. 
Problems to solve

Given a pseudo-linear system $\delta, \phi [A]$,

1) to decide whether $\delta, \phi [A]$ is regular or not

2) and in case it is regular, to compute a transformation $T$ such that $\nu(T_{\delta, \phi} [A]) \geq \omega(\delta)$.

3) More generally, to compute a gauge transformation $T$ such that $\nu(T_{\delta, \phi} [A])$ be as close as possible to $\omega(\delta)$. 
An analogue of Moser irreducible form

We associate with the system $\delta,\phi[A]$ the following rational numbers:

$$m_{\delta,\phi}(A) = \begin{cases} 
\omega(\delta) - \nu(A) + \frac{\text{rank}(A_0)}{n} & \text{if } \nu(A) < \omega(\delta) \\
0 & \text{if } \nu(A) \geq \omega(\delta)
\end{cases}$$

and

$$\mu_{\delta,\phi}(A) = \min \{ m_{\delta,\phi}(T_{\delta,\phi}[A]) \mid T \in \text{GL}(n, F) \}.$$ 

**Definition**

The matrix $A$ is called **irreducible** w.r.t. $(\delta, \phi)$ if $m_{\delta,\phi}(A) = \mu_{\delta,\phi}(A)$, otherwise it is called **reducible**.

**Remark** : The system $\delta,\phi[A]$ is regular if and only if $\mu_{\delta,\phi}(A) = 0$. 
A Reducibility Criterion

The following result is the analogue of the Moser’s Theorem which gives a reducibility criterion in the differential case.

**Theorem** [BBP, ISSAC 2008]
Suppose that $\nu(A) < \omega(\delta)$. Then the system $\delta, \phi[A]$ is reducible if and only if the polynomial

$$\theta(\lambda) = \pi \left( t^{\text{rank}(A_0)} \det (t^{-1}A_0 + A_1 - \lambda I) \right)$$

vanishes identically in $\lambda$. 
Reduction Algorithm

When $\delta,\phi[A]$ is reducible, how to compute a transformation $T$ that reduces $m_{\delta,\phi}(A)$?

- We could simply “copy” (or adapt) Moser algorithm and this would work (in fact it works).
- But we propose to proceed in a different way: more “economic” and interesting in itself.
Let $F = K((x))$ equipped with the $x$-adic valuation. The residue field of $F$ can be identified with $K$. We take $t = x$ as local parameter.

We fix $\phi$ and $\delta$ and assume that $\phi \neq 1_F$, $\delta \neq 0$.

Since $\delta = \gamma(1_F - \phi)$ for some nonzero $\gamma \in F$ we can assume that $\omega(\delta) = -1$.

Let $A \in F^{n \times n}$ with $v(A) < -1 = \omega(\delta)$

$$A = x^{v(A)} \sum_{j=0}^{\infty} A_j x^j, \quad A_j \in K^{n \times n} \quad \text{with} \quad A_0 \neq 0$$

and consider the pseudo-linear system $\delta, \phi[A]$. 
The matrix $A$ defines also a differential system $\partial,1_F[A]$.

Our assumption $\omega(\delta) = -1 = \omega(\partial)$ implies that

$$m_{\delta,\phi}(A) = m_{\partial,1_F}(A).$$

We shall refer to this quantity simply by $m(A)$.

Notice also that the two polynomials $\theta(\lambda)$ associated with the systems $\delta,\phi[A]$ and $\partial,1_F[A]$ coincide.

Therefore we have the following interesting property:

*the matrix $A$ is reducible w.r.t. $(\delta, \phi)$ if and only if it is w.r.t. $(\partial, 1_F)$.*
Suppose now that $A$ is reducible and let $T$ be a transformation such that $m(T_{\partial,1_F}[A]) < m(A)$.

One question arises naturally:

is it true that we have also $m(T_{\delta,\phi}[A]) < m(A)$?

The answer to this question is “NO”, in general.

However, the answer is “YES” for transformations $T$ of the form $T = CS$ where $C$ is a constant matrix and $S = \text{diag}(x^{d_1}, \ldots, x^{d_n})$ with $d_i \in \{0, 1\}$.

This is precisely the type of transformation used in each reduction step in the algorithm for the Moser-reduction in the differential case.
Pseudo_Linear_Reduction_1(\(A, \phi, \delta\))

Input:
\begin{itemize}
  \item \(A \in F^{n \times n}\), for some valuation field \(F\),
  \item \(\phi\) an isometry of \(F\),
  \item \(\delta\) a \(\phi\)-derivation.
\end{itemize}

Output:
\begin{itemize}
  \item Transformation \(T\) such that \(T_{\phi,\delta}[A]\) is Moser–irreducible
\end{itemize}

1. \(T := I_n;\)

2. \textbf{while} \((\nu(A) < \omega(\delta))\) \textbf{and} \((\theta(\lambda) \equiv 0)\) \textbf{do}
   \begin{enumerate}
     \item Compute a transformation \(CS\) as in the differential case;
     \item \(A := S_{\phi,\delta}[A]; T := TCS;\)
   \end{enumerate}

3. \textbf{return} \(T;\)
Super–irreducible Forms

- Consider a system $\delta,\phi[A]$ with $\nu(A) < \omega(\delta)$.
- For $k = 1, \ldots, \omega(\delta) - \nu(A)$, put

  \[ m_{\delta,\phi}^k(A) = \omega(\delta) - \nu(A) + \frac{n_0}{n^0} + \frac{n_1}{n^1} + \cdots + \frac{n_{k-1}}{n^k} \]

  where $n_i = \# \text{ of rows of } A \text{ with valuation } \nu(A) + i$.
- Define

  \[ \mu_{\delta,\phi}^k(A) = \min\{ m_{\delta,\phi}^k(T_{\delta,\phi}[A]) \mid T \in \text{GL}(n, F) \} \].

- The matrix $A$ is said to be $k$-irreducible w.r.t. $(\delta, \phi)$ if $m_{\delta,\phi}^k(A) = \mu_{\delta,\phi}^k(A)$. Otherwise $A$ is called $k$-reducible.
- The matrix $A$ is said to be super–irreducible, if it is $k$-irreducible for every $k$, or equivalently if

  \[ m_{\delta,\phi}^{\omega(\delta) - \nu(A)}(A) = \mu_{\delta,\phi}^{\omega(\delta) - \nu(A)}(A). \]
A Criterion for $k$–reducibility

- One defines

$$s_k = s_k(A) := kn_0 + (k - 1)n_1 + \cdots + n_{k-1}$$

and

$$\Theta_k(\lambda) := t^{s_k} \det(t^{-\nu(A)-k}A - \lambda I_n)$$

- One verifies that $\Theta_k(\lambda)$ belongs to $\mathcal{O}[\lambda]$.

- One can define then the polynomial $\theta_k(\lambda) \in \mathbb{R}[\lambda]$ as

$$\theta_k(\lambda) = \pi\left(t^{s_k} \det(t^{-\nu(A)-k}A - \lambda I_n)\right).$$

- One can prove the following

**Theorem** The matrix $A$ is $k$–irreducible, if and only if the polynomials $\theta_j(\lambda)$, $(j = 1, \ldots, k)$, do not vanish identically in $\lambda$. 
In [BP 2007] it was shown that, in the differential case, the computation of a super-irreducible system can be reduced to the computation of several Moser-irreducible systems of smaller size.

In [BBP 2008] we show that this algorithm can be adapted for systems of linear functional equations in order to obtain a second reduction algorithm which can be characterised as computing a new system of the form $\delta Y = B\phi Y$ where $B$ is a block-triangular super-reduced matrix where each diagonal block is Moser-reduced.
Our reduction algorithms are implemented in the Maple package ISOLDE.

New input parameters required in ISOLDE:

- Procedures defining $\phi$ and $\delta$ respectively.
- The degree $\omega$ of $\delta$.
- Calling procedure in ISOLDE: \texttt{super\_reduce}(A, x, x_0, lambda, phi, delta, omega);

?\texttt{super\_reduce} displays additional user information.
Example 1

- Let $F = \mathbb{C}((x^{-1}))$, $\phi$ the $\mathbb{C}$–automorphism defined by $\phi x = x - 1$ and $\delta = 1 - \phi$. One has $\omega(\delta) = 1$.

- Consider the difference system $\delta(Y) = A\phi(Y)$ where

$$A = \begin{pmatrix}
5 x^{-1} & x^2 & 0 \\
0 & 0 & x^{-3} \\
x^{-3} & 0 & 3 x^{-1} + 5 x^{-2}
\end{pmatrix}.$$ 

- The valuation of $A$ is $v(A) = -2 < \omega(\delta) = 1$ and

$$m_{\delta,\phi}(A) = \omega(\delta) - v(A) + \frac{\text{rank} \ A_0}{n} = \frac{10}{3}.$$ 

- We assign $\phi$ and $\delta$ in Maple:

```maple
> phi:= proc(a,x) return subs(x=x-1,a) end:
> delta:=proc(a,x) return a-phi(a,x) end:
```
Calling the super-reduction on this example gives a list of results, the first entry being the transformed system that is super-irreducible.

\[
\begin{pmatrix}
\frac{6x^2 - 9x + 4}{(x-1)^3} & 0 & (x - 1)^{-1} \\
\frac{x^2}{(x-1)^3} & 5x^{-1} & 0 \\
-3 \frac{2x^3 - 6x^2 + 10x - 5}{x(x-1)^3} & x^{-2} & \frac{5+x}{x(x-1)}
\end{pmatrix}
\]

This resulting system has \( v(A) = 1 = \omega(\delta) \) hence, it is a regular system.

As part of the third entry of the results, one obtains in this example the **indicial equation** of the system:

\[
[[0, \lambda^3 - 12 \lambda^2 + 47 \lambda - 60]]
\]
Example 2

Let \( F = \mathbb{C}((x^{-1})) \), \( \phi \) the \( \mathbb{C} \)-automorphism defined by \( \phi x = qx \), \( q \in \mathbb{C} \setminus \{0, 1\} \) and \( \delta = 1 - \phi \). In this case \( \omega(\delta) = 0 \).

Consider the \( q \)-difference system \( \delta(Y) = A\phi(Y) \) where

\[
A = \begin{pmatrix}
q^3x^4 & -x \\
x^3 & -1
\end{pmatrix}.
\]

We keep \( \delta = 1 - \phi \) and define \( \phi \) by the following:

\[
> \text{phi} := \text{proc}(a,x) \text{ return subs}(x=q*x,a) \text{ end:}
\]

This system is Moser-irreducible but not super-irreducible.
We call the super-reduction on this example:

\[
> \text{tmp := super\_reduce(A, x, 1/x, lambda, phi, delta, 1, S, invS): tmp[1];}
\]

\[
\begin{pmatrix}
q^3 x^4 - 1 & 0 \\
x^3 & -\frac{\lambda - 1 + q^3}{q^3}
\end{pmatrix}
\]

Additional output returns a list of polynomials:

\[
> \text{tmp[3];}
\]

\[
[[0, 1 - q^3 - q^3 \lambda], [1, 1], [2, 1], [3, 1], [4, \lambda - q^3]].
\]
Polynomial Solutions

- Let $q \in K$ with $q \neq 1$ and consider a linear $q$–difference system with coefficients in $K(x)$:

$$Y(qx) = M(x)Y(x), \quad M(x) \in K(x)^{n \times n}.$$  \hfill (1)

- We are interested in the problem of computing all the polynomial solutions of a system of the form (1).

- Algorithms for solving this problem in the differential and the difference cases have been proposed in [Bar99, Abramov&Bar98].

- In [BBP2008] we show that the same approach remains valid for the $q$–difference case.

- Note that a different approach for finding polynomial solutions of systems of this type is given in [Abramov95].
**Polynomial Solutions**

- A first important step consists in computing a bound $N$ on the degree of polynomial solutions.

- Such a bound can be obtained from the so-called **indicial equation** (at $x = \infty$) of the system.

- But the indicial equation is not immediately apparent for a given system.

- However when the system is super-irreducible then its indicial equation (at $\infty$) is given by the polynomial $\theta_r(\lambda)$ where $r := \max(-\nu(A), 0)$.

- Every system can be reduced to an equivalent super-irreducible one using a polynomial gauge transformation.
Idea:

- Work with $F = K((x^{-1}))$, the completion of $K(x)$ w.r.t. to the $t$–adic valuation (here $t = x^{-1}$).
- Define $\phi$ and $\delta$ by $\phi(t) = qt$ and $\delta = 1_F - \phi$.
- The original system (1) can then be written as the pseudo-linear system:

\[ \delta Y = A(t) \phi Y \quad (2) \]

where $A(t) = M(q^{-1} t^{-1}) - I_n \in M_n(F)$.
- Apply our super-reduction algorithm to make $A$ super-irreducible.

**Proposition** If $Y \in K[x]^n$ is a nonzero polynomial solution of degree $\nu$ then $\theta_r(q^{-\nu} - 1) = 0$.

The degree of polynomial solutions can be bounded by the biggest nonnegative integer $\nu$ such that $q^{-\nu} - 1$ is a root of $\theta_r(\lambda)$. 
Example

- Let $A = \begin{pmatrix} q^3 x^4 & -x \\ x^3 & -1 \end{pmatrix}$.

- $> \text{tmp} := \text{super\_reduce}(A, x, 1/x, \lambda, \phi, \delta, 1, S, \text{invS})$; \text{tmp}[1];

  \[
  \begin{pmatrix}
  q^3 x^4 - 1 & 0 \\
  x^3 & \frac{-1+q^3}{q^3}
  \end{pmatrix}
  \]

- $> \text{tmp}[3]$;

  $[[0, 1 - q^3 - q^3 \lambda], [1, 1], [2, 1], [3, 1], [4, \lambda - q^3]]$.

- This system admits a polynomial solution $Y = \begin{pmatrix} 1 \\ x^3 \end{pmatrix}$ of degree $\nu = 3$ and one indeed verifies that $\theta_r(q^{-3} - 1) = 0$ where $\theta_r(\lambda)$ is given by the first polynomial in the above output.