Local Reduced Forms of Systems of Linear Functional Equations and Applications

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- Moser and super-irreducible forms play a central role in the local analysis (and also in the global study) of liner systems of differential equations

  [Moser 1960, Hilali&Wazner 1987, Bark 1989, Bark 1997, Bark 2004, Pfluegel 2000, Bark&Pfluegel 2007]

- Algorithms for constructing such forms have been developed and implemented in the Maple package ISOLDE `http://isolde.sourceforge.net`.

- This talk is about the generalization of these notions to the class of *systems of linear functional equations* which comprises common types of systems :
  - Difference systems $Y(x+1) = M(x)Y(x)$
  - q-Difference systems $Y(qx) = M(x)Y(x)$
  - Differential systems $Y'(x) = M(x)Y(x)$

- Introduce a unifying framework that permits to treat, simultaneously, all types of linear functional systems.
- Use the language of pseudo-linear derivations over a field of discrete valuation.
- Derive a definition of regularity and develop an algorithm for recognizing regular systems inspired by Moser's work on differential equations.
- Extend the notion of super-irreducible forms to these class of systems and develop an algorithm to compute it.
- Application: the approach developed in [Bar 1999, Abramov &Bark 1998] for computing polynomial solution for differential and difference systems is also applicable for $q-$difference systems.

Notation

$\mathbb{Q} \subset \mathrm{K} \subset \mathbb{C}$

$\mathcal{O} = \mathrm{K}[[x]]$ the ring of formal power series in x over K

$\mathrm{F} = \mathrm{K}((x)) = \mathrm{K}[[x]][x^{-1}]$ the quotient field of $\mathcal{O}$

F is equipped with the $x-$*adic* valuation v :
If

$$f = \sum_i f_i x^i \in \mathrm{F} \setminus \{0\}$$

then

$$v(f) = \inf \{i \text{ such that } f_i \neq 0\}.$$

For a matrix $A = (a_{ij})$ with coefficients in F put

$$v(A) = \min v(a_{ij}).$$

$\partial = \frac{d}{dx} = \ ' =$ the standard derivation of F.

Linear differential systems with a singularity at $x = 0$

Can be written as

$$_\partial[A] \qquad\qquad \partial Y = AY$$

where Y vector of lenght n and $A \in \mathrm{F}^{n\times n}$ with $v(A) \leq -1$.

Gauge transformations

Given $T \in GL(n, \mathrm{F})$, the change of variable

$$Y = TZ$$

yields the equivalent system

$$_\partial[B] \qquad\qquad \partial Z = BZ$$

where

$$B = T_\partial[A] := T^{-1}AT - T^{-1}\partial T.$$

Classification of Singularities

Consider a system

$$_{\partial}[A] \qquad\qquad \partial Y = A(x) Y$$

$$A = x^{v(A)} \sum_{j=0}^{\infty} A_j x^j, \quad A_j \in \mathrm{K}^{n \times n} \quad \text{with} \;\; A_0 \neq 0$$

Definition $_{\partial}[A]$ is

- regular singular if $\exists T \in GL(n, \mathrm{F})$ s.t. $v(T_{\partial}[A]) \geq -1$.
- irregular singular if not regular singular.

How to recognize a regular singular system?

- **Problem 1**: Given a system $[A]_\partial$ with $v(A) < -1$, to decide whether it is regular singular or not and in case it is regular to compute a gauge transformation T *s. t.* $v(T_\partial[A]) \geq -1$.

- **Problem 2**: Given a system $[A]_\partial$ with $v(A) < -1$, to decide whether there exists $T \in \mathrm{GL}(n, \mathrm{F})$ such that $v(T_\partial[A]) > v(A)$.

- There is an algorithm due to Moser (1960) which transforms a given system $[A]_\partial$ to an equivalent one with maximal valuation.

Moser Reduced Systems

- **Moser rank**:

$$m(A) = \begin{cases} -1 - v(A) + \frac{\mathrm{rank}\,(A_0)}{n} & \text{if} \quad v(A) < -1 \\ 0 & \text{if} \quad v(A) \geq -1 \end{cases}$$

- **Moser invariant**:

$$\mu(A) = \min_{T \in \mathrm{GL}(n, K((x)))} (m(T_\partial[A]))$$

- System $[A]$ is regular singular iff $\mu(A) = 0$.
- **Definition:** $[A]$ is said to be Moser reducible if $m(A) > \mu(A)$.

A Reducibility Creterion

- **Theorem.** [Moser 1960]

  If $v(A) < -1$ then $[A]$ is Moser reducible iff the polynomial

  $$\Theta_A(\lambda) := x^{rank(A_0)} \det\left(\lambda I - A_0/x - A_1\right)_{|x=0} \equiv 0.$$

- If $[A]$ is Moser reducible then the reduction can be carried out with a transformation of the form

  $$S(x) = C\ \mathrm{diag}(x^{d_1}, \cdots, x^{d_n})$$

  where C is constant matrix and where $d_i \in \{0, 1\}$.

Moser Algorithm

- Applying the above Theorem several times, if necessary, $\mu(A)$ can be determined.
- Further, a matrix polynomial $T \in GL(n, \mathrm{F})$ such that $m(T[A]) = \mu(A)$ can be computed in this way.
- This leads to an efficient algorithm which transforms a given system into a Moser reduced one [Barkatou&Pfluegel 2007].
- This algorithm and its extensions (super-reduced forms) plays a central role even in the irregular singular case.

Generalization to other linear functional systems

- To adapt existing algorithms and their implementations for use with *other common types of linear functional system*
- Find a unifying framework that allows to treat, simultaneously, all types of linear functional systems.
- Need the language of pseudo-linear derivations over a field of discrete valuation.
- Derive a definition of regularity and develop a method for recognizing regular systems.
- Concepts of Moser- and Super-irreducible forms extended to linear functional systems (Barkatou, Broughton and Pfluegel (ISSAC 2008)).

Discrete valuation fields – Basic Objects

- F a commutative discrete valuation field of characteristic zero and v the valuation of F.

  $v : \mathrm{F} \longrightarrow \mathbb{Z} \cup \{+\infty\}$ is a map with the following properties :
  (i) $v(f) = +\infty \iff f = 0$,
  (ii) $v(fg) = v(f) + v(g)$,
  (iii) $v(f+g) \geq \min(v(f), v(g))$

- $\mathcal{O} = \{f \in \mathrm{F} \ : \ v(f) \geq 0\}$ is the *valuation ring* of F.
- $\mathfrak{M} = \{f \in \mathrm{F} : v(f) > 0\}$ is the (unique) maximal ideal of $\mathcal{O}$.
- $\bar{\mathrm{F}} := \mathcal{O}/\mathfrak{M}$ is the residue field of F.
- π the canonical homomorphism from $\mathcal{O}$ onto $\bar{\mathrm{F}}$.
- An element $t \in \mathcal{O}$ is a local parameter if $v(t) = 1$.

- The valuation ring $\mathcal{O}$ is a principal ideal ring.
- Proper ideals of $\mathcal{O}$ are of the form $t^m\mathcal{O}$ $m \in \mathbb{N}^*$. In particular, $\mathfrak{M} = t\mathcal{O}$.
- We suppose that F is complete with respect to the valuation v and fix a local parameter t of F.
- Let R be a set of representatives of $\mathcal{O}$ in $\bar{\mathrm{F}}$, that is a subset R of $\mathcal{O}$ such that $0 \in R$ and the map: $a \in R \mapsto \pi(a) \in \mathcal{O}/\mathfrak{M} = \bar{\mathrm{F}}$ is bijective.
- Every $f \in \mathrm{F}$ can be uniquely expanded as

$$f = \sum_{i=\nu}^{+\infty} f_i t^i,$$

where $\nu = v(f)$ and $f_i \in R$ with $f_\nu \neq 0$.

Example

$\mathrm{F} = \mathbb{C}((x^{-1}))$ equipped with the x^{-1}-adic valuation :
if

$$f = \sum_{i=i_0}^{+\infty} f_i x^{-i}$$

with $f_{i_0} \neq 0$ then $v(f) = i_0$.

- The valuation ring of F is $\mathcal{O} = \mathbb{C}[[x^{-1}]]$.
- The residue field of F is $\mathbb{C}$.
- $\pi(f) = f(\infty)$ for $f \in \mathcal{O}$.
- $t = x^{-1}$ is a local parameter of F.

Pseudo-derivations on discrete valuation fields

- Let ϕ be an isometry: automorphism of F *s. t.* $v(\phi f) = v(f)$ for all $f \in \mathrm{F}$.

- A *Pseudo-derivation* or $\phi-$derivation is a map $\delta : \mathrm{F} \rightarrow \mathrm{F}$ satisfying, for all $a, b \in \mathrm{F}$

  $\delta(a+b) = \delta a + \delta b$ and $\delta(ab) = \phi a \, \delta b + \delta a \, b$ (Leibniz rule)

- The field of *constants* of F is

$$\mathcal{C} = \{f \in \mathrm{F} \ : \ \phi f = f \text{ and } \delta f = 0\}.$$

- When $\phi = 1_{\mathrm{F}}$ the identity map of F then δ is simply a standard derivation.
- When $\phi \neq 1_{\mathrm{F}}$ then δ is of the form $\delta = \gamma(1_{\mathrm{F}} - \phi)$ for some $\gamma \in \mathrm{F}$.

Examples of Pseudo-derivations

1. Differential case:
   - $\mathrm{F} = \mathbb{C}((x - x_0))$, $x_0 \in \mathbb{C}$, or $\mathrm{F} = \mathbb{C}((x^{-1}))$
   - $\phi = 1_{\mathrm{F}}$ and $\delta = \partial = \frac{d}{dx}$
2. Difference case:
   - $\mathrm{F} = \mathbb{C}((x^{-1}))$
   - ϕ is the $\mathbb{C}-$automorphism defined by $\phi x = x - 1$
   - $\delta = 1 - \phi$
3. $q-$Difference case:
   - $\mathrm{F} = \mathbb{C}((x))$ or $\mathrm{F} = \mathbb{C}((x^{-1}))$,
   - ϕ the $\mathbb{C}-$automorphism defined by $\phi x = qx$, $q \in \mathbb{C} \setminus \{0, 1\}$
   - $\delta = 1 - \phi$.

Systems of Pseudo-Linear Equations

A system of *pseudo-linear* equations over $(\mathrm{F}, \phi, \delta)$ is a system of the form

$$_{\delta,\phi}[A] \qquad\qquad \delta(Y) = A\phi(Y)$$

where A is an $n \times n$ matrix in F and where Y is an unknown $n-$dimensional column vector.

Gauge Equivalent Systems

The *gauge transformation* $Y = TZ$, $T \in \mathrm{GL}(n, \mathrm{F})$ leads to the *equivalent* system

$$\delta(Z) = B\phi(Z)$$

where

$$B = T^{-1}A\phi(T) - T^{-1}\delta(T) =: T_{\delta,\phi}[A].$$

Degree of a derivation

Given $\delta \neq 0$ fixed, we define its *degree* as

$$\omega(\delta) = \inf_{f \in \mathrm{F}, f \neq 0} v(f^{-1}\delta(f)).$$

Proposition If $\delta \neq 0$ is continuous then

$$\omega(\delta) = v(t^{-1}\delta(t))$$

for some local parameter t.

Examples

1. $\mathrm{F} = \mathbb{C}((x))$, $t = x$, $\phi = 1_{\mathrm{F}}$ and $\delta = \frac{d}{dx}$.

   One has $\omega(\delta) = -1 = v(x^{-1}\delta x)$

2. $\mathrm{F} = \mathbb{C}((x^{-1}))$, $t = x^{-1}$, $\phi = 1_{\mathrm{F}}$ and $\delta = \frac{d}{dx}$.

   One has $\omega(\delta) = +1 = v(t^{-1}\delta(t)) = v(x\frac{-1}{x^2})$.

3. $\mathrm{F} = \mathbb{C}((x^{-1}))$, ϕ the $\mathbb{C}-$automorphism defined by $\phi x = x - 1$ and $\delta = 1 - \phi$.

   One has $\omega(\delta) = 1 = v(t^{-1}\delta(t)) = v(x(\frac{1}{x} - \frac{1}{x-1}))$.

4. $\mathrm{F} = \mathbb{C}((x^{-1}))$, ϕ the $\mathbb{C}-$automorphism defined by $\phi x = qx$, $q \in \mathbb{C} \setminus \{0, 1\}$ and $\delta = 1 - \phi$.

   In this case $\omega(\delta) = 0 = v(t^{-1}\delta(t)) = v(x(\frac{1}{x} - \frac{1}{qx}))$.

Regular Systems

Consider a pseudo-linear system ${}_{\delta,\phi}[A]$ with

$$A = t^{v(A)} \sum_{i=0}^{+\infty} A_i t^i$$

where the A_i are matrices with entries in R with $A_0 \neq 0$.

Definition

A pseudo-linear system ${}_{\delta,\phi}[A]$ is called regular if there exists a gauge transformation $T \in \mathrm{GL}(n, \mathrm{F})$ such that $v(T_{\delta,\phi}[A]) \geq \omega(\delta)$.

Problems to solve

Given a pseudo-linear system ${}_{\delta,\phi}[A]$,

1) to decide whether ${}_{\delta,\phi}[A]$ is regular or not

2) and in case it is regular, to compute a transformation T such that $v(T_{\delta,\phi}[A]) \geq \omega(\delta)$.

3) More generally, to compute a gauge transformation T such that $v(T_{\delta,\phi}[A])$ be as close as possible to $\omega(\delta)$.

An analogue of Moser irreducible form

We associate with the system ${}_{\delta,\phi}[A]$ the following rational numbers:

$$m_{\delta,\phi}(A) = \begin{cases} \omega(\delta) - v(A) + \frac{rank(A_0)}{n} & \text{if} \quad v(A) < \omega(\delta) \\ 0 & \text{if} \quad v(A) \geq \omega(\delta) \end{cases}$$

and

$$\mu_{\delta,\phi}(A) = \min \{ m_{\delta,\phi}(T_{\delta,\phi}[A]) \mid T \in \mathrm{GL}(n, \mathrm{F}) \}.$$

Definition

The matrix A is called irreducible w.r.t. (δ, ϕ) if $m_{\delta,\phi}(A) = \mu_{\delta,\phi}(A)$, otherwise it is called reducible.

Remark : The system ${}_{\delta,\phi}[A]$ is regular if and only if $\mu_{\delta,\phi}(A) = 0$.

A Reducibility Criterion

The following result is the analogue of the Moser's Theorem which gives a reducibility criterion in the differential case.

Theorem [BBP, ISSAC 2008]
Suppose that $v(A) < \omega(\delta)$. Then the system $_{\delta,\phi}[A]$ is reducible if and only if the polynomial

$$\theta(\lambda) = \pi\left(t^{\mathrm{rank}\,(A_0)}\,\mathrm{det}\left(t^{-1}A_0 + A_1 - \lambda I\right)\right)$$

vanishes identically in λ.

Reduction Algorithm

When ${}_{\delta,\phi}[A]$ is reducible, how to compute a transformation T that reduces $m_{\delta,\phi}$ (A) ?

- We could simply “copy” (or adapt) Moser algorithm and this would work (in fact it works)
- But we propose to proceed in a different way : more “economic ” and interesting in itself.

- Let $\mathrm{F} = \mathrm{K}((x))$ equipped with the x-adic valuation.
  The residue field of F can be identified with K.
  We take $t = x$ as local parameter.

- We fix ϕ and δ and assume that $\phi \neq 1_{\mathrm{F}}$, $\delta \neq 0$.

- Since $\delta = \gamma(1_{\mathrm{F}} - \phi)$ for some nonzero $\gamma \in \mathrm{F}$ we can assume that $\omega(\delta) = -1$.

- Let $A \in \mathrm{F}^{n\times n}$ with $v(A) < -1 = \omega(\delta)$

$$A = x^{v(A)} \sum_{j=0}^{\infty} A_j x^j, \quad A_j \in \mathrm{K}^{n\times n} \quad \text{with} \;\; A_0 \neq 0$$

  and consider the pseudo-linear system ${}_{\delta,\phi}[A]$.

- The matrix A defines also a differential system $_{\partial,1_{\mathrm{F}}}[A]$.
- Our assumption $\omega(\delta) = -1 = \omega(\partial)$ implies that

$$m_{\delta,\phi}(A) = m_{\partial,1_{\mathrm{F}}}(A).$$

- We shall refer to this quantity simply by $m(A)$.
- Notice also that the two polynomials $\theta(\lambda)$ associated with the systems $_{\delta,\phi}[A]$ and $_{\partial,1_{\mathrm{F}}}[A]$ coincide.
- Therefore we have the following interesting property:

the matrix A is reducible w.r.t. (δ, ϕ) if and only if it is w.r.t. $(\partial, 1_{\mathrm{F}})$.

- Suppose now that A is reducible and let T be a transformation such that $m(T_{\partial, 1_{\mathrm{F}}}[A]) < m(A)$.
- One question arises naturally :
  is it true that we have also $m(T_{\delta,\phi}[A]) < m(A)$?
- The answer to this question is "NO" , in general.
- However, the answer is "YES" for transformations T of the form $T = CS$ where C is a constant matrix and $S = \mathrm{diag}(x^{d_1}, \ldots, x^{d_n})$ with $d_i \in \{0, 1\}$.
- This is precisely the type of transformation used in each reduction step in the algorithm for the Moser-reduction in the differential case.

Pseudo_Linear_Reduction_1(A, ϕ, δ)

Input:

- $A \in \mathrm{F}^{n\times n}$, for some valuation field F,
- ϕ an isometry of F,
- δ a ϕ-derivation.

Output:

- Transformation T such that $T_{\phi,\delta}[A]$ is Moser–irreducible

1. $T := I_n$;
2. **while** $(v(A) < \omega(\delta))$ **and** $(\theta(\lambda) \equiv 0)$ **do**
    2.1 Compute a transformation CS as in the differential case;
    2.2 $A := S_{\phi,\delta}[A]$; $T := TCS$;
3. **return** T;

Super–irreducible Forms

- Consider a system ${}_{\delta,\phi}[A]$ with $v(A) < \omega(\delta)$.
- For $k = 1, \ldots, \omega(\delta) - v(A)$, put

$$m^k_{\delta,\phi}(A) = \omega(\delta) - v(A) + \frac{n_0}{n} + \frac{n_1}{n^2} + \cdots + \frac{n_{k-1}}{n^k}$$

where $n_i = \#$ of rows of A with valuation $v(A) + i$.

- Define

$$\mu^k_{\delta,\phi}(A) = \min\{m^k_{\delta,\phi}(T_{\delta,\phi}[A]) \mid T \in \mathrm{GL}(n, \mathrm{F})\}.$$

- The matrix A is said to be k-irreducible w.r.t. (δ, ϕ) if $m^k_{\delta,\phi}(A) = \mu^k_{\delta,\phi}(A)$. Otherwise A is called k-reducible.
- The matrix A is said to be super–irreducible, if it is k-irreducible for every k, or equivalently if

$$m^{\omega(\delta)-v(A)}_{\delta,\phi}(A) = \mu^{\omega(\delta)-v(A)}_{\delta,\phi}(A).$$

A Criterion for $k-$reducibility

- One defines

$$s_k = s_k(A) := kn_0 + (k-1)n_1 + \cdots + n_{k-1}$$

and

$$\Theta_k(\lambda) := t^{s_k} \det(t^{-v(A)-k} A - \lambda I_n)$$

- One verifies that $\Theta_k(\lambda)$ belongs to $\mathcal{O}[\lambda]$.
- One can define then the polynomial $\theta_k(\lambda) \in R[\lambda]$ as

$$\theta_k(\lambda) = \pi\left(t^{s_k} \det(t^{-v(A)-k} A - \lambda I_n)\right).$$

- One can prove the following

Theorem The matrix A is k–irreducible, if and only if the polynomials $\theta_j(\lambda)$, $(j = 1, \ldots, k)$, do not vanish identically in λ.

- In [BP 2007] it was shown that, in the differential case, the computation of a super-irreducible system can be reduced to the computation of several Moser-irreducible systems of smaller size.

- In [BBP 2008] we show that this algorithm can be adapted for systems of linear functional equations in order to obtain a second reduction algorithm which can be characterised as computing a new system of the form $\delta Y = B\phi Y$ where B is a block-triangular super-reduced matrix where each diagonal block is Moser-reduced.

- Our reduction algorithms are implemented in the Maple package ISOLDE.
- New input parameters required in ISOLDE:
  - Procedures defining ϕ and δ respectively.
  - The *degree* ω of δ.
  - Calling procedure in ISOLDE : `super_reduce(`A`, `x`, `x_0`, lambda, phi, delta, omega);`
- `?super_reduce` displays additional user information.

Example 1

- Let $\mathrm{F} = \mathbb{C}((x^{-1}))$, ϕ the $\mathbb{C}-$automorphism defined by $\phi x = x - 1$ and $\delta = 1 - \phi$. One has $\omega(\delta) = 1$.

- Consider the difference system $\delta(Y) = A\phi(Y)$ where

$$A = \begin{pmatrix} 5\,x^{-1} & x^2 & 0 \\ 0 & 0 & x^{-3} \\ x^{-3} & 0 & 3\,x^{-1} + 5\,x^{-2} \end{pmatrix}.$$

- The valuation of A is $v(A) = -2 < \omega(\delta) = 1$ and

$$m_{\delta,\phi}(A) = \omega(\delta) - v(A) + \frac{\mathrm{rank}\ A_0}{n} = \frac{10}{3}.$$

- We assign ϕ and δ in Maple:

```
> phi:= proc(a,x) return subs(x=x-1,a) end:
> delta:=proc(a,x) return a-phi(a,x) end:
```

- Calling the super-reduction on this example gives a list of results, the first entry being the transformed system that is super-irreducible.
- `> tmp := super_reduce(A, x, 1/x, lambda, phi, delta, 1, S, invS)`: tmp[1];

$$\begin{pmatrix} \frac{6\,x^2-9\,x+4}{(x-1)^3} & 0 & (x-1)^{-1} \\ \frac{x^2}{(x-1)^3} & 5\,x^{-1} & 0 \\ -3\,\frac{2\,x^3-6\,x^2+10\,x-5}{x(x-1)^3} & x^{-2} & \frac{5+x}{x(x-1)} \end{pmatrix}$$

- This resulting system has $v(A) = 1 = \omega(\delta)$ hence, it is a regular system.
- As part of the third entry of the results, one obtains in this example the indicial equation of the system:
  `>tmp[3];`

$$[[0, \lambda^3 - 12\,\lambda^2 + 47\,\lambda - 60]]$$

Example 2

- Let $\mathrm{F} = \mathbb{C}((x^{-1}))$, ϕ the $\mathbb{C}-$automorphism defined by $\phi x = qx$, $q \in \mathbb{C} \setminus \{0,1\}$ and $\delta = 1 - \phi$. In this case $\omega(\delta) = 0$.

- Consider the q-difference system $\delta(Y) = A\phi(Y)$ where

$$A = \begin{pmatrix} q^3x^4 & -x \\ x^3 & -1 \end{pmatrix}.$$

- We keep $\delta = 1 - \phi$ and define ϕ by the following:

```
> phi:= proc(a,x) return subs(x=q*x,a) end:
```

- This system is Moser-irreducible but not super-irreducible.

- We call the super-reduction on this example:

```
> tmp := super_reduce(A, x, 1/x, lambda,
phi, delta, 1, S, invS): tmp[1];
```

$$\begin{pmatrix} q^3x^4-1 & 0 \\ x^3 & -\frac{-1+q^3}{q^3} \end{pmatrix}$$

- Additional output returns a list of polynomials:

```
> tmp[3];
```

$$[[0, 1-q^3-q^3\lambda], [1,1], [2,1], [3,1], [4, \lambda-q^3]].$$

Polynomial Solutions

- Let $q \in \mathrm{K}$ with $q \neq 1$ and consider a linear $q-$difference system with coefficients in $\mathrm{K}(x)$:

$$Y(qx) = M(x)Y(x), \quad M(x) \in K(x)^{n \times n}. \tag{1}$$

- We are interested in the problem of computing all the polynomial solutions of a system of the form (1).
- Algorithms for solving this problem in the differential and the difference cases have been proposed in [Bar99, Abramov&Bar98].
- In [BBP2008] we show that the same approach remains valid for the $q-$difference case.
- Note that a different approach for finding polynomial solutions of systems of this type is given in [Abramov95].

Polynomial Solutions

- A first important step consists in computing a bound N on the degree of polynomial solutions.
- Such a bound can be obtained from the so-called indicial equation (at $x = \infty$) of the system.
- But the indicial equation is not immediately apparent for a given system.
- However when the system is super-irreducible then its indicial equation (at ∞) is given by the polynomial $\theta_r(\lambda)$ where $r := \max(-v(A), 0)$.
- Every system can be reduced to an equivalent super-irreducible one using a polynomial gauge transformation

- Idea:
  - Work with $\mathrm{F} = \mathrm{K}((x^{-1}))$, the completion of $\mathrm{K}(x)$ w.r.t. to the $t-$adic valuation (here $t = x^{-1}$).
  - Define ϕ and δ by $\phi(t) = qt$ and $\delta = 1_{\mathrm{F}} - \phi$.
  - The original ystem (1) can then be written as the pseudo-linear system:

$$\delta Y = A(t)\phi Y \tag{2}$$

    where $A(t) = M(q^{-1}t^{-1}) - I_n \in \mathcal{M}_n(\mathrm{F})$.
  - Apply our super-reduction algorithm to make A super-irreducible.
- **Proposition** If $Y \in \mathrm{K}[x]^n$ is a nonzero polynomial solution of degree ν then $\theta_r(q^{-\nu} - 1) = 0$.

- The degree of polynomial solutions can be bounded by the biggest nonnegative integer ν such that $q^{-\nu} - 1$ is a root of $\theta_r(\lambda)$.

Example

- Let $A = \begin{pmatrix} q^3x^4 & -x \\ x^3 & -1 \end{pmatrix}$.
- `> tmp := super_reduce(A, x, 1/x, lambda, phi, delta, 1, S, invS):` tmp[1];

$$\begin{pmatrix} q^3x^4 - 1 & 0 \\ x^3 & -\frac{-1+q^3}{q^3} \end{pmatrix}$$

- `> tmp[3];`

$$[[0, 1 - q^3 - q^3\lambda], [1, 1], [2, 1], [3, 1], [4, \lambda - q^3]].$$

- This system admits a polynomial solution $Y = \begin{pmatrix} 1 \\ x^3 \end{pmatrix}$ of degree $\nu = 3$ and one indeed verifies that $\theta_r(q^{-3} - 1) = 0$ where $\theta_r(\lambda)$ is given by the first polynomial in the above output.