ACA, July 2008

Relative Gröbner bases and dimension polynomials in difference-differential modules

Franz Winkler
RISC, JKU

We report on joint work with M. Zhou of Beihang University, Beijing

Supported by Chinese NSFC projects 60473019 and KLMM0705, and by the Austrian FWF projects P16357-N04 and SFB F1304.

This talk is based on

M.Zhou, F.Winkler,
"Computing difference-differential dimension polynomials by relative Gröbner bases in difference-differential modules",
J. Symbolic Computation 43/10, 726–745 (2008)

and

M.Zhou, F.Winkler,
"Gröbner bases in difference-differential modules",
Proc. ISSAC 2006, 353–360

1. Introduction

- we introduce the concept of generalized term order on $\mathbb{N}^m \times \mathbb{Z}^n$ and on difference-differential modules
- we introduce the concept of relative Gröbner basis w.r.t. two different generalized term orders
- based on relative Gröbner bases we present a new algorithmic approach for computing the Hilbert function and Hilbert polynomial (or dimension polynomial) of difference-differential modules

Difference-differential modules

Def.1.1:

- R a commutative Noetherian ring
- $\Delta = \{\delta_1, \cdots, \delta_m\}$ a set of derivations on R
  $\Sigma = \{\sigma_1, \cdots, \sigma_n\}$ a set of automorphisms of R
  s.t. $\alpha \circ \beta = \beta \circ \alpha$ for any $\alpha, \beta \in \Delta \cup \Sigma$
- Then R is called a *difference-differential ring* with the basic set of derivations Δ and the basic set of automorphisms Σ, or shortly a *Δ-Σ-ring*;
- if R is a field, then it is called a *Δ-Σ-field*.

Ex.1.1: Let $R = K[x_1, \ldots, x_n]$ for a field K, $\delta_i = \partial/\partial x_i$ and σ_i the automorphism which maps x_i to $x_i - 1$. Then R is a Δ-Σ-ring for $\Delta = \{\delta_1, \ldots, \delta_n\}$ and $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$.

Def.1.2:

- R a Δ-Σ-ring
- Λ the free commutative semigroup of words over Δ and $\tilde{\Sigma}$ (containing the elements of Σ and their inverses)
- then a *difference-differential operator* (or shortly a *Δ-Σ-operator*) over R is

$$\sum_{\lambda \in \Lambda} a_\lambda \lambda, \tag{1.2}$$

where $a_\lambda \in R$ and only finitely many coefficients a_λ are different from 0 fundamental relations:

$$
\begin{array}{rcl}
\sum_{\lambda\in\Lambda} a_\lambda\lambda + \sum_{\lambda\in\Lambda} b_\lambda\lambda & = & \sum_{\lambda\in\Lambda}(a_\lambda + b_\lambda)\lambda, \\
a(\sum_{\lambda\in\Lambda} a_\lambda\lambda) & = & \sum_{\lambda\in\Lambda}(aa_\lambda)\lambda, \\
(\sum_{\lambda\in\Lambda} a_\lambda\lambda)\mu & = & \sum_{\lambda\in\Lambda} a_\lambda(\lambda\mu), \\
\delta a = a\delta + \delta(a), & & \sigma a = \sigma(a)\sigma,
\end{array}
\tag{1.3}
$$

for $a_\lambda, b_\lambda \in R$, $\lambda, \mu \in \Lambda$, $a \in R$, $\delta \in \Delta$, $\sigma \in \tilde{\Sigma}$

Def.1.3: The (non-commutative) ring of all Δ-Σ-operators over a Δ-Σ-ring R is called the *ring of difference-differential operators* (or shortly the *ring of Δ-Σ-operators*) over R. It will be denoted by D.

A left D-module M is called a *difference-differential module* (or a Δ-Σ-*module*).

- for $\Sigma = \emptyset$: $D = R[\delta_1, \cdots, \delta_m]$, the ring of differential operators
- if $R = K[x_1, \ldots, x_m]$ over a field K and $\delta_i = \partial / \partial x_i$ for $1 \leq i \leq m$, then D is the Weyl algebra $\mathcal{A}_m(K)$.

In the ring of Δ-Σ-operators the terms are of the form

$$\lambda = \delta_1^{k_1} \cdots \delta_m^{k_m} \sigma_1^{l_1} \cdots \sigma_m^{l_n}$$

and the exponents of σ_i are in $\mathbb{Z}$. The notion of term order, as commonly used in Gröbner basis theory, is no longer valid. We need to generalize the concept of term order.

2. Generalized term order

Def.2.1: Let $\mathbb{Z}^n = \bigcup_{j=1}^{k} \mathbb{Z}^n_j$, where

(i) $(0, \cdots, 0) \in \mathbb{Z}^n_j$, and $\mathbb{Z}^n_j$ does not contain any pair of invertible elements $c = (c_1, \cdots, c_n) \neq 0$ and $-c = (-c_1, \cdots, -c_n)$,

(ii) $\mathbb{Z}^n_j$ is isomorphic to $\mathbb{N}^n$ as a semigroup,

(iii) the group generated by $\mathbb{Z}^n_j$ is $\mathbb{Z}^n$.

Then $\{\mathbb{Z}^n_j \mid j = 1, \cdots, k\}$ is called an *orthant decomposition* of $\mathbb{Z}^n$ and $\mathbb{Z}^n_j$ is called the *j-th orthant* of the decomposition.

Ex.2.1: Let $\mathbb{Z}^n_1, \cdots, \mathbb{Z}^n_{2^n}$ be all distinct Cartesian products of n sets, each of which is either $\mathbb{Z}_+$ or $\mathbb{Z}_-$. Then this is an orthant decomposition of $\mathbb{Z}^n$. The set of generators of $\mathbb{Z}^n_j$ as a semigroup is

$$\{(c_1, 0, \cdots, 0), (0, c_2, 0, \cdots, 0), \cdots, (0, \cdots, 0, c_n)\},$$

where c_j is either 1 or -1, $j = 1, \cdots, n$.

This is the *canonical orthant decomposition* of $\mathbb{Z}^n$.

Ex.2.2: The decomposition $\mathbb{Z}^2 = \mathbb{Z}_0^2 \cup \mathbb{Z}_1^2 \cup \mathbb{Z}_2^2$, where

$$\mathbb{Z}_0^2 = \{(a,b)|a \geq 0, b \geq 0, a, b \in \mathbb{Z}\},$$

$$\mathbb{Z}_1^2 = \{(a,b)|a \leq 0, b \geq a, a, b \in \mathbb{Z}\},$$

$$\mathbb{Z}_2^2 = \{(a,b)|b \leq 0, a \geq b, a, b \in \mathbb{Z}\},$$

is an orthant decomposition of $\mathbb{Z}^2$.

Def. 2.4: Let $\{\mathbb{Z}_j^n \mid j = 1, \cdots, k\}$ be an orthant decomposition of $\mathbb{Z}^n$. Let $E = \{e_1, \cdots, e_q\}$ be a set of q distinct elements. A total order $\prec$ on $\mathbb{N}^m \times \mathbb{Z}^n \times E$ is called a *generalized term order* on $\mathbb{N}^m \times \mathbb{Z}^n \times E$ w.r.t. the decomposition, if the following conditions hold:

(i) $(0, \cdots, 0, e_i)$ is the smallest element in $\mathbb{N}^m \times \mathbb{Z}^n \times \{e_i\}$ for any $e_i \in E$,

(ii) if $(a, e_i) \prec (b, e_j)$, then $(a + c, e_i) \prec (b + c, e_j)$ for any c similar to b.

Example 2.3: Given the canonical orthant decomposition of $\mathbb{Z}^n$, an order "$\prec'$" in $E = \{e_1, \cdots, e_q\}$, for two elements

$$\begin{aligned} a &= (k_1, \cdots, k_m, l_1, \cdots, l_n, e_i) \text{ and} \\ b &= (r_1, \cdots, r_m, s_1, \cdots, s_n, e_j) \end{aligned}$$

of $\mathbb{N}^m \times \mathbb{Z}^n \times E$ define:

$$|a|_1 = \sum_{j=1}^{m} k_j, \qquad |a|_2 = \sum_{j=1}^{n} |l_j|.$$

$$\begin{aligned} &(a, e_i) \prec (b, e_j) :\Longleftrightarrow \\ &\quad (|a|_1, |a|_2, e_i, k_1, \cdots, k_m, |l_1|, \cdots, |l_n|, l_1, \cdots, l_n) \\ &< (|b|_1, |b|_2, e_j, r_1, \cdots, r_m, |s_1|, \cdots, |s_n|, s_1, \cdots, s_n) \end{aligned}$$

in lexicographic order.
Then "$\prec$" is a generalized term order on $\mathbb{N}^m \times \mathbb{Z}^n \times E$.

□

Lemma 2.1: Let $\{\mathbb{Z}_j^n \mid j = 1, \cdots, k\}$ be an orthant decomposition of $\mathbb{Z}^n$ and "$\prec$" be a generalized term order on $\mathbb{N}^m \times \mathbb{Z}^n$ with respect to the orthant decomposition.

(a) Every strictly descending sequence in $\mathbb{N}^m \times \mathbb{Z}^n$ is finite. In particular, every subset of $\mathbb{N}^m \times \mathbb{Z}^n$ contains a smallest element.

(b) Every strictly descending sequence in $\mathbb{N}^m \times \mathbb{Z}^n \times E$ is finite. In particular, every subset of $\mathbb{N}^m \times \mathbb{Z}^n \times E$ contains a smallest element.

3. Relative Gröbner bases

In the following let $\{\mathbb{Z}_j^n, j = 1, \ldots, k\}$ be a fixed orthant decomposition of $\mathbb{Z}^n$.

Definition 3.1: The subset Λ_j of Λ,

$$\Lambda_j = \{\lambda = \delta_1^{k_1} \cdots \delta_m^{k_m} \sigma_1^{l_1} \cdots \sigma_n^{l_n} \;\mid\; (l_1, \cdots, l_n) \in \mathbb{Z}_j^n\},$$

is called j-th orthant of Λ.
Let F be a finitely generated free D-module and ΛE be the set of terms of F. Then

$$\Lambda_j E = \{\lambda e_i \;\mid\; \lambda \in \Lambda_j, \;\; e_i \in E\}$$

is called j-th orthant of ΛE. □

Lemma: Let F be a finitely generated (with generators in E) free D-module and $f \in F \setminus \{0\}$. Let $\lambda \in \Lambda$ and $a \in R$, "$\prec$" be a generalized term order on $\Lambda E \subseteq D$. Then

(i) $\lambda a = a'\lambda + \xi$, where $a' = \alpha(a)$ for some $\alpha \in \Sigma^*$ (if $a \neq 0$ then $a' \neq 0$), $\xi \in D$ with $lt(\xi) \prec \lambda$ and all terms of ξ are in the same orthant as λ.

(ii) If $\lambda \in \Lambda$, then $lt(\lambda f) = \max_{\prec}\{\lambda u_i\}$ where u_i are terms of f, and $lt(\lambda f) = \lambda u$ for a unique term u of f.

(iii) If $lt(f) \in \Lambda_j E$ then $lt(\lambda f) = \lambda lt(f) \in \Lambda_j E$ holds for any $\lambda \in \Lambda_j$.

(iv) For each j there exists some $\lambda \in \Lambda$ and a term u_j of f such that

$$lt(\lambda f) = \lambda u_j \in \Lambda_j E.$$

Furthermore, the term u_j of f is unique: if $lt(\lambda_1 f) = \lambda_1 u_{j_1} \in \Lambda_j E$ and $lt(\lambda_2 f) = \lambda_2 u_{j_2} \in \Lambda_j E$ then $u_{j_1} = u_{j_2}$. We will write $lt_j(f)$ for the term u_j.

Theorem 3.1: Let $\prec$ and $\prec'$ be two generalized term orders on ΛE. Let $g_1, \cdots, g_p \in F \setminus \{0\}$ and $f \in F$. Then

$$f = h_1 g_1 + \cdots + h_p g_p + r \tag{3.2}$$

for some elements $h_1, \cdots, h_p \in D$ and $r \in F$ such that

(i) $h_i = 0$ or $lt_\prec(h_i g_i) \preceq lt_\prec(f)$, $i = 1, \cdots, p$;

(ii) $r = 0$ or $lt_\prec(r) \preceq lt_\prec(f)$ and

$$lt_\prec(r) \notin \{lt_\prec(\lambda g_i) \mid lt_{\prec'}(\lambda g_i) \preceq' lt_{\prec'}(r), \lambda \in \Lambda, i = 1, \cdots, p\}$$

Definition 3.2: Let $\prec$ and $\prec'$ be two generalized term orders on ΛE. Let $g_1, \cdots, g_p \in F \setminus \{0\}$ and $f \in F$. Suppose that the equation (3.2) holds and that the conditions (i), (ii) in Theorem 3.1 are satisfied. If $r \neq f$ we say that f is $\prec$-reducible to r modulo $\{g_1, \cdots, g_p\}$ relative to $\prec'$.
In case $r = f$ and $h_i = 0$, $i = 1, \cdots, p$, we say that f is $\prec$-reduced modulo $\{g_1, \cdots, g_p\}$ relative to $\prec'$. $\square$

Example 3.1: Let the sets Δ and Σ consist of a single δ and a single σ, and let D be the ring of Δ-Σ-operators over R. Choose the canonical orthant decomposition on $\mathbb{Z}$ as in Example 2.1 and define the generalized term orders $\prec$ and $\prec'$ on terms of D as follows:

$$\delta^k\sigma^l \prec \delta^r\sigma^s \Longleftrightarrow (|l|, k, l) <_{lex} (|s|, r, s),$$

$$\delta^k\sigma^l \prec' \delta^r\sigma^s \Longleftrightarrow (k, |l|, l) <_{lex} (r, |s|, s).$$

Given $f = \delta^3\sigma - \sigma^{-1}$, $g = \delta^2 + \sigma$, then

$$lt_{\prec}(f) = \delta^3\sigma = lt_{\prec}(\delta^3 g) = lt_{\prec}(\delta^5 + \delta^3\sigma).$$

But

$$lt_{\prec'}(\delta^3 g) = \delta^5 \succ' lt_{\prec'}(f) = \delta^3\sigma.$$

So f is not $\prec$-reduced modulo g in the usual meaning, but f is $\prec$-reduced modulo g relative to $\prec'$. □

Definition 3.3: Let

- W be a submodule of the finitely generated free D-module F,
- $\prec$ and $\prec'$ two generalized term orders on ΛE,
- $G = \{g_1, \cdots, g_p\}$ a subset of $W \backslash \{0\}$.

Then G is called a $\prec$-Gröbner basis of W relative to $\prec'$ iff every $f \in W \backslash \{0\}$ is $\prec$-reducible to 0 modulo G relative to $\prec'$. We will call it shortly a relative Gröbner basis of W if no confusion is possible. □

Proposition 3.1: Let $\prec$ and $\prec'$ be two generalized term orders on ΛE, $G \subset W \setminus \{0\}$ a $\prec$-Gröbner basis of W relative to $\prec'$, and $f \in F$. Then

(i) G is a Gröbner basis of W w.r.t. $\prec$ and $\prec'$. So G generates the D-module W.

(ii) $f \in W$ if and only if $f = 0$ or f can be $\prec$-reduced to 0 modulo G relative to $\prec'$.

(iii) $f \in W$ is $\prec$-reduced modulo G relative to $\prec'$ if and only if $f = 0$.

Definition 3.4: Let F be a finitely generated free D-module and $f,\ g \in F \setminus \{0\}$. Let $\prec$ be a generalized term order on ΛE. For every Λ_j let $V(j, f, g)$ be a finite system of generators of the $R[\Lambda_j]$-module

$$_{R[\Lambda_j]}\langle lt(\lambda f) \in \Lambda_j E \mid \lambda \in \Lambda\rangle \cap {}_{R[\Lambda_j]}\langle lt(\eta g) \in \Lambda_j E \mid \eta \in \Lambda\rangle.$$

Then for every generator $v \in V(j, f, g)$

$$S(j, f, g, v) \;=\; \frac{v}{lt_j(f)} \frac{f}{lc_j(f)} \;-\; \frac{v}{lt_j(g)} \frac{g}{lc_j(g)}$$

is called an S-polynomial of f and g with respect to j and v. $\square$

Theorem 3.2 (Generalized Buchberger Theorem in Zhou and Winkler (2006)):
Let F be a free D-module and $\prec$ be a generalized term order on ΛE, G be a finite subset of $F \backslash \{0\}$ and W be the submodule in F generated by G. Then G is a Gröbner basis of W if and only if for all Λ_j, for all $g_i,\ g_k \in G$ and for all $v \in V(j, g_i, g_k)$, the S-polynomials $S(j, g_i, g_k, v)$ can be reduced to 0 by G. $\square$

Theorem 3.3: Let F be a free D-module, $\prec$ and $\prec'$ be two generalized term orders on ΛE, G be a finite subset of $F \backslash \{0\}$ and W be the submodule in F generated by G. Then G is a $\prec$-Gröbner basis of W relative to $\prec'$ if and only if G is a Gröbner basis with respect to $\prec'$ of W and for all Λ_j, for all $g_i,\ g_k \in G$ and for all $v \in V(j, g_i, g_k)$, the S-polynomials $S(j, g_i, g_k, v)$ with respect to $\prec$ can be $\prec$-reduced to 0 modulo G relative to $\prec'$.
In other words, G is a $\prec$-Gröbner basis relative to $\prec'$ if and only if all $S'(j, g_i, g_k, v)$ can be reduced (w.r.t. $\prec'$) to 0 by G and all $S(j, g_i, g_k, v)$ can be $\prec$-reduced to 0 modulo G relative to $\prec'$.

Theorem 3.4 (Buchberger's Algorithm for computing relative Gröbner bases):
Let F be a free D-module, $\prec$ and $\prec'$ be two generalized term order on ΛE, G be a finite subset of $F \backslash \{0\}$ and W be the submodule in F generated by G. For each Λ_j and $f, g \in F \backslash \{0\}$ let $V(j, f, g)$, $S(j, f, g, v)$ and $S'(j, f, g, v)$ be as in Definition 3.4 w.r.t. $\prec$ and $\prec'$, respectively. Then by the following algorithm a $\prec$-Gröbner basis of W relative to $\prec'$ can be computed:

input: $G = \{g_1, \cdots, g_\mu\}$, a set of generators of W
$\prec$ and $\prec'$, two generalized term orders on ΛE
output: $G'' = \{g''_1, \cdots, g''_\nu\}$, a $\prec$-GB of W rel. to $\prec'$
begin
  $G' := G$ **;**
  while exist $f, g \in G'$, $v \in V(j, f, g)$ s.t.
    $S'(j, f, g, v)$ is reduced (w.r.t. $\prec'$) to $r \neq 0$ by G'
    do $G' := G' \cup \{r\}$
  endwhile ;
  $G'' := G'$ **;**
  while exist $f, g \in G''$, $v \in V(j, f, g)$ s.t.
    $S(j, f, g, v)$ is $\prec$-reduced to $r \neq 0$ by G'' rel. to $\prec'$
    do $G'' := G'' \cup \{r\}$
  endwhile
end □

4. Computing difference-differential dimension polynomials

Let

- R be a Δ-Σ-field,
- D the ring of Δ-Σ-operators over R,
- F a finitely generated free Δ-Σ-module,
- M a finitely generated Δ-Σ-module (i.e. a finitely generated difference-differential-module).

Now we consider difference-differential dimension polynomials $\psi_M(t_1, t_2)$ in two variables t_1 and t_2 by the approach of relative difference-differential Gröbner bases.

Consider the canonical orthant decomposition of $\mathbb{Z}^n$ as in Example 2.1 and define the generalized term orders "$\prec$" and "$\prec'$" on ΛE of the terms of F as follows:
for $\lambda e = \delta_1^{k_1} \cdots \delta_m^{k_m} \sigma_1^{l_1} \cdots \sigma_n^{l_n} e$ we set

$$|\lambda|_1 := k_1 + \cdots + k_m \quad \text{and} \quad |\lambda|_2 := |l_1| + \cdots + |l_n|;$$

furthermore

$$\begin{aligned} \langle \lambda e \rangle &:= (|\lambda|_2, |\lambda|_1, e_i, k_1, \cdots, k_m, |l_1|, \cdots, |l_n|, l_1, \cdots, l_n) \\ \langle \lambda e \rangle' &:= (|\lambda|_1, |\lambda|_2, e_i, k_1, \cdots, k_m, |l_1|, \cdots, |l_n|, l_1, \cdots, l_n) \end{aligned}$$

Now we define

$$\begin{aligned} \lambda e_i \prec \mu e_j \quad &:\Longleftrightarrow \quad \langle \lambda e_i \rangle <_{lex} \langle \mu e_j \rangle \\ \lambda e_i \prec' \mu e_j \quad &:\Longleftrightarrow \quad \langle \lambda e_i \rangle' <_{lex} \langle \mu e_j \rangle' \end{aligned}$$

For $u = \sum_{\lambda \in \Lambda} a_\lambda \lambda \in D$ and $i \in \{1, 2\}$ we define

$$|u|_i := \max\{|\lambda|_i \mid a_\lambda \neq 0\}.$$

D is a bifiltered ring with the bifiltration $(D_{rs})_{r,s\in\mathbb{Z}}$ s.t.

$$D_{rs} = \{u \in D \mid |u|_1 \leq r,\ |u|_2 \leq s\} \text{ for } (r, s) \in \mathbb{N}^2$$

$$\text{and } D_{rs} = \{\} \text{ for } (r, s) \in \mathbb{Z}^2 \setminus \mathbb{N}^2.$$

Obviously

$$\bigcup\{D_{rs} \mid r, s \in \mathbb{Z}\} = D,$$

$$D_{rs} \subseteq D_{r+1,s}, \quad D_{rs} \subseteq D_{r,s+1} \text{ for } r, s \in \mathbb{Z}, \text{ and}$$

$$D_{kl}D_{rs} = D_{r+k,s+l} \text{ for } r, s, k, l \in \mathbb{Z}.$$

Let M be a finitely generated left D-module with generators $h_1, \cdots, h_q$. Let

$$M_{rs} = D_{rs}h_1 + \cdots + D_{rs}h_q$$

for any $r, s \in \mathbb{Z}$. Then $(M_{rs})_{r,s\in\mathbb{Z}}$ is an excellent bifiltration of M, i.e. every (M_{rs}) is a finitely generated R-module and $D_{kl}M_{rs} = M_{r+k,s+l}$.

Definition 4.1: A polynomial $\psi(t_1, t_2)$ in $\mathbb{Q}[t_1, t_2]$ is called a (bivariate) numerical if $\psi(t_1, t_2) \in \mathbb{Z}$ for all sufficiently large $(r_1, r_2) \in \mathbb{Z}^2$, i.e. there exists a tuple $(s_1, s_2) \in \mathbb{Z}^2$ such that $\psi(r_1, r_2) \in \mathbb{Z}$ for all integers $r_1, r_2 \in \mathbb{Z}$ with $r_i \geq s_i$ $(1 \leq i \leq 2)$.
The numerical polynomial $\psi(t_1, t_2)$ is called (bivariate) difference-differential dimension polynomial associated with M, iff

(i) $deg\ \psi \leq m + n$, $deg_{t_1}\psi \leq m$, and $deg_{t_2}\psi \leq n$ and

(ii)$\psi(t_1, t_2) = dim_R M_{t_1,t_2}$ for all sufficiently large t_1, t_2.

Theorem 4.1: Let R be a Δ-Σ-field, D and M be as above, in particular let M have the generators $h_1, \cdots, h_q$. Let F be a free Δ-Σ-module with a basis $e_1, \cdots, e_q$ and $\pi : F \longrightarrow M$ the natural Δ-Σ-epimorphism of F onto M ($\pi(e_i) = h_i$ for $i = 1, \cdots, q$).

Let $\prec$ and $\prec'$ be the generalized term orders on ΛE of the terms of F defined above. Consider the submodule $N = ker(\pi)$ of F and let $G = \{g_1, \cdots, g_p\}$ be a $\prec$-Gröbner basis of N relative to $\prec'$. Let

$$\begin{aligned} U_{r,s} = \ & \{w \in \Lambda E \mid |w|_1 \leq r, |w|_2 \leq s, \text{ and} \\ & \qquad w \neq lt_{\prec}(\lambda g_i) \text{ for all } \lambda \in \Lambda, g_i \in G\} \\ & \cup \\ & \{w \in \Lambda E \mid |w|_1 \leq r, |w|_2 \leq s, \text{ and} \\ & \qquad |lt_{\prec'}(\lambda g_i)|_1 > r \text{ for all } \lambda \in \Lambda, \\ & \qquad\qquad g_i \in G \text{ s.t. } w = lt_{\prec}(\lambda g_i)\} \ . \end{aligned}$$

Then the bivariate difference-differential dimension polynomial ψ associated with M is the cardinality of U, i.e.

$$\psi(r, s) = |U_{r,s}| \ .$$

The difference-differential dimension polynomial $\psi(t_1, t_2)$ carries more information than the "one variable" dimension polynomial $\phi(t)$. From the point of view of strength of systems of difference-differential equations, the polynomial $\psi(t_1, t_2)$ determines the strength of systems w.r.t. each of the sets of operators Δ and Σ while the polynomial $\phi(t)$ determines just the general strength of the systems w.r.t. the set $\Delta \bigcup \Sigma$.

Example 4.1: Let R be a difference-differential field whose basic sets Δ and Σ consist of a single δ and a single σ. Furthermore, let D be the ring of Δ-Σ-operators over R and $M = Dh$ be a Δ-Σ-module whose generator h satisfies the defining equation

$$(\delta\sigma + \sigma^{-2})h = 0 \ .$$

In other words, M is isomorphic to the factor module of a free Δ-Σ-module F with a free generator e by its Δ-Σ-submodule N which is a cyclic submodule with a generator $\{g = \delta\sigma + \sigma^{-2}\}$. We compute the difference-differential dimension polynomial $\psi(r, s)$.

By Theorem 4.1, we need to compute a relative Gröbner basis of N and then $\psi(r,s) = |U_{r,s}|$. Clearly the relative Gröbner basis is

$$\{g = \delta\sigma + \sigma^{-2}\}.$$

We have $lt(g) = \sigma^{-2} \in \Lambda_2$. Because the leading term of $\sigma g = \delta\sigma^2 + \sigma^{-1}$ is $\delta\sigma^2 \in \Lambda_1$, we have

$$lt(\lambda g) = \Lambda_1 \delta\sigma^2 \bigcup \Lambda_2 \sigma^{-2} \ .$$

Put

$$\begin{aligned} U'_{r,s} := \ & \{w \in \Lambda \mid |w|_1 \le r, |w|_2 \le s, \text{ and} \\ & \qquad w \ne lt_{\prec}(\lambda g) \text{ for all } \lambda \in \Lambda\} \ , \end{aligned}$$

$$\begin{aligned} U''_{r,s} := \ & \{w \in \Lambda \mid |w|_1 \le r, |w|_2 \le s, \text{ and} \\ & \qquad |lt_{\prec'}(\lambda g)|_1 > r \text{ for all } \lambda \in \Lambda \text{ s.t.} \\ & \qquad w = lt_{\prec}(\lambda g)\} \ . \end{aligned}$$

Then

$$|U_{r,s}| = |U'_{r,s}| + |U''_{r,s}| = (3r + s + 2) + (s - 1)$$

and

$$\psi(r,s) = dim_R M_{r,s} = |U_{r,s}| = 3r + 2s + 1 \ .$$