Integro-Differential Algebras as a Natural Setting for Boundary Problems

$$u'' = f$$

$$u(0) = u(1) = 0$$

Boundary problems

$$\partial(fg) = f\,\partial(g) + g\,\partial(f) \qquad \textstyle\int f \cdot \int g = \int f \int g + \int g \int f$$

Leibniz *Baxter*

Georg Regensburger and Markus Rosenkranz
Johann Radon Institute for Computational and Applied Mathematics (RICAM)
Austrian Academy of Sciences
{georg.regensburger,markus.rosenkranz}@oeaw.ac.at

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Differential Algebras and Equations

$(\mathcal{F}, \partial)$ commutative differential algebra over a field K

$\partial\colon \mathcal{F} \to \mathcal{F}$ $\quad K$-linear $\qquad f' = \partial(f) \qquad \mathbb{Q} \le K$

Leibniz rule $\qquad \partial(fg) = f\,\partial(g) + g\,\partial(f)$

$\mathcal{F}[\partial]$ differential operators $\quad$ Multiplication: $\partial f = f' + f\partial$

Differential equation (DE) $\quad Tu = f \qquad T \in \mathcal{F}[\partial], f \in \mathcal{F}$

Solution operators $\qquad G\colon f \mapsto u$

Simplest equation $\qquad u' = f$

Solution operators $\int$ are sections (right inverses) of ∂

$$\partial \int = 1$$

(sections do not always exist)

Sections

Characterization of sections from Linear Algebra:

Every section $\int$ of ∂ $\qquad \partial \int = 1$

corresponds to a projector $P = 1 - \int \partial$

and to a direct sum $\qquad \mathcal{F} = \mathcal{C} \dotplus \mathcal{I}$

where $\mathcal{C} = \mathrm{Ker}(\partial) = \mathrm{Im}(P)$ $\qquad \mathcal{I} = \mathrm{Im}(\int) = \mathrm{Ker}(P)$

constants $\qquad$ *initialized*

Example: $\mathcal{F} = C^\infty[a, b]$ with usual derivation

Fix $\alpha \in [a, b]$ section $\int f = \int_\alpha^x f(\xi)\, d\xi$

$$f \in \mathcal{I} \Leftrightarrow f(\alpha) = 0$$

Baxter Algebras

Iterate integral for higher order DE

Baxter axiom $\int f \cdot \int g = \int f \int g + \int g \int f$

Integration by parts

where $\int f \int g = \int (f \int g)$

$(\mathcal{F}, \int)$ *Rota-Baxter algebra (of weight 0)* Baxter [Bax60], Rota [Rot69]

"flattens" products "integrate out" coefficients

Diff. Baxter axiom $\int f g = f \int g - \int f' \int g$

Diff. Baxter axiom $\overset{f = \int f}{\Rightarrow}$ Baxter axiom

$\not\Leftarrow$

Integro-Differential Algebras

Baxter axiom + Homogeneity over the constants:

$$\textstyle\int cf = c \int f \qquad c \in \mathcal{C}, f \in \mathcal{F}$$

$\Leftrightarrow$ Diff. Baxter axiom

A section $\int$ of ∂ is an *integral* if it satisfies the diff. Baxter axiom

$(\mathcal{F}, \partial, \int)$ *integro-differential algebra* [RR08b]

differential Rota-Baxter algebras Guo and Keigher [GK08]

Examples: $\mathcal{F} = C^\infty[a, b]$ or exponential polynomials (usual ∂)

$\int^*: f \mapsto \int_a^x f(\xi)\, d\xi$ $\quad$ $\int_*: f \mapsto \int_x^b f(\xi)\, d\xi$ $\quad$ $f \mapsto \int_a^b \int_\tau^x f(\xi)\, d\xi\, d\tau$

$(\mathcal{F}, \partial, \int^*)$ $\quad$ $(\mathcal{F}, \partial, -\int_*)$ $\quad$ section

$P^*: f \mapsto f(a)$ $\quad$ $P_*: f \mapsto f(b)$ $\quad$ $P: f \mapsto \int_a^b f(\xi)\, d\xi$

multiplicative $\quad$ nonmultiplicative

Power series, Hurwitz series Keigher and Pritchard [KP00]

Multiplicative Projectors

A section $\int$ of ∂ is an integral

$\Leftrightarrow \; P = 1 - \int\partial$ is multiplicative

$$P(fg) = P(f)P(g)$$

$\Leftrightarrow \; \mathcal{I} = \mathrm{Im}(\int) = \mathrm{Ker}(P)$ is an ideal

Linear structure and algebra structure fit together

Variant of differential Baxter axiom (Integration by parts):

$$\int fg' = fg - \int f'g - P(f)P(g)$$

Ordinary Integro-Differential Algebras

Ordinary differential Equations $\dim_K \operatorname{Ker}(\partial) = 1$

$(\mathcal{F}, \partial, \int)$ *Ordinary (integro)-differential algebra*

$$\mathcal{C} = \operatorname{Ker}(\partial) = K$$

Diff. Baxter axiom $\Leftrightarrow$ Baxter axiom

Projection: $\mathtt{E} = 1 - \int \partial$

Evaluation

Multiplicative linear functional (character)

$\mathcal{M}(\mathcal{F})$ Multiplicative functionals φ, ψ

Differential Operators

$\mathcal{F}[\partial]$ differential operators:

Free K-algebra generated by the symbol(s) ∂

the „functions“ $f \in \mathcal{F}$

modulo the rewrite system

$$\boxed{fg \quad \to \quad f \cdot g \;\Big|\; \partial f \quad \to \quad \partial \cdot f + f\partial}$$

· action on $\mathcal{F}$

Normal forms $\sum_i f_i \partial^i$

Noncommutative Gröbner Basis Bergman [Ber78]

Proposition. The rewrite system is Noetherian and confluent

Integro-Differential Operators

$\mathcal{F}[\partial, \int]$ integro-differential operators:

Free K-algebra generated by the symbol(s) ∂ and $\int$

the „functions" $f \in \mathcal{F}$ and the „functionals" $\varphi \in \mathcal{M}(\mathcal{F})$

modulo the rewrite system

$$
\begin{array}{|lcl|lcl|}
\hline
fg & \to & f \cdot g & \partial f & \to & \partial \cdot f + f\partial \\
\varphi\psi & \to & \psi & \partial\varphi & \to & 0 \\
\varphi f & \to & (\varphi \cdot f)\,\varphi & \partial\int & \to & 1 \\
\hline
\int f \int & \to & \multicolumn{4}{l|}{(\int \cdot f)\int - \int(\int \cdot f)} \\
\int f \partial & \to & \multicolumn{4}{l|}{f - \int(\partial \cdot f) - (\mathtt{E} \cdot f)\,\mathtt{E}} \\
\int f \varphi & \to & \multicolumn{4}{l|}{(\int \cdot f)\,\varphi} \\
\hline
\end{array}
$$

$\cdot$ action on $\mathcal{F}$

Noncommutative Gröbner Basis Bergman [Ber78]

Proposition. The rewrite system is Noetherian and confluent

Normal Forms for Integro-Differential Operators

Proposition. Every normal form can be uniquely written as

$$T + G + B$$

differential integral boundary operator

$$T = \sum f_i \partial^i \qquad G = \sum f_i \textstyle\int g_i \qquad B = \sum f_i \beta_i$$

$$\beta_i \in \mathcal{S}(\mathcal{F}) = \mathcal{M}(\mathcal{F})\,\mathcal{F}[\partial, \textstyle\int]$$

Stieltjes boundary conditions

Subalgebras

$$\mathcal{F}[\partial] \qquad \mathcal{F}[\textstyle\int] \qquad \mathcal{F}[\mathtt{E}]$$

Stieltjes boundary condition

$$\beta = \sum_{\varphi} \Big(\sum a_{\varphi,i}\, \varphi \partial^i + \varphi \textstyle\int f_\varphi \Big) \qquad u \mapsto u(0) + u''(0) + \int_0^1 u(\xi)\, d\xi$$

Boundary Problems

$T \in \mathcal{F}[\partial]$ monic with degree n

$\beta_1, \ldots, \beta_n \in \mathcal{S}(\mathcal{F})$ Stieltjes boundary conditions

Given $f \in \mathcal{F}$ find $u \in \mathcal{F}$

$$Tu = f$$

$$\beta_1 u = \ldots = \beta_n u = 0$$

regular $\forall f \;\; \exists! u$

Criterion $[\beta_i(u_j)]$ regular

$\mathrm{Ker}(T) = [u_1, \ldots, u_n]$

fundamental system

Boundary Problem:

$(T, \mathcal{B})$

$\mathcal{B} \leq \mathcal{S}(\mathcal{F})$ Dimension n

$$\mathrm{Ker}(T) \dot{+} \mathcal{B}^{\perp} = \mathcal{F}$$

Abstract setting:
Boundary problem = Surjective linear Map + Subspace of dual
(Includes LPDEs, Systems) [RR08]

Green's Operators

Green's operator: $G\colon \mathcal{F} \to \mathcal{F}$ $\quad TG = 1$

$f \mapsto u$ $\quad \operatorname{Im}(G) = \mathcal{B}^\perp$

$$G = (T, \mathcal{B})^{-1}$$

Theorem. Given a fundamental system with regular Wronskian, the Green's operator of a regular boundary problem can be computed as integro-differential operator. Its normal form gives the Green's function.

State and solve boundary problems
with integro-differential operators.

Composing and Factoring Boundary Problems

Composition of boundary problems:

$$(T_1, \mathcal{B}_1) \cdot (T_2, \mathcal{B}_2) = (T_1 T_2, \mathcal{B}_1 T_2 + \mathcal{B}_2)$$

(Semidirect product of monoids)

Composition of regular problems is regular

$$((T_1, \mathcal{B}_1) \cdot (T_2, \mathcal{B}_2))^{-1} = (T_2, \mathcal{B}_2)^{-1} \cdot (T_1, \mathcal{B}_1)^{-1}$$

Theorem. Given a regular boundary problem $(T, \mathcal{B})$,

every factorization $T = T_1 T_2$ can be lifted to a factorization

$$(T_1, \mathcal{B}_1) \cdot (T_2, \mathcal{B}_2) = (T, \mathcal{B})$$ into regular problems.

unique

$\mathcal{B}_2 \leq \mathcal{B}$ such that it is regular

abstract setting

Integro-Differential Polynomials

$(\mathcal{F}, \partial, \int)$ Integro-differential algebra

adjoin indeterminate u

Example:

$$x^3 u(0) \int (x^4 u u''^2 \int (u''(0)^2 \; x e^{3x} u'^3 \int u)) \cdot (x^2 \int x u'' \int x u'(0)^2 u)$$

Baxter rule $\int f \cdot \int g = \int f \int g + \int g \int f$ Section $\partial \int = 1$

Evaluation $1 - \int \partial$ Notation: $(1 - \int \partial)(u) = u(0)$

Sum of terms of the form

$$f u(0)^\alpha u^\beta \int f_1 u^{\gamma_1} \int \ldots \int f_n u^{\gamma_n}$$ n depth

$u^\beta = u_0^{\beta_0} \cdots u_k^{\beta_k}$ differential monomial $\beta_k \neq 0$

u_n nth derivative

Canonical Simplifier

Represent a polynomial as sum of terms of the form

$$f u(0)^{\alpha} u^{\beta} \int f_1 u^{\gamma_1} \int \ldots \int f_n u^{\gamma_n}$$

Is this representation unique (a canoncical form)?

Do we have a canonical simplifier?

Buchberger, Loos [BL83]

Integration by parts:

$$\int f u' \equiv f u - \int f' u - f(0) u(0)$$

Highest derivative appears linearly:

u^{β} with order > 0 *quasilinear*

and *functional* otherwise

Canonical Forms

$\mathcal{R}$ set of all linear combinations of terms

$$bu(0)^{\alpha}u^{\beta}\int b_1 u^{\gamma_1}\int \ldots \int b_n u^{\gamma_n}$$

b_i basis elements $\quad u^{\gamma_i}$ functional

Theorem. Every integro-differential polynomial can be uniquely represented by a term in $\mathcal{R}$

Main part of the proof:

Show that different terms represent different polynomials

Integro-Differential Algebra on Canonical Forms

Strategy:

Equip $\mathcal{R}$ with the structure of an integro-differential algebra

Substitutionhomomorphism polynomials in universal algebra

Coproduct of free algebra and cofficient algebra

Lausch, Nöbauer [LN73] Buchberger, Loos [BL83]

Multiplication: shuffle product (Li Guo's talk) [Ree58]

Derivation: Leibniz rule, section axiom and basis expansion

Integral: Recursively for quasiconstant, quasilinear, functional

Difficult part is verifying the Baxter axiom

Conclusion

Algebraic setting with differential and integral operator

Allows us to treat local conditions (evaluations)

Integro-differential operators (state and solve BPs)

Integro-differential polynomials (extensions, quotients)

Implementation in Theorema based on functors
Buchberger, R., R., Tec [BRRT08]

Extension to several indeterminates

Systems of integro-differential polynomial equations

PDEs

Thank you!

References

[BN98] Baader, F. and Nipkow, T., *Term rewriting and all that*, Cambridge University Press, 1998

[Bax60] Baxter, G., “An Analytic Problem Whose Solution Follows from a Simple Algebraic Identity”, Pacific J. Math., 1960, 10, 731-742

[Berg78] Bergman, G. M., The Diamond Lemma for Ring Theory, Advances in Mathematics, 1978, 29, 179-218

[BRRT08] Buchberber, B. Regensburger, G., Rosenkranz, M., Tec, L. “General Polynomial Reduction with THEOREMA Functors: Applications to Integro-Differential Operators and Polynomials”, Extended Abstract and Poster ISSAC’08

[BL83] Buchberger, B. and Loos, R., “Algebraic simplification”, Springer, 1983, 11-43

[GK08] Guo, L. and Keigher, W., “On Differential Rota-Baxter Algebras”, J. Pure Appl. Algebra, 2008, 212, 522-540

[Hul69] Hule, H., “Polynome über universalen Algebren”, Monatsh. Math., 1969, 73, 329-340

[KP00] Keigher, W. F. and Pritchard, F. L., “Hurwitz series as formal functions” J. Pure Appl. Algebra, 2000, 146, 291-304

References

[LN73] Lausch, H. and Nöbauer, W., *Algebra of polynomials*, North-Holland Publishing Co., 1973

[Ree58] Ree, R., “Lie elements and an algebra associated with shuffles” Ann. of Math. (2), 1958, 68, 210-220

[RR08c] Regensburger, G. and Rosenkranz, M., “An Algebraic Foundation for Factoring Linear Boundary Problems”, Ann. Mat. Pura Appl. (4), 2008.

[RR08a] Rosenkranz, M. and Regensburger, G., “Integro-Differential Polynomials and Operators”, In: ISSAC ’08, 2008, 261-268.

[RR08b] Rosenkranz, M. and Regensburger, G., “Solving and Factoring Boundary Problems for Linear Ordinary Differential Equations in Differential Algebras“, J. Symbolic Comput., 2008, 43, 515-544

[Ros05] Rosenkranz, M., “A new symbolic method for solving linear two-point boundary value problems on the level of operators”, J. Symbolic Comput., 2005, 39 , 171-199

[Rot69] Rota, G.-C., “Baxter algebras and combinatorial identities (I, II)”, Bull. Amer. Math. Soc., 1969, 75, 325-334