Holonomic Function Identities

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Algebraic and Algorithmic Aspects
of
Differential and Integral Operators

History

1990: D. Zeilberger's "Holonomic Systems Approach"

1998: extensions and refinements by F. Chyzak

2008: Mathematica implementation by CK

Notation

- $\mathbb{K}$: field of characteristic 0
- $A_n = A_n(\mathbb{K})$: the n-th Weyl algebra
- D_x: differential operator w.r.t. x, i.e., $D_x \bullet f(x) = f'(x)$
- S_n: shift operator w.r.t. n, i.e., $S_n \bullet f(n) = f(n+1)$
- $\mathbb{O}$: an Ore algebra
- $\operatorname{Ann}_{\mathbb{O}} f$: the ideal of annihilating operators of f in $\mathbb{O}$, i.e., $\operatorname{Ann}_{\mathbb{O}} f = \{P \in \mathbb{O} \mid P \bullet f = 0\}$

Definition: Ore Algebra (1)

Let $\mathcal{F}$ be a $\mathbb{K}$-algebra (of "functions"), and let $\sigma, \delta \in \operatorname{End}_{\mathbb{K}} \mathcal{F}$ with

$$\delta(fg) = \sigma(f)\delta(g) + \delta(f)g \quad \text{for all } f, g \in \mathcal{F} \quad \text{(skew Leibniz law)}.$$

The endomorphism δ is called a *σ-derivation*.

Let $\mathbb{A}$ be a $\mathbb{K}$-subalgebra of $\mathcal{F}$ (e.g., $\mathbb{A} = \mathbb{K}[x]$ or $\mathbb{A} = \mathbb{K}(x)$) and assume that σ, δ restrict to a σ-derivation on $\mathbb{A}$.

Define the skew polynomial ring $\mathbb{O} := \mathbb{A}[\partial; \sigma, \delta]$:

- polynomials in ∂ with coefficients in $\mathbb{A}$
- usual addition
- product that makes use of the commutation rule

$$\partial a = \sigma(a)\partial + \delta(a) \text{ for all } a \in \mathbb{A}$$

Definition: Ore Algebra (2)

We turn $\mathcal{F}$ into an $\mathbb{O}$-module by defining an action of elements in $\mathbb{O}$ on a function $f \in \mathcal{F}$:

$$\begin{aligned} a \bullet f &:= a \cdot f, \\ \partial \bullet f &:= \delta(f). \end{aligned}$$

Remark: In special cases we define the action $\partial \bullet f := \sigma(f)$.

Example 1: $\mathbb{A} = \mathbb{K}[x]$, $\sigma = 1$, $\delta = \frac{\mathrm{d}}{\mathrm{d}x}$.

Then $\mathbb{K}[x][D_x; 1, \frac{\mathrm{d}}{\mathrm{d}x}] = \mathbb{K}[x][D_x; 1, D_x]$ is the Weyl algebra A_1.

Example 2: $\mathbb{A} = \mathbb{K}[n]$, $\sigma(n) = n + 1$, $\sigma(c) = c$ for $c \in \mathbb{K}$, $\delta = 0$.

Then $\mathbb{K}[n][S_n; S_n, 0]$ is a shift algebra.

Example 3: $\mathbb{K}(n, x, y)[S_n; S_n, 0][D_x; 1, D_x][D_y; 1, D_y]$

Holonomic functions

Definition:
A function $f(x_1, \ldots, x_n) \in \mathcal{F}$ is said to be holonomic if $A_n / \operatorname{Ann}_{A_n} f$ is a holonomic module.

Definition:
A sequence $f(k_1, \ldots, k_r) \in \mathbb{C}^{\mathbb{N}^r}$ is holonomic if its multivariate generating function

$$F(x_1, \ldots, x_r) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_r=0}^{\infty} f(k_1, \ldots, k_r) x_1^{k_1} \cdots x_r^{k_r}.$$

is a holonomic function.

Properties of holonomic functions

Closure properties:

- sum
- product
- definite integration

Elimination property:
Given an ideal I in A_n s.t. A_n/I is holonomic; then for any choice of $n+1$ among the $2n$ generators of A_n there exists a nonzero operator in I that depends only on these. In other words, we can eliminate $n-1$ variables.

Integration via elimination

Given: $\mathrm{Ann}_{\mathbb{O}}\, f$, the annihilator of a holonomic function $f(x,y)$ in the Ore algebra $\mathbb{O} = \mathbb{K}[x,y][D_x;1,D_x][D_y;1,D_y]$.
Task: Compute $F(y) = \int_a^b f(x,y)\,\mathrm{d}x$

Since $\mathrm{Ann}_{\mathbb{O}}\, f$ is holonomic, there exists $P \in \mathrm{Ann}_{\mathbb{O}}\, f$ that does not contain x ("elimination property"):

$$P(y, D_x, D_y) = Q(y, D_y) + D_x R(y, D_x, D_y)$$

Apply $\int_a^b \ldots \,\mathrm{d}x$ to $P \bullet f = 0$:

$$Q(y, D_y) \bullet F(y) + \Big[R(y, D_x, D_y) \bullet f(x,y) \Big]_{x=a}^{x=b}$$

Summation via elimination

Given: $\operatorname{Ann}_{\mathbb{O}} f$, the annihilator of a holonomic sequence $f(k,n)$ in the Ore algebra $\mathbb{O} = \mathbb{K}[k,n][S_k; S_k, 0][S_n; S_n, 0]$.
Task: Compute $F(n) = \sum_{k=a}^{b} f(k,n)$

By the elimination property there exists $P \in \operatorname{Ann}_{\mathbb{O}} f$ that does not contain k:

$$P(n, S_k, S_n) = Q(n, S_n) + (S_k - 1)R(n, S_k, S_n)$$

Sum over this equation:

$$Q(n, S_n) \bullet F(n) + \Big[R(n, S_k, S_n) \bullet f(k,n) \Big]_{k=a}^{k=b}$$

Example: Orthogonality of Hermite Polynomials (1)

Compute the integral

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x)\,\mathrm{d}x = \delta_{m,n}\sqrt{\pi}2^n n!$$

First we compute an annihilator of the integrand:

```
ann = Annihilator[Exp[-x^2]*HermiteH[m,x]*HermiteH[n,x],
                  {S[m], S[n], Der[x]}]
```

$$\{-2x + D_x + S_m + S_n,$$
$$D_x^2 + 2m - 2n + (-2x)D_x + 2S_nD_x - 2,$$
$$S_n^2 + 2n - 2xS_n + 2,$$
$$D_x^3 + 4mx - 4nx - 4x + (4n-4m)S_n + \left(-4x^2 + 2m + 6n + 4\right) D_x\}$$

Example: Orthogonality of Hermite Polynomials (2)

Next step is to compute a Gröbner basis w.r.t. lexicographical order in order to eliminate x:

```
gb = OreGroebnerBasis[
        ann, OreAlgebra[x, m, n, S[m], S[n], Der[x]],
        MonomialOrder -> Lexicographic]
```

$$\begin{aligned}\{&2n - S_m S_n - S_n D_x + 2,\\ &2m - S_m D_x - S_m S_n + 2,\\ &D_x - 2x + S_m + S_n \qquad\}\end{aligned}$$

Example: Orthogonality of Hermite Polynomials (3)

In the first operator, the part $R = -S_n D_x$, in the second $R = -S_m D_x$. We have to check that $[R \bullet f]_{x=-\infty}^{x=\infty}$ indeed vanishes:

```
Limit[D[Exp[-x^2]*HermiteH[m,x]*HermiteH[n+1,x], x],
      x -> Infinity]
```

$$0$$

Hence we take the first two operators (which do not involve the integration variable x) and set R to 0:

```
OrePolynomialSubstitute[Take[gb, 2], {Der[x] -> 0}]
```

$$\{2n - S_m S_n + 2, 2m - S_m S_n + 2\}$$

Example: Orthogonality of Hermite Polynomials (4)

By computing a last Gröbner basis, we get the result:

```
OreGroebnerBasis[%, OreAlgebra[m, n, S[m], S[n]]]
```

$$\{m - n, 2n - S_m S_n + 2\}$$

This proves that the right hand side can only be nonzero if $m = n$.
By similar computations we obtain the recurrence

$$\left(4n^2 + 8n + 4\right) f(n) + (-4n - 6) f(n + 1) + f(n + 2) = 0$$

for the right hand side when we set m to n.
Together with the initial values $f(0) = \sqrt{\pi}$ and $f(1) = 2\sqrt{\pi}$ we have a full and simple description of the desired result.

Definite integration with Takayama

Given: $\operatorname{Ann}_{\mathbb{O}} f$, the annihilator of a holonomic function $f(x,y)$ (with natural boundaries) in the Ore algebra
$\mathbb{O} = \mathbb{K}[x,y][D_x;1;D_x][D_y;1,D_y]$.
Find: The annihilator of $F(y) = \int_a^b f(x,y)\mathrm{d}x$ in the Ore algebra
$\mathbb{O}' = \mathbb{K}[y][D_y;1,D_y]$
Find $P \in \operatorname{Ann}_{\mathbb{O}} f$ which can be written in the form

$$P(x,y,D_x,D_y) = Q(y,D_y) + D_x R(x,y,D_x,D_y)$$

Apply $\int_a^b \ldots \mathrm{d}x$ to $P \bullet f = 0$:

$$\int_a^b Q(y,D_y) f(x,y)\,\mathrm{d}x + \int_a^b D_x R(x,y,D_x,D_y) f(x,y)\,\mathrm{d}x$$

Hence $Q(y,D_y)F(y) = 0$

The operator Q can be computed with Takayama's algorithm.

Comparison

Zeilberger:

1. eliminate x
2. reduce modulo $D_x\mathbb{O}$

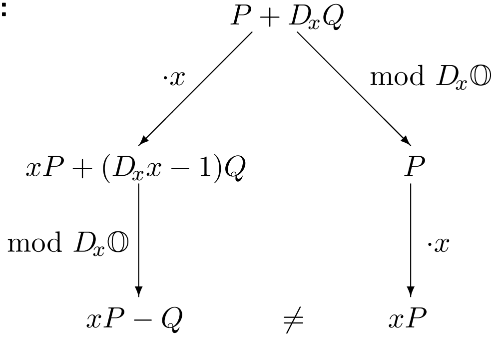
Takayama (variant due to Chyzak/Salvy):

1. reduce modulo $D_x\mathbb{O}$
2. eliminate x

How to eliminate x?

Problem: After reducing modulo $D_x\mathbb{O}$, no multiplication by x is allowed!

Example:



Takayama's algorithm

Eliminate x by computing a Groebner basis in the $\mathbb{O}'$-module w.r.t. the basis $x^\alpha, \alpha \in \mathbb{N}$:

$$x^2(1+y) + xD_x = x^2(1+y) + D_x x - 1 \equiv x^2(1+y) - 1 \mod D_x\mathbb{O}$$
$\longrightarrow$ gives $(-1, 0, 1+y, 0, \dots)$

$$x + D_x D_y + y \equiv x + y \mod D_x\mathbb{O}$$
$\longrightarrow$ gives $(y, 1, 0, \dots)$

We have to include also multiples by x^α to the generators of $\mathrm{Ann}_{\mathbb{O}} f$:

$$x^2 + xD_x D_y + xy = x^2 + D_x x D_y - xD_y + xy \equiv x^2 - x(D_y + y)$$
$\longrightarrow$ gives $(0, D_y + y, 1, 0, \dots)$

Takayama's algorithm

Input: a set of generators $\{G_1, \ldots, G_m\}$ for $\operatorname{Ann}_{\mathbb{O}} f$
Output: $\operatorname{Ann}_{\mathbb{O}'} F$

1. set $d = \max_{1 \leq i \leq m} \deg_x G_i$
2. set $A' = \{G_1, \ldots, G_m\} \cup \bigcup_{i=1}^{m} \{x^\alpha G_i \mid 1 \leq \alpha \leq \deg_x G_i\}$
3. reduce all elements in A' modulo $D_x \mathbb{O}$
4. compute a Groebner basis in the corresponding module eliminating x
5. if no $(P, 0, \ldots, 0)$ is found, increase d

Since f is holonomic the algorithm is guaranteed to terminate.

Example: Victor Moll's irresistible integral (1)

Compute a closed form for the integral

$$\int_0^\infty \frac{1}{(x^4 + 2ax^2 + 1)^{m+1}}\,\mathrm{d}x$$

Again, we start by computing annihilating operators for the integrand:

```
ann = Annihilator[1/(x^4 + 2*a*x^2 + 1)^(m+1),
                  {Der[x], S[m], Der[a]}]
```

$$\{2mx^2 + 2x^2 + \left(x^4 + 2ax^2 + 1\right) D_a,$$
$$\left(x^4 + 2ax^2 + 1\right) S_m - 1,$$
$$4mx^3 + 4x^3 + 4ax + 4amx + \left(x^4 + 2ax^2 + 1\right) D_x\}$$

Example: Victor Moll's irresistible integral (2)

Although the integral does not have natural boundaries (e.g. the integrand does not vanish for $x = 0$), Takayama's algorithm gives the correct result here:

```
Takayama[ann, {x}, Saturate -> True]
```

$$\{4m + (2a)D_a + (-4m - 4)S_m + 3,\\ 4m + \left(4a^2 - 4\right) D_a^2 + (8ma + 12a)D_a + 3\}$$

Example: Victor Moll's irresistible integral (3)

The second operator is an ordinary differential equation in a:

```
de = ApplyOreOperator[%[[2]], Int[a]]
```

$$(4m+3)\mathsf{Int}(a) + (8ma+12a)\mathsf{Int}'(a) + \left(4a^2-4\right)\mathsf{Int}''(a)$$

This ODE can be automatically solved by using Mathematica's `DSolve` command (also the initial values can be computed automatically), giving the final result

$$\frac{(1+i)i^m 2^{-m-2}\left(a^2-1\right)^{-\frac{m}{2}-\frac{1}{4}}\pi\Gamma\left(2m+\frac{3}{2}\right)P_m^{-m-\frac{1}{2}}(a)}{\Gamma(m+1)}$$

∂-finite functions

Definition: A function $f(x_1, \ldots, x_m)$ is called ∂-finite w.r.t. $\mathbb{O} = \mathbb{K}(x_1, \ldots, x_m)[\partial_1; \sigma_1, \delta_1] \cdots [\partial_m; \sigma_m, \delta_m]$ if $\mathbb{O}/\operatorname{Ann}_{\mathbb{O}} f$ is a finite-dimensional $\mathbb{K}(x_1, \ldots, x_m)$-vector space.

In other words, f is ∂-finite if its "derivatives" span a finite-dimensional $\mathbb{K}(x_1, \ldots, x_m)$-vector space.

Example: All derivatives (w.r.t. x and y) of $\sin\left(\frac{x+y}{x-y}\right)$ are of the form

$$r_1(x,y)\sin\left(\frac{x+y}{x-y}\right) + r_2(x,y)\cos\left(\frac{x+y}{x-y}\right),$$

e.g.,

$$\begin{aligned} D_x^3 D_y^2 \bullet \sin\left(\tfrac{x+y}{x-y}\right) \quad &= \quad \frac{32\left(3x^4+12yx^3-30y^2x^2-4y^3x+9y^4\right)\sin\left(\frac{x+y}{x-y}\right)}{(x-y)^9} \\ &\quad - \frac{16\left(6x^5-33yx^4+80y^3x^2-54y^4x+3y^5\right)\cos\left(\frac{x+y}{x-y}\right)}{(x-y)^{10}} \end{aligned}$$

Chyzak's extension of Zeilberger's fast algorithm

Given: $\operatorname{Ann}_{\mathbb{O}} f$, the annihilator of a ∂-finite function $f(x,y)$ in the rational Ore algebra $\mathbb{O} = \mathbb{K}(x,y)[D_x;1;D_x][D_y;1,D_y]$.
Find: $Q(y,D_y)$ and $R(x,y,D_x,D_y)$ such that $Q + D_x R \in \operatorname{Ann}_{\mathbb{O}} f$.

1. compute a Gröbner basis G of $\operatorname{Ann}_{\mathbb{O}} f$ in order to know the set $U = \{u_1, \ldots, u_k\}$ of monomials that can not be reduced by $\operatorname{Ann}_{\mathbb{O}} f$, i.e., the elements under the stairs.
2. make an ansatz for $Q(y,D_y) = \sum_{i=0}^{d} \eta_i(y) D_y^i$ and $R(x,y,D_x,D_y) = \sum_{j=1}^{k} \phi_j(x,y) u_j$
3. reduce the ansatz with G and set all coefficients to zero
4. solve the corresponding coupled system of differential equations (for rational solutions)

Example: Victor Moll's irresistible integral (4)

As mentioned before, Moll's integral does not have natural boundaries, hence we should not use Takayama's algorithm but Chyzak's creative telescoping algorithm:

```
CreativeTelescoping[1/(x^4 + 2*a*x^2 + 1)^(m+1),
                         Der[x], Der[a]]
```

$$\left\{ \{4m + \left(4\left(a^2-1\right)\right)D_a^2 + (4(2ma+3a))D_a + 3\} \atop \{\frac{x^5-2ax^3-4amx^3-4mx-3x}{x^4+2ax^2+1}\} \right\}$$

The first operator corresponds to Q and the second operator to R as before, meaning that $Q + D_x R \in \mathrm{Ann}_{\mathbb{O}}\, f$ where f is the integrand.

Example: Victor Moll's irresistible integral (5)

We now have to check what $[R \bullet f]_{x=0}^{x=\infty}$ gives:

```
ApplyOreOperator[%[[2,1]], 1/(x^4 + 2*a*x^2 + 1)^(m+1)]
```

$$\left(x^4 + 2ax^2 + 1\right)^{-m-2}\left(x^5 - 2ax^3 - 4amx^3 - 4mx - 3x\right)$$

```
Limit[%, x -> Infinity, Assumptions -> m >= 0]
```

$$0$$

and also for $x = 0$ the value of $R \bullet f$ is 0. Hence the first operator annihilates the whole integral (observe that it is the same as obtained by Takayama's algorithm).

Thanks for your attention!