

On Differential Rota-Baxter Algebras ¹

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¹Joint work with W. Keigher and W. Sit

► **1. Rota-Baxter algebras:**

Fix λ in the base ring \mathbf{k} . A **Rota-Baxter operator** or a **Baxter operator of weight λ** on a \mathbf{k} -algebra R is a linear map $P : R \rightarrow R$ such that

$$P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy), \quad \forall x, y \in R.$$

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$$P : R \rightarrow R, P[f](x) := \int_0^x f(t)dt.$$

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$$F(x) := P[f](x) = \int_0^x f(t) dt, \quad G(x) := P[g](x) = \int_0^x g(t) dt.$$

Then the **integration by parts** formula states

$$\int_0^x F(t)G'(t)dt = F(x)G(x) - \int_0^x F'(t)G(t)dt$$

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$$P[P[f]g](x) = P[f](x)P[g](x) - P[fP[g]](x).$$

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$$\begin{aligned} P[f](x)P[g](x) &= \left(\sum_{n \geq 1} f(x+n) \right) \left(\sum_{m \geq 1} g(x+m) \right) \\ &= \sum_{n \geq 1, m \geq 1} f(x+n)g(x+m) \\ &= \left(\sum_{n > m \geq 1} + \sum_{m > n \geq 1} + \sum_{m=n \geq 1} \right) f(x+n)g(x+m) \\ &= \sum_{m \geq 1} \left(\sum_{k \geq 1} f(x + \underbrace{k+m}_{=n}) \right) g(x+m) + \sum_{n \geq 1} \left(\sum_{k \geq 1} g(x + \underbrace{k+n}_{=m}) \right) f(x+n) \\ &+ \sum_{n \geq 1} f(x+n)g(x+n) \\ &= P(P(f)g)(x) + P(fP(g))(x) + P(fg)(x). \end{aligned}$$

- **Partial sum:** Let R be the set of sequences $\{a_n\}$ with values in \mathbf{k} . Then R is a \mathbf{k} -algebra with termwise addition, multiplication and scalar product. Define

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- **Matrices:** On the algebra of upper triangular $n \times n$ matrices $M_n^u(\mathbf{k})$, define

$$P((c_{kl}))_{ij} = \delta_{ij} \sum_{k \geq i} c_{ik}.$$

Then P is a Rota-Baxter operator of weight -1 .

- **Scalar product:** Let R be a \mathbf{k} -algebra. For a given $\lambda \in \mathbf{k}$, define

$$P_\lambda : R \rightarrow R, x \mapsto -\lambda x, \forall x \in R.$$

Then (R, P_λ) is a Rota-Baxter algebra of weight λ . In particular, id is a Rota-Baxter operator of weight -1 and any \mathbf{k} -algebra is a Rota-Baxter algebra.

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- **QFT dimensional regularization:** Let $R = \mathbb{C}[t^{-1}, t]$ be the ring of Laurent series $\sum_{n=-k}^{\infty} a_n t^n$, $k \geq 0$. Define

$$P\left(\sum_{n=-k}^{\infty} a_n t^n\right) = \sum_{n=-k}^{-1} a_n t^n.$$

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- **Classical Yang-Baxter equation:** Let \mathfrak{g} be a Lie algebra with a self-duality $\mathfrak{g}^* := \text{Hom}(\mathfrak{g}, \mathbf{k}) \cong \mathfrak{g}$. Then $\mathfrak{g}^{\otimes 2} \cong \mathfrak{g} \otimes \mathfrak{g}^* \cong \text{End}(\mathfrak{g})$. Let $r_{12} \in \mathfrak{g}^{\otimes 2}$ be anti-symmetric. Then r_{12} is a solution (r-matrix) of the classical Yang-Baxter equation (CYB)

$$\text{CYB}(r) := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$$

if and only if the corresponding $P \in \text{End}(\mathfrak{g})$ is a (Lie algebra) Rota-Baxter operator of weight 0:

$$[P(x), P(y)] = P[P(x), y] + P[x, P(y)].$$

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- **Associative Yang-Baxter equations** (Aguiar) Let A be an associative algebra and let $r := \sum_j u_j \otimes v_j \in R \otimes R$ be a solution of the **associative Yang-Baxter equation**

$$r_{13}r_{12} - r_{12}r_{23} + r_{23}r_{13} = 0.$$

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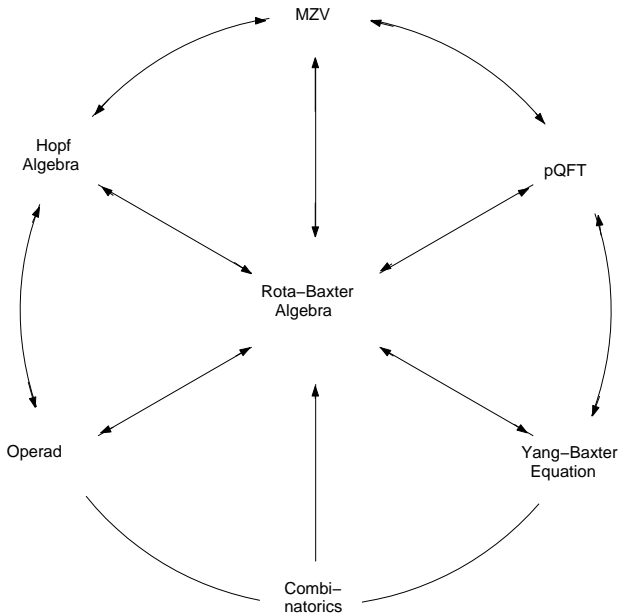
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- ▶ **Others** Divided powers, Hochschild homology ring, dendriform algebras, rooted trees, quasi-shuffles, Chen integral symbols,



► **2. Free commutative Rota-Baxter algebras**

Let A be a commutative \mathbf{k} -algebra. Let $\mathbb{H}^+(A) = \bigoplus_{n \geq 0} A^{\otimes n} (= T(A))$. Consider the following products on $\mathbb{H}^+(A)$. Define $\mathbf{1}_{\mathbf{k}} \in \mathbf{k}$ to be the unit. Let $\mathfrak{a} = a_1 \otimes \cdots \otimes a_m \in A^{\otimes m}$ and $\mathfrak{b} = b_1 \otimes \cdots \otimes b_n \in A^{\otimes n}$.

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- **Quasi-shuffle product:** Hoffman (2000) on multiple zeta values. Write $a = a_1 \otimes a'$, $b = b_1 \otimes b'$. Recursively define

$$(a_1 \otimes a') * (b_1 \otimes b') = a_1 \otimes (a' * (b_1 \otimes b')) + b_1 \otimes ((a_1 \otimes a') * b') + a_1 b_1 \otimes (a' * b'),$$

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► **Example.**

$$\begin{aligned} a_1 * (b_1 \otimes b_2) &= a_1 \otimes (a' * (b_1 \otimes b_2)) + b_1 \otimes (a_1 * b_2) + (a_1 b_1) \otimes (a' * b_2) \\ &= a_1 \otimes (b_1 \otimes b_2) + b_1 \otimes (a_1 * b_2) + (a_1 b_1) \otimes b_2. \\ &= a_1 \otimes b_1 \otimes b_2 + b_1 \otimes a_1 \otimes b_2 + b_1 \otimes b_2 \otimes a_1 + b_1 \otimes a_1 b_2 + a_1 b_1 \otimes b_2. \end{aligned}$$

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 - ▶ A **mixable shuffle** is a shuffle in which some pairs $a_i \otimes b_j$ are merged into $a_i b_j$.
- Define $(a_1 \otimes \dots \otimes a_m) \diamond (b_1 \otimes \dots \otimes b_n)$ to be the sum of mixable shuffles of $a_1 \otimes \dots \otimes a_m$ and $b_1 \otimes \dots \otimes b_n$.

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- ▶ **Example:**

$$\begin{aligned}
 & a_1 \diamond (b_1 \otimes b_2) \\
 &= a_1 \otimes b_1 \otimes b_2 + b_1 \otimes a_1 \otimes b_2 + b_1 \otimes b_2 \otimes a_1 \quad (\text{shuffles}) \\
 &+ a_1 b_1 \otimes b_2 + b_1 \otimes a_1 b_2 \quad (\text{merged shuffles}).
 \end{aligned}$$

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- ▶ Define a product

$$a \diamond b = \sum_{(k, \alpha, \beta) \in \overline{\mathcal{S}}_c(m, n)} a_{\alpha^{-1}(1)}b_{\beta^{-1}(1)} \otimes \cdots \otimes a_{\alpha^{-1}(k)}b_{\beta^{-1}(k)}$$

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Example:

$a = a_1, m = 1,$
 $b = b_1 \otimes b_2,$
 $n = 2.$

k	α	β	stuffles
3	$\alpha(1) = 1$	$\beta(1) = 2, \beta(2) = 3$	$a_1 \otimes b_1 \otimes b_2$
3	$\alpha(1) = 2$	$\beta(1) = 1, \beta(2) = 3$	$b_1 \otimes a_1 \otimes b_2$
3	$\alpha(1) = 3$	$\beta(1) = 1, \beta(2) = 2$	$b_1 \otimes b_2 \otimes a_1$
2	$\alpha(1) = 1$	$\beta(1) = 1, \beta(2) = 2$	$a_1 b_1 \otimes b_2$
2	$\alpha(1) = 2$	$\beta(1) = 1, \beta(2) = 2$	$b_1 \otimes a_1 b_2$

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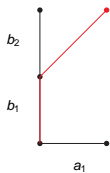
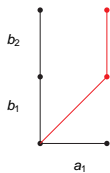
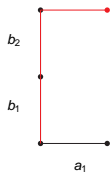
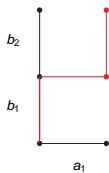
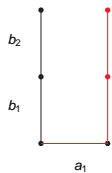
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- ▶ Let $D(m, n)$ be the set of lattice paths from $(0, 0)$ to (m, n) consisting steps either to the right, to the above, or to the above-right. For $d \in D(m, n)$ define $d(\mathbf{a}, \mathbf{b})$ to be the path d with $\mathbf{a} = (a_1, \dots, a_m)$ (resp. $\mathbf{b} = (b_1, \dots, b_n)$) sequentially labeling the horizontal (resp. vertical) and diagonal segments of d . Define

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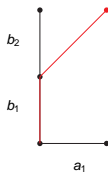
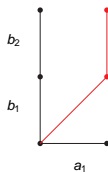
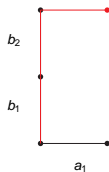
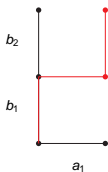
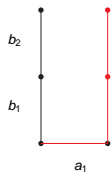
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- ▶ **Theorem** All the above products define the same algebra on $\mathbb{H}^+(A)$ (of weight $\lambda = 1$).

- A **free commutative Rota-Baxter algebra over another commutative algebra A** is a commutative Rota-Baxter algebra $\mathbb{III}(A)$ with an algebra homomorphism $j_A : A \rightarrow \mathbb{III}(A)$ such that for any commutative Rota-Baxter algebra R and algebra homomorphism $f : A \rightarrow R$, there is a unique Rota-Baxter algebra homomorphism making the diagram commute

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- ▶ When $A = \mathbf{k}[X]$, we have the free commutative Rota-Baxter algebra over X .
- ▶ Recall $(\mathbb{III}^+(A), \diamond)$ is a commutative algebra. Then the tensor product algebra (scalar extension) $\mathbb{III}(A) := A \otimes \mathbb{III}^+(A)$ is a commutative A -algebra.

Theorem (Guo-Keigher) $\mathbb{III}(A)$ with the shift operator $P(a) := 1 \otimes a$ is the free commutative RBA over A .

- Let $\lambda \in \mathbf{k}$ be given. A differential operator of weight λ on a \mathbf{k} -algebra R is a linear map $\delta : R \rightarrow R$ such that

$$\delta(xy) = \delta(x)y + x\delta(y) + \lambda\delta(x)\delta(y), \quad x, y \in R.$$

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- ▶ A **differential Rota-Baxter algebra (DRBA)** is a triple (R, δ, P) where (R, δ) is differential algebra of weight λ , (R, P) is a Rota-Baxter algebra of weight λ and $\delta P = \text{id}_R$.

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- ▶ **Examples.** The fundamental theorem of calculus shows that $(\text{Cont}(\mathbb{R}), \int_0^x, d/dx)$ is a DRBA of weight 0.
- ▶ An integral-differential algebra (R, ∂, \int) (from last talk) is a DRBA of weight 0.
- ▶ For fixed $0 < \lambda \in \mathbb{R}$, define $\delta_\lambda(f)(x) = (f(x + \lambda) - f(x))/\lambda$ and $P_\lambda(f)(x) = -\lambda \sum_{n \geq 0} f(x + n\lambda)$ on a suitable algebra R of functions (say $f \searrow 0$ and $\int_1^\infty f(t)dt < \infty$). Then $(R, \delta_\lambda, P_\lambda)$ is a DRBA of weight λ .

- Consider $R = A^{\mathbb{N}} = \{f : \mathbb{N} \rightarrow A\}$ with multiplication

$$(fg)(n) = \sum_{k=0}^n \sum_{j=0}^{n-k} \binom{n}{k} \binom{n-k}{j} \lambda^k f(n-j)g(k+j).$$

Define the operators $\delta(f)(n) = f(n+1)$, $n \geq 0$ and $P(f)(n) = f(n-1)$, $n \geq 1$, $P(f)(0) = 0$. Then (R, δ, P) is a DRBA of weight λ .

- ▶ **Free commutative DRBA** is defined by the same kind of universal property as for free commutative RBA.
- ▶ **Construction.** Let X be a set of variables. Let $A = \mathbf{k}\{X\} = \mathbf{k}[X \times \mathbb{N}] = \mathbf{k}[x^{(n)} \mid x \in X, n \in \mathbb{N}]$ be the ring of differential polynomials (of weight λ) with derivation $d(x^{(n)}) = x^{(n+1)}$. Define the free commutative RBA $(\mathbb{I}\mathbb{I}(A), P_A)$ on A as before, namely, $\mathbb{I}\mathbb{I}(A) = A \otimes \mathbb{I}\mathbb{I}^+(A)$ where $\mathbb{I}\mathbb{I}^+(A) = \bigoplus_{n \geq 1} A^{\otimes n}$ with its product defined by the mixable shuffle (or quasi-shuffle, or stuffle, or Delannoy paths). Define $\delta(a_0 \otimes a_1 \otimes \cdots \otimes a_k) = d(a_0) \otimes a_1 \otimes \cdots \otimes a_k + (a_0 a_1) \otimes a_2 \otimes \cdots \otimes a_k + \lambda(d(a_0) a_1) \otimes a_2 \otimes \cdots \otimes a_k$.
- ▶ **Theorem.** $(\mathbb{I}\mathbb{I}(A), \delta_A, P_A)$ is the free commutative DRBA on X (or differential Rota-Baxter polynomial on X).

- **Enumeration** in free commutative RBA $\mathbb{III}(\{x\})$ generated by x . A linear basis of $\mathbb{III}(\{x\})$ is the set

$$R := \{\mathfrak{x} := x^{i_0} \otimes \cdots \otimes x^{i_n} \mid i_j \geq 0, 1 \leq j \leq n, n \geq 0\}.$$

Call n the depth (degree) of \mathfrak{x} and $w := i_0 + \cdots + i_n$ the weight (arity) of \mathfrak{x} . Let $R(n, w)$ be the subset of basis elements of degree n and weight w . Let $r(n, w) = |R(n, w)|$. Then

$$\mathbf{R}(z, t) := \sum_{n, w \geq 0} r(n, w) z^n t^w = 1/(1 - z - t).$$

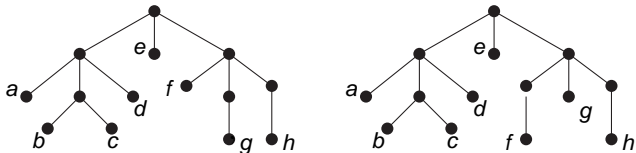
- ▶ **Free (noncommutative) Rota-Baxter algebras of rooted forests.**
Let $\mathbb{III}^{\text{NC}}(X)$ be the free Rota-Baxter algebra on a set X . Let $\mathcal{F}(X)$ be the set of rooted forests with leafs decorated by X ,

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- ▶ Define $\mathcal{F}_\ell(X)$ to be the subset of $\mathcal{F}(X)$ consisting of leaf decorated forests that do not have a vertex with adjacent non-leaf branches. Such a forest is called **leaf-spaced**.

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For example, the left tree is not leaf-spaced since the two right most branches are not separated by a leaf branch. But the right tree is leaf-spaced.

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$$F \diamond F' = \begin{cases} FF' \text{ (concatenation of trees),} & F = \bullet_x \text{ or } F' = \bullet'_x, \\ [\overline{F} \diamond F'] + [F \diamond \overline{F}'] + \lambda[\overline{F} \diamond \overline{F}'], & F = [\overline{F}], F' = [\overline{F}']. \end{cases}$$

Here the second line makes sense since a leaf decorated tree is either of the form \bullet_x for some $x \in X$, or is the [grafting](#) $[\overline{F}]$ where \overline{F} is the leaf decorated forest obtained from F by removing its root.

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- 2. If $F = F_1 \cdots F_b$ and $F' = F'_1 \cdots F'_{b'}$ are in $\mathcal{F}_\ell(X)$ with their corresponding decompositions into leaf decorated trees. Then

$$F \diamond F' = F_1 \cdots (F_b \diamond F'_1) \cdots F_{b'}.$$

$$\begin{aligned} \begin{array}{c} \bullet \\ | \\ \bullet_x \end{array} \diamond \begin{array}{c} \bullet \\ | \\ \bullet_y \end{array} &= [\bullet_x] \diamond [\bullet_y] \\ &= [\bullet_x \diamond \begin{array}{c} \bullet \\ | \\ \bullet_y \end{array}] + [\begin{array}{c} \bullet \\ | \\ \bullet_x \end{array} \diamond \bullet_y] + \lambda[\bullet_x \diamond \bullet_y] \\ &= [\bullet_x \begin{array}{c} \bullet \\ | \\ \bullet_y \end{array}] + [\begin{array}{c} \bullet \\ | \\ \bullet_x \end{array} \bullet_y] + \lambda[\bullet_x \bullet_y] \\ &= \begin{array}{c} \bullet & \bullet \\ / & \backslash \\ x & y \end{array} + \begin{array}{c} \bullet & \bullet \\ / & \backslash \\ \bullet_x & y \end{array} + \lambda \begin{array}{c} \bullet & \bullet \\ / & \backslash \\ x & y \end{array} \end{aligned}$$

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► **Theorem.** The quadruple $(\mathbf{k}\mathcal{F}_\ell(X), \diamond, \lfloor \rfloor, j_X)$ is the free nonunitary (noncommutative) Rota-Baxter algebra on X .

1. **Enumeration.** Let $r(n, d)$ be the number of leaf-spaced forests that have n non-leaf vertices and d leaf vertices. For example, in the last tree, there are six non-leaf vertices and eight leaf vertices.

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2. **Theorem** $\mathbf{R}(z, t) := \sum_{n, d \geq 0} r(n, d) z^n t^d = f\left(\frac{z}{1-z}, \frac{t}{1-t}\right)$. where

$$f(z, t) = \frac{1 - \sqrt{1 - 4zt(1+t)}}{2zt} = (1+t)\mathbf{C}(zt(1+t)).$$

Here $\mathbf{C}(x) = \frac{1 - \sqrt{1-x}}{2x}$ is the generating function of catalan numbers.

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- Free (non-unitary) DRBA generated by x is the free RBA $\mathbf{kF}_\ell(x^{(n)}, n \geq 0)$, namely the linear span of the set of leaf-spaced forests with the leaf vertices decorated by the set $\{x^{(n)}, n \geq 0\}$. Now define $\deg(x^{(n)}) = n$. Define $e(n, d)$ to be the number of such forests with n non-leaf vertices, and with total degree of the leaf vertices to be d .

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4. **Theorem**

$$\begin{aligned} E(z, t) &:= \sum_{n, d \geq 0} e(n, d) z^n s^d \\ &= f\left(\frac{z}{1-z}, \frac{s}{1-2s}\right) + \left(\frac{1}{1-z}\right) \left(\frac{1-s-s^2}{(1-s)(1-2s)}\right) - \left(\frac{1-s}{1-2s}\right). \end{aligned}$$