On algebraic simplifications of linear functional systems

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AADIOS Session

Computer algebra & Applications

- Objective of this work:

  Use computer algebra (algebraic manipulations) to simplify systems coming from mathematical physics, applied mathematics, engineering sciences or control theory

- Interest:
  - Simplify the equations of the system
    $\Rightarrow$ simplify the study of its structural properties
  - Pre-conditioner to numerical analysis methods

Outline of the talk

1. Existing works
2. Goal of the talk
3. Methodology and statement of the problem
4. Result
5. Examples
6. Implementation / Conclusion / Perspectives

I

Existing works

Bloc decomposition of a linear differential system

- Consider an ordinary differential system of the first order:

$$\partial\, y = E(t)\, y, \quad E(t) \in \mathbb{Q}(t)^{n \times n}$$

- Does-it exist an invertible change of variables

$$y = P(t)\, z,$$

such that

$$\partial\, y = E(t)\, y \quad \Leftrightarrow \quad \partial\, z = F(t)\, z,$$

where $F = P^{-1}\,(E\,P - \partial P)$ has the following form:

$$F = \begin{pmatrix} \star & 0 \\ 0 & \star \end{pmatrix} ?$$

- Many algorithms exist: Beke, Jacobson, Schwarz, Singer, Bronstein, Tsarëv, van Hoeij, Barkatou, Pflügel, Giesbrecht, ...

The eigenring method (Jacobson, Singer, Barkatou,...)

- Consider the following system :

$$\partial\, y = E(t)\, y, \quad E(t) = \begin{pmatrix} t\,(2\,t+1) & -2\,t^3 - 2\,t^2 + 1 \\ 2\,t & -t\,(2\,t+1) \end{pmatrix}$$

- Eigenring of the system: $\mathcal{E} = \{P \in \mathbb{Q}(t)^{2\times 2}\,;\, \partial P = E\,P - P\,E\}$

- We compute $\mathcal{E} = \left\{P = \begin{pmatrix} a_1 - a_2\,(t+1) & a_2\,t\,(t+1) \\ -a_2 & a_2\,t + a_1 \end{pmatrix}\,;\, a_1,\, a_2 \in \mathbb{Q}\right\}$

- Jordan form of P: $J = \mathrm{V}^{-1}\,P\,\mathrm{V} = \begin{pmatrix} a_1 & 0 \\ 0 & a_1 - a_2 \end{pmatrix}$

- Let $y = \mathrm{V}\,z$: $\partial\, y = E(t)\, y \Leftrightarrow \partial\, z = \begin{pmatrix} -t & 0 \\ 0 & t \end{pmatrix} z$

Smith canonical form

- $D = k[s]$, k field (e.g., $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$): euclidian ring.
- Theorem. $\forall\, R \in D^{q \times p}$, $\exists\, V \in \mathrm{GL}_q(D)$, $U \in \mathrm{GL}_p(D)$:

$$\overline{R} = V\,R\,U = \begin{pmatrix} \alpha_1 & 0 & \dots & \dots & 0 & \dots & 0 \\ 0 & \alpha_2 & \ddots & & \vdots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & & \vdots \\ 0 & \dots & 0 & \alpha_r & 0 & \dots & 0 \\ 0 & \dots & \dots & \dots & 0 & \dots & 0 \\ \vdots & & & & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 0 & \dots & 0 \end{pmatrix},$$

où $\alpha_1 \,|\, \alpha_2 \,|\, \dots \,|\, \alpha_r \neq 0$ et $\alpha_i \in D$, $i = 1, \dots, r$.

- Applications in control theory: polynomial approach (Rosenbrock, Kailath,...)
- Generalization: Jacobson form: $D = K\left[\frac{d}{dt}\right]$, K differential field

II

Goal of the talk

Generalization of the previous methods?

- Observation: many systems coming from mathematical physics, applied mathematics, engineering sciences or control theory can not be handled by the previous methods: partial differential equations, differential time-delay equations, ...

- Question: can we generalize these methods to handle more linear functional systems?

- Approach: constructive homological algebra

Example 1

- Model of a one-dimensional tank containing a fluid subjected to an horizontal move (Petit-Rouchon, IEEE TAC 02):

$$\begin{cases} \dot{y}_1(t) - \dot{y}_2(t-2\,h) + \alpha\, \ddot{y}_3(t-h) = 0, \\ \dot{y}_1(t-2\,h) - \dot{y}_2(t) + \alpha\, \ddot{y}_3(t-h) = 0, \end{cases} \qquad \alpha \in \mathbb{R}, \quad h \in \mathbb{R}_+.$$

- Consider $D = \mathbb{R}\left[\frac{d}{dt}, \delta\right]$, $\delta(y(t)) = y(t-h)$, and the matrix

$$R = \begin{pmatrix} \frac{d}{dt} & -\frac{d}{dt}\,\delta^2 & \alpha\,\frac{d^2}{dt^2}\,\delta \\ \frac{d}{dt}\,\delta^2 & -\frac{d}{dt} & \alpha\,\frac{d^2}{dt^2}\,\delta \end{pmatrix} \in D^{2\times 3}.$$

- Question: $\exists\ U \in \mathrm{GL}_3(D)$, $V \in \mathrm{GL}_2(D)$ such that:

$$V\,R\,U = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & \alpha_3 \end{pmatrix}, \quad \alpha_1, \alpha_2, \alpha_3 \in D\,?$$

Example 2

- Consider the 4 matrices:

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- In electromagnetism, the Dirac equations can be written :

$$\sum_{i=1}^{4} \gamma^i \frac{\partial \psi(x)}{\partial x_i} = 0,$$

where $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ and $x = (x_1, x_2, x_3, x_4)$ contains the space coordinates and the time.

Example 2

- We can then write the Dirac equations as follows:

$$\left\{\begin{array}{l} d_4\,\psi_1 - i\,d_3\,\psi_3 - (i\,d_1 + d_2)\,\psi_4 = 0, \\ d_4\,\psi_2 - (i\,d_1 - d_2)\,\psi_3 + i\,d_3\,\psi_4 = 0, \\ i\,d_3\,\psi_1 + (i\,d_1 + d_2)\,\psi_2 - d_4\,\psi_3 = 0, \\ (i\,d_1 - d_2)\,\psi_1 - i\,d_3\,\psi_2 - d_4\,\psi_4 = 0, \end{array}\right. \qquad d_i = \partial/\partial x_i.$$

- Consider $D = \mathbb{Q}(i)[d_1, d_2, d_3, d_4]$ and the matrix

$$R = \left(\begin{array}{cccc} d_4 & 0 & -i\,d_3 & -(i\,d_1 + d_2) \\ 0 & d_4 & -i\,d_1 + d_2 & i\,d_3 \\ i\,d_3 & i\,d_1 + d_2 & -d_4 & 0 \\ i\,d_1 - d_2 & -i\,d_3 & 0 & -d_4 \end{array}\right).$$

- Question: $\exists\ U \in \mathrm{GL}_4(D),\ V \in \mathrm{GL}_4(D)$ such that:

$$V\,R\,U = \left(\begin{array}{cccc} \star & \star & 0 & 0 \\ \star & \star & 0 & 0 \\ 0 & 0 & \star & \star \\ 0 & 0 & \star & \star \end{array}\right)?$$

III

Methodology and statement of the problem

Methodology

1. A linear system is defined by a matrix R with coefficients in a ring D of functional operators:

$$R\, y = 0. \quad (\star)$$

2. To $(\star)$ we associate a left D-module M (finitely presented).
3. There exists a dictionary between the properties of $(\star)$ and M.
4. Homological algebra allows to check the properties of M.
5. Effective algebra (non-commutative Gröbner/Janet bases) gives algorithms.
6. Implementation (Maple, Singular/Plural, Cocoa...).

D: Ore algebra of functional operators

- Differential operators: $A = \mathbb{Q},\ \mathbb{Q}[x_1, \ldots, x_n],\ \mathbb{Q}(x_1, \ldots, x_n)$,

$$D = A[\partial_1, \ldots, \partial_n], \quad \partial_i = \frac{\partial}{\partial x_i},$$

$$P = \sum_{0 \leq |\mu| \leq m} a_\mu(x)\, \partial^\mu \in D, \quad \partial^\mu = \partial_1^{\mu_1} \ldots \partial_n^{\mu_n}, \quad a_\mu \in A.$$

- Shift operators:

$$D = A[\sigma], \quad A = \mathbb{Q},\ \mathbb{Q}[n],\ \mathbb{Q}(n),$$

$$P = \sum_{i=0}^{m} a_i(n)\, \sigma^i \in D, \quad \sigma(a(n)) = a(n+1).$$

- Differential time-delay operators:

$$D = A[\tfrac{d}{dt}, \delta], \quad A = \mathbb{Q},\ \mathbb{Q}[t],\ \mathbb{Q}(t),$$

$$P = \sum_{0 \leq i+j \leq m} a_{ij}(t)\, \frac{d^i}{dt^i}\, \delta^j \in D, \quad \delta(a(t)) = a(t-h).$$

- Theorem. For every monomial order, there exists a Gröbner basis which can be computed by Buchberger algorithm.

The left D-module M

- Let D be an Ore algebra, $R \in D^{q \times p}$ and a left D-module $\mathcal{F}$.

- Consider $\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}$.

- As in number theory or algebraic geometry, to $\ker_{\mathcal{F}}(R.)$ we associate the finitely presented left D-module:

$$M = D^{1 \times p} / (D^{1 \times q}\, R).$$

Theorem (Malgrange)

$\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F}) = \{f : M \to \mathcal{F},\ f \text{ is left } D\text{-linear}\}$.

Statement of the problem

- Let D be a Ore algebra of functional operators
- Let $R \in D^{q \times p}$ be a matrix.
- Question:

$$\exists\, W \in \mathrm{GL}_p(D),\ V \in \mathrm{GL}_q(D) \text{ s.t. } V\,R\,W = \begin{pmatrix} \star & 0 \\ 0 & \star \end{pmatrix} ?$$

- Remark: with $M = D^{1\times p}/(D^{1\times q}\,R)$, this is equivalent to

$$\exists\, M_1,\, M_2 :\ M = M_1 \oplus M_2?$$

IV

Result

Endomorphisms of M (Cf. eigenring method)

- Let D be a Ore algebra of functional operators.
- Let $R \in D^{q \times p}$ be a matrix.
- We have the following exact commutative diagram:

$$\begin{array}{ccccccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ \downarrow .Q & & \downarrow .P & & \downarrow f & & \\ D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0. \end{array}$$

Theorem

$f \in \mathrm{end}_D(M)$ is defined by $P \in D^{p\times p}$ et $Q \in D^{q\times q}$ satisfying the relation: $R\,P = Q\,R$.

- Algorithms for computing P and Q implemented in the Maple package OREMORPHISMS based on the library OREMODULES.

Bloc decomposition theorem

Theorem

Let $R \in D^{q\times p}$, $M = D^{1\times p}/(D^{1\times q}\, R)$ *and* $f \in \mathrm{end}_D(M)$ *defined by* P *and* Q *satisfying*

$$P^2 = P, \quad Q^2 = Q \quad (\textit{idempotent matrices}) \quad \Rightarrow f^2 = f.$$

If the left D*-modules*

$$\ker_D(.P),\ \mathrm{im}_D(.P),\ \ker_D(.Q),\ \mathrm{im}_D(.Q)$$

are free, then there exist $U \in \mathrm{GL}_p(D)$, $V \in \mathrm{GL}_q(D)$ *such that*

$$\overline{R} = V\, R\, U^{-1} = \begin{pmatrix} \star & 0 \\ 0 & \star \end{pmatrix} \in D^{q\times p}.$$

- U and V can be obtained by computing bases of free left D-modules

V

Examples

Example 1: tank model (Petit-Rouchon, IEEE TAC 02)

- Consider $D = \mathbb{Q}(\alpha)\left[\frac{d}{dt}, \delta\right]$, the matrix of the system

$$R = \begin{pmatrix} \frac{d}{dt} & -\frac{d}{dt}\,\delta^2 & \alpha\,\frac{d^2}{dt^2}\,\delta \\ \frac{d}{dt}\,\delta^2 & -\frac{d}{dt} & \alpha\,\frac{d^2}{dt^2}\,\delta \end{pmatrix} \in D^{2\times 3}.$$

- The matrices $P = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ et $Q = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ satisfy:

$$R\,P = Q\,R, \quad P^2 = P, \quad Q^2 = Q.$$

- Using linear algebra, we get:

$$U = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_3(D), \quad V = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}_2(D),$$

$$\Rightarrow \overline{R} = V\,R\,U^{-1} = \begin{pmatrix} \frac{d}{dt}\,(1-\delta)\,(1+\delta) & 0 & 0 \\ 0 & \frac{d}{dt}\,(\delta^2+1) & 2\,\alpha\,\frac{d^2}{dt^2}\,\delta \end{pmatrix}.$$

Example 2: Dirac equations

$$R = \begin{pmatrix} d_4 & 0 & -i\,d_3 & -(i\,d_1 + d_2) \\ 0 & d_4 & -i\,d_1 + d_2 & i\,d_3 \\ i\,d_3 & i\,d_1 + d_2 & -d_4 & 0 \\ i\,d_1 - d_2 & -i\,d_3 & 0 & -d_4 \end{pmatrix}.$$

$$P = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \quad Q = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

$$\Rightarrow U = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix},$$

$$\Rightarrow V\,R\,U^{-1} = \begin{pmatrix} i\,d_3 - d_4 & -i\,d_1 - d_2 & 0 & 0 \\ i\,d_1 - d_2 & i\,d_3 + d_4 & 0 & 0 \\ 0 & 0 & i\,d_3 + d_4 & i\,d_1 + d_2 \\ 0 & 0 & i\,d_1 - d_2 & -i\,d_3 + d_4 \end{pmatrix}.$$

Example 3: string model (Mounier-Rudolph-Fliess-Rouchon,COCV 98)

- Consider the model of a vibrating string with interior mass:

$$\begin{cases} \phi_1(t) + \psi_1(t) - \phi_2(t) - \psi_2(t) = 0, \\ \dot{\phi}_1(t) + \dot{\psi}_1(t) + \eta_1 \, \phi_1(t) - \eta_1 \, \psi_1(t) - \eta_2 \, \phi_2(t) + \eta_2 \, \psi_2(t) = 0, \\ \phi_1(t - 2\,h_1) + \psi_1(t) - u(t - h_1) = 0, \\ \phi_2(t) + \psi_2(t - 2\,h_2) - v(t - h_2) = 0, \end{cases}$$

where h_1 and $h_2 \in \mathbb{R}_+$ satisfy $\dim_{\mathbb{Q}}(\mathbb{Q}\,h_1 + \mathbb{Q}\,h_2) = 2$.

- Consider $D = \mathbb{Q}(\eta_1, \eta_2)\left[\frac{d}{dt}, \sigma_1, \sigma_2\right]$, $M = D^{1\times 6}/(D^{1\times 4}\,R)$,

$$R = \begin{pmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ \dfrac{d}{dt} + \eta_1 & \dfrac{d}{dt} - \eta_1 & -\eta_2 & \eta_2 & 0 & 0 \\ \sigma_1^2 & 1 & 0 & 0 & -\sigma_1 & 0 \\ 0 & 0 & 1 & \sigma_2^2 & 0 & -\sigma_2 \end{pmatrix} \in D^{4\times 6}.$$

Example 3: string model

- The following matrices satisfy $R\,P = Q\,R$, $P^2 = P$ and $Q^2 = Q$:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\sigma_1^2 & 0 & 0 & 0 & \sigma_1 & 0 \\ 0 & 0 & 0 & -\sigma_2^2 & 0 & \sigma_2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, Q = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -\frac{d}{dt} + \eta_1 & \eta_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

- The modules $\ker_D(.P)$, $\mathrm{im}_D(.P)$, $\ker_D(.P)$, $\mathrm{im}_D(.P)$ are free:

$$U = \begin{pmatrix} \sigma_1^2 & 1 & 0 & 0 & -\sigma_1 & 0 \\ 0 & 0 & 1 & \sigma_2^2 & 0 & -\sigma_2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, V = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & -1 & \frac{d}{dt} - \eta_1 & -\eta_2 \end{pmatrix}.$$

Example 3: string model

- R is then equivalent to the bloc-diagonal matrix:

$$\overline{R} = V\,R\,U^{-1} = \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\sigma_1^2 & \sigma_2^2-1 & \sigma_1 & -\sigma_2 \\ 0 & 0 & \sigma_1^2\left(\frac{d}{dt}-\eta_1\right)-\left(\frac{d}{dt}+\eta_1\right) & -\eta_2\,(\sigma_2^2+1) & -\sigma_1\left(\frac{d}{dt}+\eta_1\right) & \eta_2\,\sigma_2 \end{array}\right).$$

- Consider the second diagonal bloc

$$S = \left(\begin{array}{cccc} 1-\sigma_1^2 & \sigma_2^2-1 & \sigma_1 & -\sigma_2 \\ \sigma_1^2\left(\frac{d}{dt}-\eta_1\right)-\left(\frac{d}{dt}+\eta_1\right) & -\eta_2\,(\sigma_2^2+1) & -\sigma_1\left(\frac{d}{dt}+\eta_1\right) & \eta_2\,\sigma_2 \end{array}\right),$$

and the D-module $N = D^{1\times 4}/(D^{1\times 2}\,S)$.

- An idempotent $g \in \mathrm{end}_D(N)$ is defined by the matrices

$$P' = \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ a & 0 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right), \quad Q' = \frac{1}{2}\left(\begin{array}{cc} \sigma_2^2+1 & (\sigma_2^2-1)/\eta_2 \\ -\eta_2\,(\sigma_2^2+1) & -\sigma_2^2+1 \end{array}\right),$$

$$\left\{\begin{array}{l} a = (\sigma_1^2\,\left(\frac{d}{dt}-(\eta_1+\eta_2)\right)-\frac{d}{dt}+(\eta_2-\eta_1))/(2\,\eta_2), \\ b = -\sigma_1\,(\frac{d}{dt}-(\eta_1+\eta_2))/(2\,\eta_2). \end{array}\right.$$

Example 3: string model

- The modules $\ker_D(.P)$, $\mathrm{im}\,(.P)$, $\ker_D(.Q)$ et $\mathrm{im}\,(.Q)$ are free.

$$U' = \begin{pmatrix} \sigma_1^2\,\left(\frac{d}{dt}-\eta_1-\eta_2\right)-\left(\frac{d}{dt}+\eta_1-\eta_2\right) & -2\,\eta_2 & -\sigma_1\,\left(\frac{d}{dt}-\eta_1-\eta_2\right) & 0 \\ 1 & 0 & 0 & 0 \\ -\sigma_1 & 0 & 1 & 0 \\ \sigma_1^2\,\sigma_2\,(d-\eta_1-\eta_2)-\sigma_2\,(d+\eta_1-\eta_2) & 0 & -\sigma_1\,\sigma_2\,(d-\eta_1-\eta_2) & -2\,\eta_2 \end{pmatrix},$$

$$V' = \begin{pmatrix} \eta_2 & 1 \\ \eta_2\,(\sigma_2^2+1) & \sigma_2^2-1 \end{pmatrix}.$$

$$\Rightarrow \overline{S} = V'\,S\,U'^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{d}{dt}+\eta_1+\eta_2 & \sigma_1\,\left(\frac{d}{dt}+\eta_2-\eta_1\right) & \sigma_2 \end{pmatrix}.$$

- Let $U'' = \mathrm{diag}(I_2, U')$, $V'' = \mathrm{diag}(I_2, V')$. We then have:

$$\overline{\overline{R}} = (V''\,V)\,R\,(U''\,U)^{-1} = \mathrm{diag}(I_2, \overline{S}).$$

Example 3: string model

- Note $\alpha = \eta_1 + \eta_2$ et $\beta = \eta_2 - \eta_1$. We have obtained:

$$\left\{ \begin{array}{l} \phi_1(t) + \psi_1(t) - \phi_2(t) - \psi_2(t) = 0, \\ \dot{\phi}_1(t) + \dot{\psi}_1(t) + \eta_1\,\phi_1(t) - \eta_1\,\psi_1(t) - \eta_2\,\phi_2(t) + \eta_2\,\psi_2(t) = 0, \\ \phi_1(t - 2\,h_1) + \psi_1(t) - u(t - h_1) = 0, \\ \phi_2(t) + \psi_2(t - 2\,h_2) - v(t - h_2) = 0, \end{array} \right.$$

$$\Leftrightarrow$$

$$\dot{z}_1(t) + \alpha\,z_1(t) + \dot{z}_2(t - h_1) + \beta\,z_2(t - h_1) + z_3(t - h_2) = 0.$$

$\Rightarrow$ We can then easily compute a parametrization of the string.

$\Rightarrow$ The system is σ_2-free and σ_1-free...

VI

Implementations / Conclusions / Perspectives

The OreMorphisms package

- Algorithms are implemented in a Maple package called OreMorphisms based on the library OreModules developed by Q. et Robertz:

http://wwwb.math.rwth-aachen.de/OreModules

- List of functions:
  - Morphisms, MorphismsConstCoeff, MorphismsRat,
  - Idempotents, IdempotentsConstCoeff, IdempotentsRat
  - IdempotentsMat, IdempotentsMatConstCoeffs, IdempotentsMatRat
  - KerMorphism(Rat), ImMorphism(Rat), CokerMorphism(Rat), CoimMorphism(Rat),
  - TestSurj(Rat), TestInj(Rat), TestIso(Rat).

- It is freely available with a library of examples at:

http://www.ensil.unilim.fr/~cluzeau/OreMorphisms

Conclusion / Perspectives

• We have used algebraic manipulations and computer algebra to simplify systems coming from mathematical physics, applied mathematics, engineering sciences or control theory.

$\Rightarrow$ Algorithms & Implementation & Open Questions.

• Future works:

- Use these techniques in the study of generalized Smith forms
- Study the links with the quadratic conservation laws studied in engineering sciences, the integrability of Hamiltonian systems...
- Study the algebraic structure of $end_D(M)$ by means of non-commutatives Gröbner bases (regular elements, idempotents,...)