Various algorithms for the computation of Bernstein-Sato polynomial

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Let K be a field and let

$$D = K\langle \boldsymbol{x}, \boldsymbol{\partial} \rangle = K\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \mid \{x_i \partial_j = \partial_j x_i + \delta_{ij}\}\rangle$$

denote the n-th Weyl algebra over K.

Definition (Initial Form)

Let $0 \neq w \in \mathbb{R}^n$ be a weight vector. For $p = \sum_{\alpha,\beta} c_{\alpha\beta} \boldsymbol{x}^\alpha \boldsymbol{\partial}^\beta \in D$ put $m = \max_{\alpha,\beta}\{-w\alpha + w\beta \mid c_{\alpha\beta} \neq 0\}$.
We call

$$\mathrm{in}_{(-w,w)}(p) := \sum_{\alpha,\beta : -w\alpha + w\beta = m} c_{\alpha\beta} \boldsymbol{x}^\alpha \boldsymbol{\partial}^\beta \in D$$

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For an ideal I in D, $\mathrm{in}_{(-w,w)}(I) := K \cdot \{\mathrm{in}_{(-w,w)}(p) \mid p \in I\}$ is called the **initial ideal of I w.r.t. w**.

Definitions (Global b-function)

- Let I be a holonomic D-ideal and $0 \neq w \in \mathbb{R}^n$. Set $s := \sum_{i=1}^{n} w_i x_i \partial_i$. Then $\mathrm{in}_{(-w,w)}(I) \cap K[s]$ is a non-zero principal ideal in $K[s]$. We call its monic generator $b(s)$ the **global b-function of I w.r.t. w**.

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- For $f \in K[\mathbf{x}]$ put $I_f := \langle t - f, \partial_i + \frac{\partial f}{\partial x_i}\partial_t \mid i = 1, \ldots, n \rangle \subset D\langle t, \partial_t \rangle$. Let $w = (1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$ be a weight vector such that the weight of ∂_t is 1, and let $B(s)$ be the global b-function of I_f w.r.t. w.
  We call $b(s) := B(-s-1)$ the **global b-function** (or the **Bernstein-Sato polynomial**) **of f**.

Some applications of the *b*-function

The *b*-function, respectively its minimal/maximal integer roots, are needed in other areas of *D*-module theory:

- in computing restrictions
- in computing integrations
- in computing localizations
- in computing operators and annihilators

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The *b*-function, respectively its minimal/maximal integer roots, are needed in other areas of *D*-module theory:

- in computing restrictions
- in computing integrations
- in computing localizations
- in computing operators and annihilators:
  $b(s)$ is the uniquely determined monic polynomial satisfying the identity

  $$P(\boldsymbol{x}, \boldsymbol{\partial}, s) \bullet f^{s+1} = b(s) \cdot f^s \in D[s]/\operatorname{Ann}(f^s)$$

  for some $P(\boldsymbol{x}, \boldsymbol{\partial}, s) \in D[s]$.

Annihilator based methods

The idea:

The equation

$$P(\boldsymbol{x}, \boldsymbol{\partial}, s) \bullet f^{s+1} = b(s) \cdot f^s$$

implies

$$P(\boldsymbol{x}, \boldsymbol{\partial}, s) \bullet f - b(s) \in \mathrm{Ann}(f^s).$$

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1. Compute $\mathrm{Ann}(f^s)$
(e.g. by using the algorithms by Oaku-Takayama, Levandovskyy-Oaku-Takayama or Briançon-Maisonobe)
2. Compute $\langle b(s) \rangle = \langle \mathrm{Ann}(f^s) \cap D[s], f \rangle \cap K[s]$
(e.g. by using preimages)

Cf. the talk of Viktor Levandovskyy and Jorge Morales.

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Oaku, Takayama:

1. Homogenize and compute a Gröbner basis G of J
2. Compute $J' = J \cap K[x_1\partial_1, \ldots, x_n\partial_n]$ by Gröbner basis elimination,
   compute $J' \cap K[s]$ by commutative methods

Computation of $\mathrm{in}_{(-w,w)}(I)$

Lemma

Let $<$ be a term order and let $<_{(-w,w)}$ be the (non-term) monomial order defined by

$$
\begin{aligned}
& \boldsymbol{x}^{\alpha}\boldsymbol{\partial}^{\beta} <_{(-w,w)} \boldsymbol{x}^{\gamma}\boldsymbol{\partial}^{\delta} \\
\Leftrightarrow \quad & -w\alpha + w\beta < -w\gamma + w\delta \\
\text{or} \quad & -w\alpha + w\beta = -w\gamma + w\delta \text{ and } \boldsymbol{x}^{\alpha}\boldsymbol{\partial}^{\beta} < \boldsymbol{x}^{\gamma}\boldsymbol{\partial}^{\delta}.
\end{aligned}
$$

If G is a Gröbner basis of I w.r.t. $<_{(-w,w)}$, then

$$G_{(-w,w)} = \{\mathrm{in}_{(-w,w)}(g) \mid g \in G\}$$

is a Gröbner basis of $\mathrm{in}_{(-w,w)}(I)$ w.r.t. $<$.

Weighted homogenization

Let $u, v \in \mathbb{R}^n_{>0}$. Consider $D^{(h)}_{(u,v)} = K\langle \mathbf{x}, \boldsymbol{\partial}, h\rangle$ with non commutative relations $\partial_i x_i = x_i \partial_i + h^{u_i+v_i}, 1 \le i \le n$.

For $p = \sum c_{\lambda\alpha\beta} h^{\lambda} \mathbf{x}^{\alpha} \boldsymbol{\partial}^{\beta}$ define the *weighted total degree* of p:

$$\deg_{(u,v)}(p) = \max\{\lambda + u\alpha + v\beta \mid c_{\lambda\alpha\beta} \neq 0\}$$

For $p = \sum c_{\alpha\beta} \mathbf{x}^{\alpha} \boldsymbol{\partial}^{\beta}$ define the *weighted homogenization* of p:

$$H_{(u,v)}(p) = \sum c_{\alpha\beta} h^{\deg_{(u,v)}(p) - (u\alpha + v\beta)} \mathbf{x}^{\alpha} \boldsymbol{\partial}^{\beta}$$

For a monomial order $<$ in D define a term order $<^h$ in $D^{(h)}_{(u,v)}$:

$$\begin{array}{rl} & p <^h q \\ \Leftrightarrow & \deg_{(u,v)}(p) < \deg_{(u,v)}(q) \\ \text{or} & \deg_{(u,v)}(p) = \deg_{(u,v)}(q) \text{ and } p\,|_{h=1} < q\,|_{h=1} \end{array}$$

Theorem

Let F be a non-empty subset of D.
If G^h is a Gröbner basis of $\langle H_{(u,v)}(F)\rangle$ w.r.t. $<^h$, then $G^h\,|_{h=1}$ is a Gröbner basis of $\langle F\rangle$ w.r.t. $<$.

Computation of $\mathrm{in}_{(-w,w)}(I) \cap K[s]$

The idea: Interpret $s = \sum_{i=1}^{n} w_i x_i \partial_i$ as a left D-endomorphism

$$D/\,\mathrm{in}_{(-w,w)}(I) \to D/\,\mathrm{in}_{(-w,w)}(I),\ \ p \mapsto p \cdot s.$$

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Well definition: For every $p \in \mathrm{in}_{(-w,w)}(I)$,

$$p \cdot s = (s + m) \cdot p \in \mathrm{in}_{(-w,w)}(I),$$

where $m = -w\alpha + w\beta$ for any non-zero term $c_{\alpha,\beta}\boldsymbol{x}^{\alpha}\boldsymbol{\partial}^{\beta}$ in p.

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Known fact: For a holonomic D-ideal J, $\mathrm{Hom}_D(D/J, D/J)$ is a finite dimensional K-vector space.

Hence, s has a well defined minimal polynomial.

Algorithm: MinimalPolynomial(s, G)

Input: $s \in D$ such that $\langle G \rangle \cap K[s] \neq \{0\}$ for a GB G of J
Output: $b(s) \in K[s]$ such that $J \cap K[s] = \langle b(s) \rangle$

$i \leftarrow 1$
while 1 **do**
  if there exist $a_0, \ldots, a_{i-1} \in K$ such that
    $\mathrm{NF}(s^i, G) + \sum_{j=0}^{i-1} a_j \, \mathrm{NF}(s^j, G) = 0$ **then**
    return $s^i + \sum_{j=0}^{i-1} a_j s^j$
    break
  else
    $i \leftarrow i + 1$
  end if
end while

A modular alteration

Consider $K = \mathbb{Q}$. For a prime p let $\mathbb{Z}_{(p)} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \notin p\mathbb{Z}\}$ and let $\phi_p : \mathbb{Z}_{(p)} \to \mathbb{Z}_p$ be the canonical projection.

Lemma (Noro)

For a prime p such that $G \subset \mathbb{Z}_{(p)}[\mathbf{x}, \boldsymbol{\partial}]$, $\phi_p(G)$ is a Gröbner basis of $\langle \phi_p(G) \rangle$ w.r.t. $<$ and $\phi_p(b(s)) \in \langle \phi_p(G) \rangle$.

Theorem (Noro)

Let $b_p(s)$ be the minimal polynomial of $\phi_p(s)$ in $\phi_p(D)/\langle \phi_p(G) \rangle$. If there exists $f \in \mathbb{Z}[s]$ such that $\deg(f(s)) = \deg(b_p(s))$ and $f(s) \in \langle G \rangle$, then $f(s) = b(s)$.

The implementation

To appear: `bfct.lib`

http://www.singular.uni-kl.de/

Example	Input
uw_1	$-xyz(y-z)(y+z)$
uw_2	$-xyz(x+y+z)(y-z)$
uw_3	$-xyz(x+z)(y-z)$
uw_4	$xyz(x+y+z)(3x+2y+z)$
uw_5	$-xyz(y-z)(2y+z)(y+z)$
uw_{11}	$xyz(x-z)(-x+y)(y+z)$
uw_{12}	$-xyz(x-z)(-x+y)(y-z)$
uw_{13}	$xyz(4x+2y+z)(9x+3y+z)(x+y+z)$
uw_{14}	$xyz(2y+z)(y+z)(4y+z)(3y+z)$
uw_{15}	$-xyz(-x+y-z)(3y+z)(2y+z)(y+z)$
uw_{16}	$-xyz(x-z)(2y+z)(3y+z)(y+z)$
uw_{19}	$-xyz(-x+y+2z)(x+y+2z)(y-z)(y+z)$
uw_{20}	$xyz(x-z)(x+z)(y-z)(y+z)$
uw_{22}	$-xyz(x+z)(-x+y)(y-z)(y+z)$
uw_{32}	$xyz(x-z)(x-y-z)(x-y)(y-z)$

		Asir	Singular	
Example	$\deg(b(s))$	`bfct`	`bernsteinBM`	`bfct`
uw_1	7	**0:01**	**0:01**	**0:01**
uw_2	10	10:52	0:15	**0:07**
uw_3	9	0:04	**0:02**	0:03
uw_4	8	1h:09	0:40	**0:04**
uw_5	9	**0:01**	**0:01**	0:05
uw_{11}	11	2h:51	13h:29†	**0:22**
uw_{12}	11	17:00	42h:46†	**0:19**
uw_{13}	10	37h:42$^\times$	n/a	**0:22**
uw_{14}	11	0:04	**0:03**	0:35
uw_{15}	18	27h:07$^\times$	n/a	**32:55**
uw_{16}	19	23h:55$^\times$	n/a	**13:28**
uw_{19}	18	33h:44$^\times$	n/a	**2:13**
uw_{20}	15	06h:47	55h:51†	**0:33**
uw_{22}	17	28h:43$^\times$	n/a	**3:25**
uw_{32}	13	24h:24$^\times$	n/a	**2:52**

$t^\dagger$: out of memory after time t, $t^\times$: killed by user after time t

Ongoing Work

- `bfct` is not yet able to compute the *b*-function of
$$(xz+y)(x^4+y^5+xy^4)$$
- Hyperplane arrangements in more variables
Yet unknown: the *b*-function of
$$xyzw(x+y)(x+z)(x+w)(y+z)(y+w)$$
- Other types of polynomials
Yet unknown: the *b*-function of
$$(z^3+w^4)(3z^2x+4w^3y)$$
- Applications in theoretical physics: „kind of“ ζ-functions

Thank you

Thank you for your attention!