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Stability and Buchberger Criterion for Standard Bases in Power Series Rings

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Abstract We prove that the property of a finite set of formal power series to be a standard basis of the ideal it generates is locally stable in the space of admissible term orders. Consequently, universal standard bases exist. We give a criterion for an ideal basis to be a standard basis which is an analogue to Buchberger's "critical pair criterion" for Groebner bases.

1. Introduction Let $K[[X_1,\ldots,X_n]]$ be the ring of formal power series in the indeterminates $X_1,\ldots,X_n$ with coefficients in the field K. We denote by T the set of all terms (i.e. power products) in $X_1,\ldots,X_n$, and by an *admissible order* on T we mean an ordering of T that satisfies $1 < t$ for all $t \in T$, and $t_1 < t_2$ implies $st_1 < st_2$ for all $s, t_1, t_2 \in T$. *Dixon's lemma* states that every subset S of T has a finite subset F such that for each $t \in S$, there is $s \in F$ with $s|t$. One concludes easily that every admissible term order $<$ is a well-ordering of T. For $f \in K[[X_1,\ldots,X_n]]$, we let $T(f)$ be the set of all terms that occur in f, i.e. whose coefficient in f is not zero, and $LT_<(f)$ the $<$-least element of $T(f)$. We drop the subscript $<$ if there is no danger of confusion. The following theorem summarizes known facts about standard bases.

Theorem (HIRONAKA THEOREM) *Let $<$ be an admissible order on T, I an ideal of $K[[X_1,\ldots,X_n]]$, $S = \{g_1,\ldots,g_m\}$ a finite subset of I. Then the following are equivalent.*

(i) For each $f \in I$, there is $1 \le i \le m$ such that $LT(g_i)|LT(f)$.

(ii) S is a basis of I, and for each $f \in K[[X_1,\ldots,X_n]]$, there exists $r \in K[[X_1,\ldots,X_n]]$ which is unique with the following two properties: $r =$

$f - \sum_{i=1}^{m} q_i g_i$ for some $q_1, \ldots, q_m \in K[[X_1, \ldots, X_n]]$, and $LT(g_i) \nmid t$ for all $t \in T(r)$, $1 \leq i \leq m$.

(iii) For each $f \in I$, there exist $q_1, \ldots, q_m \in K[[X_1, \ldots, X_n]]$ such that $f = \sum_{i=1}^{n} q_i g_i$ and $LT(f) \leq LT(q_i) LT(g_i)$ for $1 \leq i \leq m$.

Moreover, every ideal I of $K[[X_1, \ldots, X_n]]$ has a finite subset that satisfies the equivalent conditions above. Such a subset is then called a **standard basis** *of I (w.r.t. $<$).*

The above theorem goes back to [4] and [3]. For more details on the history of the theorem and its contributors we refer the reader to [2]. If, in the Hironaka theorem, one replaces power series by polynomials and lowest terms by leading terms, then one obtains the basic (non-algorithmic) facts about Groebner bases. The aim of this paper is to "transfer" two more results from the theory of Groebner bases to standard bases.

It is shown in [7] (non-constructively) and [5] and [9] (constructively) that the property of a finite set of polynomials to be a Groebner basis is locally stable in the space of admissible term orders, and that consequently every ideal in the polynomial ring has a finite basis which is a Groebner basis w.r.t. every admissible term order. In section 2 of this paper, we show that the analogous statement holds for standard bases in power series rings. Section 3 contains a few remarks on admissible orders of order type ω which will be needed later on. In section 4, we give a criterion for a finite set of power series to be a standard basis of the ideal it generates. This criterion is the obvious "translation" of Buchberger's "critical pair criterion" for Groebner bases.

The Hironaka theorem as stated above together with the results of sections 1 and 3 of this paper can be interpreted as saying that all basic *non-algorithmic* facts concerning Groebner bases translate into valid results on standard bases in rings of power series. Naturally, questions concerning computability raise additional difficulties in the case of power series. Section 5 of this paper contains a few remarks on the state of the art and the nature of the problems that occur.

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2. Stability of standard bases The set of all admissible orders on the set T of terms can be turned into a topological space by taking as a neighbourhood basis of any $<$ all sets of the form $\{ <' | <'$ is an admissible order on T, and $s <' t \Leftrightarrow s < t$ for all $s, t \in F \}$, where F ranges over all finite subsets of T. This topological space will from now on be denoted by $AO(T)$.

Lemma 2.1 *Let $< \in AO(T)$, $S_1, \ldots, S_m \subseteq T$. Then there exists a neighbourhood U of $<$ in $AO(T)$ such that $\min_{<'}(S_i) = \min_{<}(S_i)$ for all $<' \in U$, $1 \leq i \leq m$.*

Proof It obviously suffices to consider the case $i = 1$, $S_1 = S$. Let $s = \min_{<}(S)$, and assume for a contradiction that for every neighbourhood U of $<$ in $AO(T)$, there exists $<' \in U$ and $t \in S$ with $t <' s$. Denote by T_n the set of all terms whose total degree is $\leq n$ ($n \in \mathbf{N}$). For each $n \in \mathbf{N}$, we can thus pick $<_n \in AO(T)$ such that $<_n \cap (T_n \times T_n) = < \cap (T_n \times T_n)$, and $t_n \in S$ with $t_n <_n s$. By Dixon's lemma, there exist $n_1, \ldots, n_k \in \mathbf{N}$ such that for all $n \in \mathbf{N}$, there is $1 \leq j \leq k$ with $t_{n_j} | t_n$, and thus $t_{n_j} <_n t_n <_n s$. But if we pick $n \in \mathbf{N}$ such that n is greater than or equal to the total degrees of s and t_{n_j} ($1 \leq j \leq k$), then $s <_n t_{n_j}$ for $1 \leq j \leq k$, a contradiction. □

Theorem 2.2 *Let S be a finite subset of $K[[X_1, \ldots, X_n]]$, $< \in AO(T)$ such that S is a standard basis of the ideal it generates w.r.t. $<$. Then there exists a neighbourhood U of $<$ in $AO(T)$ such that S is a standard basis of the ideal it generates w.r.t. each $<' \in U$.*

Proof By 2.1, we can find a neighbourhood U of $<$ in $AO(T)$ such that for all $<' \in U$ and $g \in S$, $LT_{<'}(g) = LT_{<}(g)$. It is now obvious that condition (ii) of the Hironaka theorem as stated in the introduction (with the ideal generated by S taken for I) holds w.r.t $<$ if and only if it holds w.r.t. $<'$ for all $<' \in U$. □

The following corollary follows immediately from the fact that $AO(T)$ is compact (see e.g. [7]), and from the trivial observation that a standard basis w.r.t. some admissible order may be arbitrarily enlarged by finitely many elements.

Corollary 2.3 *Let I be an ideal of $K[[X_1, \ldots, X_n]]$. Then I has a* **universal standard basis**, *i.e. a finite basis which is a standard basis w.r.t. every admissible term order.*

3. Admissible orders of order type ω We will later make use of the fact that every open set in $AO(T)$ contains an admissible order of order type ω. This fact seems to belong to the general folklore of the theory. For clarity, we give a brief discussion here.

Lemma 3.1 *Let $a_1, \ldots, a_n$ be strictly positive real numbers. For each term $t = X_1^{\nu_1} \cdot \ldots \cdot X_n^{\nu_n}$, we write $w(t) = \sum_{i=1}^n \nu_i a_i$. Let $<\in AO(T)$ such that for all $s, t \in T$, $w(s) < w(t)$ implies $s < t$. Then $<$ is of order type ω.*

Proof Assume for a contradiction that there exists $t \in T$ and an ascending sequence $\{t_n\}_{n\in\mathbf{N}}$ of terms such that $t_n < t$ for all $n \in \mathbf{N}$. Using Dixon's lemma, it is not hard to define by induction an ascending sequence $\{n_i\}_{i\in\mathbf{N}}$ such that $t_{n_i} | t_{n_{i+1}}$ for all $i \in \mathbf{N}$. Then $w(t_{n_{i+1}}) - w(t_{n_i}) \geq \min\{\, a_i \mid 1 \leq i \leq n\,\}$, and thus there exists $j \in \mathbf{N}$ with $w(t_{n_j}) > w(t)$. This implies $t < t_{n_j}$, a contradiction. □

Let $\mathbf{R}(t)$ be a simple transcendental extension of the reals, ordered by setting $\mathbf{R} < t$. It is shown in [8] that an ordering $<$ of T is an admissible term order if and only if there exist strictly positive $a_1, \ldots, a_n \in \mathbf{R}(t)$ such that

$$X_1^{\mu_1} \cdot \ldots \cdot X_n^{\mu_n} < X_1^{\nu_1} \cdot \ldots \cdot X_n^{\nu_n} \iff \sum_{i=1}^n \mu_i a_i < \sum_{i=1}^n \nu_i a_i.$$

(The a_i can actually be chosen to be in $\mathbf{R}[t]$).

Proposition 3.2 *The admissible orders of order type ω form a dense subset of $AO(T)$.*

Proof Let $<\in AO(T)$, F a finite subset of T, $a_1, \ldots, a_n \in \mathbf{R}(t)$ (t transcendental over $\mathbf{R}$, $\mathbf{R} < t$) such that

$$X_1^{\mu_1} \cdot \ldots \cdot X_n^{\mu_n} < X_1^{\nu_1} \cdot \ldots \cdot X_n^{\nu_n} \iff \sum_{i=1}^n \mu_i a_i < \sum_{i=1}^n \nu_i a_i.$$

By the model-completeness of the theory of real-closed fields, we can find strictly positive $b_1, \ldots, b_n \in \mathbf{R}$ such that $\sum_{i=1}^n \mu_i b_i < \sum_{i=1}^n \nu_i b_i$ for all $X_1^{\mu_1} \cdot \ldots \cdot X_n^{\mu_n}, X_1^{\nu_1} \cdot \ldots \cdot X_n^{\nu_n} \in F$ with $X_1^{\mu_1} \cdot \ldots \cdot X_n^{\mu_n} < X_1^{\nu_1} \cdot \ldots \cdot X_n^{\nu_n}$. By varying the b_i in a suitable manner, we may assume that they are algebraically independent over $\mathbf{Q}$. We can now define $<' \in AO(T)$ by setting

$$X_1^{\mu_1} \cdot \ldots \cdot X_n^{\mu_n} <' X_1^{\nu_1} \cdot \ldots \cdot X_n^{\nu_n} \Longleftrightarrow \sum_{i=1}^{n} \mu_i b_i < \sum_{i=1}^{n} \nu_i b_i.$$

Then $<'$ lies in the neighbourhood of $<$ defined by F and has order type ω by 3.1. □

In the above proof, we used the fact that if $a_1, \ldots, a_n \in \mathbf{R}$ are strictly positive and algebraically independent, then by 3.1, the corresponding linear form induces an admissible order of order type ω on T. In the view of Weispfenning's characterization of admissible term orders via positive linear forms in [8], one is tempted to conjecture that the converse is true ,too, that every admissible term order of order type ω is induced by a positive linear form whose coefficients are strictly positive real numbers. Inspection of Weispfenning's proof shows that even in the case of order type ω, the coefficients of his linear form will in general end up being in $\mathbf{R}(t) \setminus \mathbf{R}$. Indeed, the conjecture is false: let $n = 2$, $a_1, a_2 \in \mathbf{R}$ such that $a_1 < a_2$ and a_1 and a_2 are algebraically independent over $\mathbf{Q}$. Then there exists $n \in \mathrm{N}$ such that $((n+1)/n))a_1 < a_2$ and thus $(n+1)a_1 < na_2$. It follows that the induced term order satisfies $X_1^{n+1} < X_2^n$. This shows that the total degree orders on T (e.g. the one where ties are broken lexicographically), which have order type ω, can never be induced in this way.

4. Buchberger criterion for standard bases In order to be able to formulate and prove a "critical pair criterion" for standard bases we need to introduce some terminology and notation. Let $<$ be a fixed admissible term order. If $g_1, \ldots, g_m, f \in K[[X_1, \ldots, X_n]]$, then we say that f has a *standard representation* in terms of $\{g_, \ldots, g_m\}$ (w.r.t. $<$), if there exist $q_1, \ldots, q_m \in K[[X_1, \ldots, X_n]]$ such that $f = \sum_{i=1}^{n} q_i g_i$ and $LT(f) \leq LT(q_i)LT(g_i)$ for $1 \leq i \leq m$. Condition (iii) of the Hironaka theorem can thus be rephrased by saying that S is a standard basis of I if and only if every element of I has a standard representation in terms of S. (Here, "has a standard representation" corresponds to "reduces to 0" in the case of Buchberger theory in polynomial rings.) If $f \in K[[X_1, \ldots, X_n]]$, $t \in T$, then we denote by $c(t, f)$ the coefficient of t in f. Now let $f, g \in K[[X_1, \ldots, X_n]]$ with

$s = LT(f)$, $t = LT(g)$, $us = vt = \text{lcm}(s,t)$. Then we define $S(f,g) = c(t,g)uf - c(s,f)vg$. Choosing analogy with polynomials over phonetic elegance, we call $S(f,g)$ the *s-series (w.r.t. <)* of f and g. We say that $S \subseteq K[[X_1,\ldots,X_n]]$ is *closed under s-series (w.r.t. <)* if for all $f,g \in S$, $S(f,g)$ has a standard representation in terms of S. Our goal in this section is to prove the following theorem.

Theorem 4.1 *Let S be a finite subset of $K[[X_1,\ldots,X_n]]$. Then, w.r.t. a fixed admissible term order, S is a standard basis of the ideal it generates if and only if S is closed under s-series.*

The next lemma shows that if S is closed under s-series and we have a representation of $f \in K[[X_1,\ldots,X_n]]$ in terms of S which is not standard, then this representation can be "improved". For $f \in K[[X_1,\ldots,X_n]]$, we denote by $LC(f)$ the lowest coefficient of f, i.e. $c(LT(f),f)$, and by $LM(f)$ the lowest monomial of f, i.e. $LC(f)LT(f)$.

Lemma 4.2 *Let $<$ be an admissible term order. Assume that $\{g_1,\ldots,g_m\}$ is a finite subset of $K[[X_1,\ldots,X_n]]$ which is closed under s-series, $f = \sum_{i=1}^m q_i g_i$ with $q_1,\ldots,q_m \in K[[X_1,\ldots,X_n]]$ such that*

$$\min_{1\leq i\leq m} LT(q_i)LT(g_i) < LT(f).$$

Then there exist $q_1',\ldots,q_m' \in K[[X_1,\ldots,X_n]]$ such that $f = \sum_{i=1}^m q_i' g_i$ and

$$\min_{1\leq i\leq m} LT(q_i)LT(g_i) < \min_{1\leq i\leq m} LT(q_i')LT(g_i).$$

Proof We proceed by induction on the number k of indices j such that

$$LT(q_j)LT(g_j) = \min_{1\leq i\leq m} LT(q_i)LT(g_i).$$

$k = 0$ or $k = 1$ is impossible. Let $k = 2$. W.l.o.g., we may assume that

$$LT(q_1)LT(g_1) = LT(q_2)LT(g_2) = \min_{1\leq i\leq m} L(q_i)LT(g_i).$$

Let $\sum_{i=1}^m q_i'' g_i$be a standard representation of $S(g_1,g_2)$. Since $k = 2$, we must have

$$LM(q_1)LM(g_1) = -LM(q_2)LM(g_2).$$

Hence we can find $u \in T$ and $a \in K$ such that

$$LM(q_1)g_1 + LM(q_2)g_2 = auS(g_1, g_2) = au\sum_{i=1}^{m} q_i''g_i.$$

Now we define

$$\begin{aligned} q_1' &= q_1 - LM(q_1) + auq_1'' \\ q_2' &= q_2 - LM(q_2) + auq_2'' \\ q_i' &= q_i + auq_i'' \qquad \text{for} \quad 3 \leq i \leq m. \end{aligned}$$

Then we have

$$\begin{aligned} \sum_{i=1}^{m} q_i'g_i &= \sum_{i=1}^{m} q_ig_i - LM(q_1)g_1 - LM(q_2)g_2 + au\sum_{i=1}^{m} q_i''g_i \\ &= \sum_{i=1}^{m} q_ig_i = f. \end{aligned}$$

Furthermore, we see that for $j = 1, 2$ and $1 \leq i \leq m$,

$$\begin{aligned} LT(q_j)LT(g_j) &< LT(LM(q_1)g_1 + LM(q_2)g_2) \\ &= LT(auS(g_1, g_2)) \\ &\leq LT(uq_i'')LT(g_i). \end{aligned}$$

From this and the definition of the q_i', we conclude that

$$\min_{1\leq i\leq m} LT(q_i)LT(g_i) < \min_{1\leq i\leq m} LT(q_i')LT(g_i),$$

which finishes the proof for $k = 2$. Now let $k > 2$, and assume w.l.o.g. that for $j = 1, 2$,

$$LT(q_j)LT(g_j) = \min_{1\leq i\leq m} LT(q_i)LT(g_i).$$

Then we write

$$\begin{aligned} f &= \sum_{i=1}^{m} q_ig_i \\ &= q_1g_1 - \frac{LC(q_1g_1)}{LC(q_2g_2)}q_2g_2 + \left(\frac{LC(q_1g_1)}{LC(q_2g_2)} + 1\right) q_2g_2 + \sum_{i=3}^{m} q_ig_i. \end{aligned}$$

We may now apply the induction hypothesis to the first two summands and also to the remaining $n-1$. Recombining, we obtain the desired result. □

Next, we show that the property of a finite subset of $K[[X_1, \ldots, X_n]]$ to be closed under s-series is locally stable in $AO(T)$.

Lemma 4.3 *Let $<$ be an admissible term order, $S = \{g_1, \ldots, g_m\}$ a finite subset of $K[[X_1, \ldots, X_n]]$ which is closed under s-series w.r.t. $<$. Then there exists a neighbourhood U of $<$ in $AO(T)$ such that S is closed under s-series w.r.t. every $<' \in U$.*

Proof Let $\{f_1, \ldots, f_k\}$ be the set of all s-series of pairs of elements of S, q_i^j ($1 \leq i \leq m$, $1 \leq j \leq k$) such that $\sum_{i=1}^{m} q_i^j g_i$ is a standard representation of f_j in terms of S w.r.t. $<$ for $1 \leq j \leq k$. By 2.1, we can find a neighbourhood U of $<$ in $AO(T)$ such that $LT_{<'}(f_j) = LT_{<}(f_j)$, $LT_{<'}(q_i^j) = LT_{<}(q_i^j)$, and $LT_{<'}(g_i) = LT_{<}(g_i)$ for all $<' \in U$, $1 \leq j \leq k$, and $1 \leq i \leq m$. It is obvious that U has the desired property. □

We are now in a position to prove the main theorem of this section.

Proof of theorem 4.1 The direction from left to right follows immediately from condition (iii) of the Hironaka theorem as stated in the introduction and the definition of s-series. Now assume that $S = \{g_1, \ldots, g_m\}$ is closed under s-series. We denote by $I(S)$ the ideal generated by S.

Case 1: $<$ has order type ω. We verify condition (iii) of the Hironaka theorem with $I = I(S)$. Assume for a contradiction that $f \in I(S)$ does not have a standard representation in terms of S w.r.t. $<$. Since there are only finitely many terms below $LT(f)$, we can find $q_1, \ldots, q_m \in K[[X_1, \ldots, X_n]]$ such that $f = \sum_{i=1}^{m} q_i g_i$ and

$$\min_{1 \leq i \leq m} LT(q_i)LT(g_i) = \max\{ \min_{1 \leq i \leq m} LT(q_i')LT(g_i) \mid q_1', \ldots, q_m' \in K[[X_1, \ldots, X_n]],\ f = \sum_{i=1}^{m} q_i' g_i \}.$$

By 4.2, there exist $q_1'', \ldots q_m'' \in K[[X_1, \ldots, X_n]]$ such that $f = \sum_{i=1}^{m} q_i'' g_i$ and

$$\min_{1 \leq i \leq m} LT(q_i)LT(g_i) < \min_{1 \leq i \leq m} LT(q_i'')LT(g_i),$$

a contradiction.

Case 2: $<$ has order type greater than ω. Here, we verify condition (ii) of the Hironaka theorem with $I = I(S)$. By 2.1 and 4.3, we can find a neighbourhood U of $<$ in $AO(T)$ such that for all $<' \in U$, $LT_{<'}(g_i) = LT_<(g_i)$ $(1 \leq i \leq m)$ and S is closed under s-series w.r.t. $<'$. By 3.2, we can find $<' \in U$ such that $<'$ is of order type ω. Then S is a standard basis of $I(S)$ w.r.t. $<'$ by the choice of U and case 1 above. Hence condition (ii) of the Hironaka theorem is satisfied w.r.t. $<'$. Since $LT(g_1), \ldots, LT(g_m)$ are the same w.r.t. $<'$ and $<$, this condition remains valid w.r.t. $<$. □

5. Remarks on computability The main thrust of Buchberger's critical pair criterion for ideal bases in polynomial rings is that it provides the correctness of an algorithm which computes a Groebner basis from any given basis of an ideal. Power series being infinitary objects by nature, things cannot be as smooth in power series rings. The following propositon provides us with an abstract procedure to produce a standard basis from a given basis. We will use proposition 2.1 of [2], which states the following: given an admissible term order $<$ and $f, g_1, \ldots, g_m \in K[[X_1, \ldots, X_n]]$, there exist $r, q_1, \ldots, q_m \in K[[X_1, \ldots, X_n]]$ such that

(i) $f = \sum_{i=1}^m q_i g_i + r$,

(ii) $LT(g_i) \nmid t$ for all $t \in T(r)$ and $1 \leq i \leq m$, and

(iii) $LT(f) \leq LT(q_i)LT(g_i)$ for $1 \leq i \leq m$.

Let us call such an r a *standard remainder* (w.r.t. $<$) of f mod $\{g_1, \ldots, g_m\}$. (The characteristic property that distinguishes a standard basis from an arbitrary basis is thus not the existence of the Hironaka remainder r, but its uniqueness.) The standard remainder is produced by successive elimination of unwanted terms similar to reduction of polynomials. This reduction process, however, has "λ many" steps, where λ is the order type of $<$.

Proposition 5.1 *Let $<$ be an admissible term order. Assume that an ideal machine were given which could store power series and calculate a standard remainder w.r.t. $<$ of any power series modulo any finite subset S of $K[[X_1, \ldots, X_n]]$. Then there would be an algorithm that computes a standard basis of any ideal of $K[[X_1, \ldots, X_n]]$ from any given finite basis.*

Proof Let I be an ideal of $K[[X_1, \ldots, X_n]]$, B a finite basis of I. We construct an ascending sequence $\{B_n\}_{n \in \mathbf{N}}$ of finite subsets of $K[[X_1, \ldots, X_n]]$ as follows. Set $B_0 = B$. Now assume that B_i have been defined for $i \leq n$. For each pair (f, g) of elements of B_n, let $r(f, g)$ be a standard remainder of the s-series $S(f, g)$ mod B_n. If all $r(f, g)$ are zero, we set $B_{n+1} = B_n$. If not, let r be any one of the non-zero $r(f, g)$ (e.g. the first one encountered in the "computation" on our ideal machine), and set $B_{n+1} = B_n \cup \{r\}$. We claim that the sequence $\{B_n\}_{n \in \mathbf{N}}$ is eventually constant: if not, then there would have to be a sequence $\{r_n\}_{n \in \mathbf{N}}$ such that for all $n \in \mathbf{N}$, $r_n \in B_{n+1} \setminus B_n$ and $LT(f) \nmid LT(r_n)$ for all $f \in B_n$, which is impossible by Dixon's lemma. Let $n_0 \in \mathbf{N}$ such that $B_n = B_{n_0}$ for all $n \geq n_0$, and set $S = B_{n_0}$. (I.e. we let our "algorithm" terminate when $B_{i+1} = B_i$.) Then S is closed under s-series by one of the properties of the standard remainder, and it is clear that all B_n generate the same ideal. □

Two observations are in order. Firstly, if we only wish to *decide* whether or not a given basis is a standard basis, rather than compute one, then by 4.1 it suffices to be able to decide wether or not any given power series has a standard representation in terms of some finite subset of $K[[X_1, \ldots, X_n]]$. Secondly, the above proposition of course provides an existence proof of standard bases. But this not really relevant: once the equivalence of conditions (i)-(iii) of the Hironaka theorem has been established, Dixon's lemma together with condition (i) of the Hironaka theorem provides an existence proof that is almost trivial.

Now as far as realization of the above "algorithm" is concerned, it is of course obvious that any actual computation with power series has to work with truncations, i.e. approximations up to a certain degree or weighted degree. Now if the admissible term order in question has order type ω, then it is tedious but rather straightforward to turn the above proposition into an actual algorithm that computes truncated standard bases. This, however, has already been achieved directly in [6]. In the general case, e.g. when $<$ is the lexicographical term order, the problem is that at any stage of the computation, it is necessary to know what the actual lowest term is of the new element that is being added to the given basis. But this cannot be inferred from any given truncation: lowest terms can have arbitrarily and unpredictably high degrees. It is not clear how an ideal basis could naturally be given by some sort of oracle that would provide this kind of information. In [2], we show how at least we can compute the Hironaka remainder w.r.t. the lexicographical order modulo a *principal* ideal. Then the assumption that I is principal eliminates the necessity to compute a standard basis. Another promising approach is presented in [1]: here, the computations are performed in the ring of those power series that are algebraic over the polynomial ring.

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