ACM SIGSAM Bulletin **33**/1 (1999), 5

On the Significance of the Theory of Convex Polyhedra for Formal Algebra

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Historical Background by Michael Abramson

This paper was originally published as Über die Bedeutung der Theorie der konvexen Polyeder für die formale Algebra in Jahresbericht d. Deutschen Math. Ver. **30** *(1922): 98-99. It is based on a talk given in Jena to the German Mathematical Society on 23 September 1921. Quoted in Ostrowski's later paper On Multiplication and Factorization of Polynomials. I Lexicographic Orderings and Extreme Aggregates of Terms. Aequationes Math.* **13** *(1975): 201-228.*

Über die Bedeutung der Theorie der konvexen Polyeder für die formale Algebra. Jahresbericht d. Deutschen Math. Ver. **30** (1922): 98-99. Translation by Michael Abramson.

The many-sided applications which the lexicographic ordering principle finds in algebra, suggest ideas for inquiring generally about all possible ways of distinguishing certain aggregates of terms in polynomials so that this designation remains invariant under multiplication, i.e. that the designated aggregates of terms reproduce themselves under multiplication, and thus behave similarly to the lead term of the polynomial under a fixed ordering of the variables. Thus to every polynomial $P(x_1, \ldots, x_n)$, a term $\overline{P}$ in P is assigned so that $\overline{PP_1} = \overline{P} \cdot \overline{P_1}$. We obtain all such assignments if we introduce a set of weight functions $G_1, G_2, \ldots, G_k$ and then distinguish every term of P which is the "heaviest" relative to G_1, and from those designated ones, distinguish the ones which are heaviest relative to G_2, and so forth. Using only a geometric construction, we obtain an overview for all possible cases of a polynomial. If we assign to every power product $x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$ the point in n-dimensional space with coordinates $m_1, \ldots, m_k$ and form the smallest convex polyhedron (the *weight polyhedron* of P) containing every point assigned to the individual terms of the polynomial P in this way, then we can immediately read off the behavior of P relative to an arbitrary set of weight functions from the linear boundary manifold of the weight polyhedron. The product of two polynomials corresponds to the sum of its weight polyhedra in the sense of Minkowski. This yields criteria for absolute irreducibility, which depend only on the weight properties of a polynomial, hence only on the exponent and not on the coefficients. In particular, the methods developed allow us to exhibit, for all polynomials with 2, 3, or 4 terms, all types of reducible polynomials.

This entire type of questioning is connected with Newton-Puiseux methods for the development of series of algebraic functions.