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Artificial Intelligence 75 (1995) 93-113

Artificial Intelligence

Geometric theorem proving by integrated logical and algebraic reasoning ★

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Abstract

Algebraic geometric reasoning by the Gröbner basis method and Wu's method has been shown to be powerful enough to prove those complex geometric theorems that could not be proved by ordinary logical reasoning methods. These algebraic reasoning methods, however, have a crucial limitation: they cannot correctly handle any geometric concepts involving *order relations* such as *between* and *congruent angles*. To overcome this limitation, we propose a novel geometric reasoning method, where both logical and algebraic reasoning methods are integrated into a unified reasoning process. In this paper, we prove the soundness of the proposed reasoning method and demonstrate its effectiveness with several illustrative examples.

1. Introduction

Geometric reasoning implies reasoning about geometric concepts, and geometric theorem proving is its typical example. There have been proposed two different approaches to geometric reasoning: logical and algebraic reasoning.

Most of the earlier systems for automated geometric theorem proving took the former approach based on the first-order predicate calculus, and many efforts have been made to improve their reasoning capabilities (e.g. [6,16]). However, those geometric theorems which could be proved were limited to rather simple ones.

On the other hand, Wu [17], Buchberger [3,4], and others proposed new geometric reasoning methods based on algebraic reasoning, where geometric concepts are represented by algebraic equations and geometric reasoning is realized by algebraic operations

★ This research was supported by the Grant-in-Aid for Scientific Research (B) (No.03452171) of the Japanese Ministry of Education, Science and Culture.

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SSDI 0004-3702(94)00064-6

on such equations. In the late 1980s, many researchers were attracted to the algebraic approach and various geometric reasoning methods and systems were developed based on Wu's method and on Buchberger's Gröbner basis method [5, 11-15].

Algebraic geometric reasoning by Wu's method and the Gröbner basis method has been shown to be powerful enough to prove those complex geometric theorems that could not be proved by ordinary logical reasoning methods. As will be discussed later, however, these algebraic reasoning methods have several crucial limitations and hence their reasoning capabilities are not perfect either.

Most geometric reasoning methods so far proposed took either a logical or algebraic approach exclusively and little attention was paid to the comparison and integration of these two approaches. But now, since we have much experience with both reasoning methods, we should investigate their characteristics in detail to develop a more capable reasoning method.

Roughly speaking, characteristics of logical and algebraic geometric reasoning can be summarized as follows. First, algebraic reasoning is suitable for representing and reasoning about

- metric information such as length of line segments,
- parallelism between multiple lines,
- collinearity of multiple point data, and
- congruence of points such as intersections among multiple lines.

This is because all these geometric concepts can be represented by simple polynomial equations.

In algebraic reasoning, however, it is difficult to handle geometric objects with complex internal structures such as polygons; since algebraic reasoning describes a geometric problem in terms of coordinate values of point data, the algebraic description is very flat and uniform and no structural organization mechanism is supported. Moreover, practically usable algebraic reasoning methods, such as Wu's method and the Gröbner basis method, cannot manage any inequalities, because their computation is defined in the complex number domain. This means that geometric reasoning by these methods cannot handle any geometric concepts involving *order relations* such as *between* and *congruent angles*. (This problem will be discussed in detail later.)

In logical reasoning, on the other hand, both the structural information and order relations can easily be represented in terms of logical formulae, while rather complicated reasoning capabilities are required to manage the metric information, parallelism, and collinearity and congruence of points. The complication comes from the fact that reasoning about equivalence and equality relations is required to handle these geometric concepts.

As can be seen from the above discussion, logical and algebraic reasoning methods are complementary to each other. Thus to realize a more capable geometric reasoning system, we should *integrate* these two reasoning methods; in the integrated reasoning system, limitations of one reasoning method can be compensated by the other method. This is the key idea of our research.

Recently, Arnon [2] and Iba et al. [9, 10] proposed geometric reasoning systems which employed both logical and algebraic reasoning methods. Their reasoning methods, however, should be regarded as a combination rather than integration. In Arnon's

Table 1
Four primitive predicates

on(P,L)	Point P is on line L
between(P,A,B)	Point P is located between a pair of points A and B
eqseg(A,B,C,D)	The length of line segment AB is the same as that of CD
eqang(A,B,C,D,E,F)	The size of ∠ABC is the same as that of ∠DEF

system, for example, geometric reasoning itself was realized by algebraic reasoning, and logical formulae were used mainly for problem description. That is, no substantial geometric reasoning was conducted by logical reasoning. The system proposed by Iba et al., on the other hand, incorporated a bi-directional transformation function between logical and algebraic problem descriptions, by which heterogeneous geometric reasoning was realized. However, substantial geometric reasoning was still conducted by algebraic reasoning and the role of logical reasoning was confined to an auxiliary one to support algebraic reasoning: e.g. the priority determination of variables to make algebraic reasoning efficient.

In this paper, we propose a novel geometric reasoning method which integrates both logical and algebraic reasoning into a unified reasoning process, and demonstrate its effectiveness with several illustrative examples. The integration here means that substantial geometric reasoning is conducted by both logical and algebraic reasoning.

In Section 2, following a summary of the Gröbner basis method, we point out its three limitations: the real versus complex number domain problem, the quantification problem, and the order relation problem. Then we discuss a method to handle order relations by the Gröbner basis method in detail. Section 3 proposes an integrated logical and algebraic geometric reasoning method, where the most crucial limitation of the Gröbner basis method, the order relation problem, is overcome with the help of logical reasoning. Then the soundness and completeness of the proposed method are discussed. Section 4 describes a geometric theorem proving system developed which is based on the integrated reasoning method, and its effectiveness is demonstrated with several illustrative examples in Section 5. Section 6 gives conclusions and future problems.

2. Algebraic geometric reasoning by the Gröbner basis method and its limitations

2.1. Refutational reasoning by the Gröbner basis method

Hilbert [7] defined an axiomatic theory of Euclidean plane geometry in terms of 15 axioms. All axioms except that of continuity can be described as logical formulae in the first-order predicate calculus. Table 1 shows four primitive predicates used to describe the axioms.

Let *Axioms* denote the set of Hilbert's axioms. [1] Then, the proof of a geometric property *Conc* under a given set of hypotheses $\{hyp_1, hyp_2, \ldots, hyp_n\}$ is formalized as follows.

[1] In practical applications, the axiom of continuity is eliminated from *Axioms*, because it is second-order and its elimination does not introduce any problems in proving most theorems in Euclidean plane geometry.

Table 2
Algebraic representation of the four primitive predicates (from [10])

`on(P,L)`	$ax_P + by_P + c = 0$
`between(P,A,B)`	$(\exists t \in \mathbb{R})(0 < t < 1 \wedge x_P = (1-t)x_A + tx_B \wedge y_P = (1-t)y_A + ty_B)$
`eqseg(A,B,C,D)`	$(x_B - x_A)^2 + (y_B - y_A)^2 - (x_D - x_C)^2 - (y_D - y_C)^2 = 0$
`eqang(A,B,C,D,E,F)`	$(x_B \neq x_A \vee y_B \neq y_A) \wedge (x_A \neq x_C \vee y_B \neq y_C) \wedge$ $\neg(\exists t \in \mathbb{R})(0 < t < 1 \wedge x_B = (1-t)x_A + tx_C \wedge y_B = (1-t)y_A + ty_C) \wedge$ $((x_A - x_B)(x_C - x_B) + (y_A - y_B)(y_C - y_B))^2$ $((x_D - x_E)^2 + (y_D - y_E)^2)((x_F - x_E)^2 + (y_F - y_E)^2) =$ $((x_D - x_E)(x_F - x_E) + (y_D - y_E)(y_F - y_E))^2$ $((x_A - x_B)^2 + (y_A - y_B)^2)((x_C - x_B)^2 + (y_C - y_B)^2) \wedge$ $((x_A - x_B)(x_C - x_B) + (y_A - y_B)(y_C - y_B))$ $((x_D - x_E)(x_F - x_E) + (y_D - y_E)(y_F - y_E)) \geqslant 0$

$$Axioms \cup \{hyp_1, hyp_2, \ldots, hyp_n\} \vdash Conc, \tag{1}$$

where $hyp_1, hyp_2, \ldots, hyp_n$, and $Conc$ denote logical formulae in the first-order predicate calculus. This formula is equivalent to

$$Axioms \models (hyp_1 \wedge hyp_2 \wedge \cdots \wedge hyp_n) \rightarrow Conc, \tag{2}$$

which implies that all logical models of *Axioms* satisfy

$$(hyp_1 \wedge hyp_2 \wedge \cdots \wedge hyp_n) \rightarrow Conc. \tag{3}$$

In refutational reasoning, we prove formula (2) by showing that the negation of the right-side logical formula

$$\neg\{(hyp_1 \wedge hyp_2 \wedge \cdots \wedge hyp_n) \rightarrow Conc\}$$
$$\equiv (hyp_1 \wedge hyp_2 \wedge \cdots \wedge hyp_n \wedge (\neg Conc) \tag{4}$$

is not satisfied by any logical models of *Axioms*. However, since it is known that *Axioms* is categorical and all its logical models are isomorphic, it is sufficient to show that logical formula (4) is not satisfied by a specific logical model of *Axioms*. To prove this unsatisfiability, the refutational reasoning by the Gröbner basis method uses the algebraic model of *Axioms* defined by analytic geometry in the two-dimensional orthogonal coordinate system.

First, the algebraic meanings of the four primitive predicates in Table 1 are described by a set of polynomial equations, inequations (negated equations), and inequalities with variables in $\mathbb{R}$ ($\mathbb{R}$: the set of real numbers) representing coordinate values of points. Table 2 shows such algebraic descriptions, where x_P and y_P represent the x–y coordinates of point P, respectively [14].

Then we can derive the algebraic representation of logical formula (4), since all logical formulae about Euclidean plane geometry can be described in terms of the four primitive predicates. Although the derived algebraic representation includes negated equality ($\neq$) and inequality relations ($<$, $\leqslant$), and disjunctive and negation operators ($\vee$ and $\neg$), we can eliminate them in the following way. First, by introducing an auxiliary variable z ($\in \mathbb{R}$) and applying the following rules, we can eliminate negated equality and inequality relations. Let p denote an algebraic formula with real-valued variables and coefficients. Then,

$$p \neq 0 \Leftrightarrow (\exists z) pz - 1 = 0, \tag{5}$$

$$p > 0 \Leftrightarrow (\exists z) pz^2 - 1 = 0, \tag{6}$$

$$p \geqslant 0 \Leftrightarrow (\exists z) p - z^2 = 0. \tag{7}$$

Disjunctive and negation operators can be eliminated by using the following rules. Let f and g denote algebraic formulae with real-valued variables and coefficients. Then,

$$f = 0 \vee g = 0 \Leftrightarrow fg = 0, \tag{8}$$

$$\neg(f = 0) \Leftrightarrow (\exists z) fz - 1 = 0. \tag{9}$$

Using these rules, we have the following conjunctive set of polynomial equations as the algebraic representation of logical formula (4).

$$h_1 = 0 \wedge h_2 = 0 \wedge \cdots \wedge h_n = 0 \wedge cz - 1 = 0, \tag{10}$$

where we assume for simplicity that those elementary logical formulae in (4), $hyp_1, hyp_2, \ldots, hyp_n$, and $\neg Conc$, are represented by polynomial equations $h_1 = 0$, $h_2 = 0, \ldots, h_n = 0$, and $cz - 1 = 0$, respectively. Note that we intentionally omitted all quantifiers from (10). The quantifier problem will be discussed in the next section.

Now we can prove the insatisfiability of logical formula (4) by showing that the simultaneous polynomial equations in (10) have no solution in $\mathbb{R}$.

Let $Z_{\mathbb{R}}^{[x_1,\ldots,x_v]}(\{f_1 R_1 0, \ldots, f_k R_k 0\})$ denote the set of solutions in $\mathbb{R}$ of the simultaneous polynomial equations, inequations, and inequalities $f_1 R_1 0, \ldots, f_k R_k 0$ ($R_i \in \{=, \neq, >, \geqslant\}$ $(1 \leqslant i \leqslant k)$) with variables $x_1, \ldots, x_v$. Similarly let $Z_{\mathbb{C}}^{[x_1,\ldots,x_v]}(\{f_1 R_1 0, \ldots, f_k R_k 0\})$ denote the set of solutions in $\mathbb{C}$ (the set of complex numbers). Then, the goal of the refutational algebraic reasoning is to prove

$$Z_{\mathbb{R}}^{[y_1,\ldots,y_m,z]}(\{h_1 = 0, h_2 = 0, \ldots, h_n = 0, cz - 1 = 0\}) = \emptyset, \tag{11}$$

where $\emptyset$ denotes the empty set and $y_1, \ldots, y_m, z$ variables included in polynomials $h_1, h_2, \ldots, h_n$, and $cz - 1$. Note that in our case all coefficients in the polynomials are real numbers.

In the Gröbner basis method, Eq. (11) is proved in the following way [4]. Let $F = \{f_1, \ldots, f_k\}$ denote a set of polynomials including variables $x_1, \ldots, x_v$ and coefficients $a_1, \ldots, a_u \in \mathbb{C}$, and let $\mathrm{GB}(F)$ denote the Gröbner bases of F. Then it is known that

$$\begin{aligned} &Z_{\mathbb{R}}^{[x_1,\ldots,x_v]}(\{f_1 = 0, \ldots, f_k = 0\}) = \emptyset \\ &\quad \Leftarrow Z_{\mathbb{C}}^{[x_1,\ldots,x_v]}(\{f_1 = 0, \ldots, f_k = 0\}) = \emptyset \\ &\quad \Leftrightarrow 1 \in \mathrm{GB}(F). \end{aligned} \tag{12}$$

Using this property, Eq. (11) can be proved as follows.

$$Z_{\mathbb{R}}^{[y_1,\ldots,y_m,z]}(\{h_1 = 0, h_2 = 0, \ldots, h_n = 0, cz - 1 = 0\}) = \emptyset$$

$$\Leftarrow 1 \in \mathrm{GB}(\{h_1, h_2, \ldots, h_n, cz - 1\}). \tag{13}$$

That is, if the Gröbner bases of $\{h_1, h_2, \ldots, h_n, cz - 1\}$ include 1, then Eq. (11) holds, which in turn implies that formula (1) holds and the geometric property *Conc* is proved under the set of hypotheses $\{hyp_1, hyp_2, \ldots, hyp_n\}$.

2.2. Limitations of the Gröbner basis method

The refutational reasoning by the Gröbner basis method described above is not complete and has the following three theoretical limitations.

(1) *Real versus complex number domain problem.* The Gröbner basis method works in the complex number domain while the algebraic representation of the Euclidean plane geometry is defined in the real number domain. This makes the implication in (13) ($\Leftarrow$) uni-directional. That is, while the Gröbner basis method is sound, it is not complete; it cannot prove all geometric theorems. Wu's method also shares this incompleteness. Although Kutzler [14] noted that this incompleteness is not significant because most Euclidean theorems are valid also in $\mathbb{C}$, the degree of the incompleteness is not clear.

(2) *Quantification problem.* The following is the strict representation of the proposition proved by the refutational reasoning by the Gröbner basis method.

$$Z_{\mathbb{R}}^{\{y_1,\ldots,y_m,z\}}(\{h_1 = 0, h_2 = 0, \ldots, h_n = 0, cz - 1 = 0\}) = \emptyset$$
$$\equiv \neg\{(\exists y_1, \ldots, y_m, z)(h_1 = 0 \wedge h_2 = 0 \wedge \cdots \wedge h_n = 0 \wedge cz - 1 = 0)\}. \tag{14}$$

Translating this back to the original logical formula, we have

$$(\forall \boldsymbol{x})\{(hyp_1 \wedge hyp_2 \wedge \cdots \wedge hyp_n) \rightarrow Conc\}, \tag{15}$$

where $\forall \boldsymbol{x}$ denotes that all variables are universally quantified. That is, the form of those geometric problems that can be proved by the Gröbner basis method is confined to this universally quantified one. In other words, geometric problems including existential quantifiers cannot be proved by the Gröbner basis method. Note, however, that some existential quantifiers such as those introduced by rules (5) and (9) into the problem hypotheses can be eliminated by using the following property of the first-order predicate calculus.

$$((\exists x)\alpha(x) \rightarrow \beta) \equiv (\forall x)(\alpha(x) \rightarrow \beta), \tag{16}$$

where $\alpha(x)$ and β denote logical formulae with and without free variable x respectively. This limitation is not so significant either, because most geometric theorems can be described by universally quantified logical formulae as shown in (15).

(3) *Order relation problem.* Since the Gröbner basis method processes a conjunctive set of polynomial equations, those algebraic representations including negated equality and inequality relations, and disjunctive and negation operators must be

Table 3
Reduced algebraic representations after the inequality elimination

between(P,A,B)	$(\exists t \in \mathbb{R})(x_P = (1-t)x_A + tx_B \wedge y_P = (1-t)y_A + ty_B)$
eqang(A,B,C,D,E,F)	$(x_B \neq x_A \vee y_B \neq y_A) \wedge (x_A \neq x_C \vee y_B \neq y_C) \wedge$ $\neg(\exists t \in \mathbb{R})(x_B = (1-t)x_A + tx_C \wedge y_B = (1-t)y_A + ty_C) \wedge$ $((x_A - x_B)(x_C - x_B) + (y_A - y_B)(y_C - y_B))^2$ $((x_D - x_E)^2 + (y_D - y_E)^2)((x_F - x_E)^2 + (y_F - y_E)^2) =$ $((x_D - x_E)(x_F - x_E) + (y_D - y_E)(y_F - y_E))^2$ $((x_A - x_B)^2 + (y_A - y_B)^2)((x_C - x_B)^2 + (y_C - y_B)^2)$

transformed into equations by using rules (5), (6), (7), (8), and (9). In applying the Gröbner basis method, however, rules (6) and (7) become meaningless; the Gröbner basis method works in $\mathbb{C}$ while these rules are originally defined in $\mathbb{R}$. That is, it is impossible to translate inequalities in the real number domain into valid equations in the complex number domain.

In this paper we call *order relations* those geometric relations whose algebraic representations include inequalities. As can be seen from Table 2, between and eqang are order relations, whose valid algebraic representations cannot be obtained in the Gröbner basis method. In fact all geometric theorems proved so far by the Gröbner basis method do not include any order relations [4, 5, 8, 11–15]. This is also the case in Wu's method. This limitation is crucial because the very primitive and important geometric relation eqang cannot be handled by the Gröbner basis method. In the next section, we will discuss the problem of reasoning about order relations in more details.

2.3. *Reasoning about order relations*

One way to handle order relations by the Gröbner basis method is to remove inequalities from their algebraic representations. Note that even if we used transformation rules (6) and (7), the resultant equations would introduce no effective constraints in the complex number domain. That is, the effect of the rule application is the same as the inequality elimination. Table 3 shows such reduced algebraic representations of between and eqang. The reduced algebraic representation of between becomes equivalent to the algebraic representation of on, and that of eqang implies that two angles θ_1 and θ_2 satisfy $\cos^2\theta_1 = \cos^2\theta_2$. That is, it means θ_1 and θ_2 are either congruent or supplementary. In this section, we will discuss the soundness and capability of the Gröbner basis method when we use these reduced algebraic representations.

The elimination of inequalities implies the expansion of the solution set of simultaneous equations. Let $\{g_1R_10, \ldots, g_qR_q0\}$ $(R_i \in \{=, \neq, >, \geqslant\}$ $(1 \leqslant i \leqslant q))$ denote a set of simultaneous equations, inequations, and inequalities including variables $x_1, \ldots, x_u$ and $\{g'_1R'_10, \ldots, g'_{q'}R'_{q'}0\}$ $(R'_j \in \{=, \neq\}$, $g'_jR'_j0 \in \{g_1R_10, \ldots, g_qR_q0\}$ $(1 \leqslant j \leqslant q'))$ a set of equations obtained by eliminating all inequalities. Then,

$$Z_{\mathbb{R}}^{[x_1,\ldots,x_u]}(\{g_1R_10, \ldots, g_qR_q0\}) \subseteq Z_{\mathbb{R}}^{[x_1,\ldots,x_u]}(\{g'_1R'_10, \ldots, g'_{q'}R'_{q'}0\}) \quad (17)$$

This expansion of the solution set introduces the following problems into the Gröbner basis method.

Before discussing the problems, we define several simplified notations. Let H and C denote the original algebraic representations of the set of hypotheses $\{hyp_1, hyp_2, \ldots, hyp_n\}$ and the geometric property to be proved *Conc*, respectively. Let H' and C' denote the reduced set of equations and inequations obtained by eliminating inequalities from H and C respectively. In what follows we abbreviate $Z_{\mathbb{R}}^{[y_1,\ldots,y_m]}(X)$ to $Z_{\mathbb{R}}(X)$ and $Z_{\mathbb{C}}^{[y_1,\ldots,y_m]}(X)$ to $Z_{\mathbb{C}}(X)$, where X denotes a set of algebraic equations, inequations, and inequalities.

First, from (17) we have

$$Z_{\mathbb{R}}(H) \subseteq Z_{\mathbb{R}}(H'), \tag{18}$$

$$Z_{\mathbb{R}}(C) \subseteq Z_{\mathbb{R}}(C'). \tag{19}$$

Secondly, since Eq. (11) implies $Z_{\mathbb{R}}(H) \cap \overline{Z_{\mathbb{R}}(C)} = \emptyset$, what is proved by the Gröbner basis method can be described by

$$Z_{\mathbb{R}}(H) \subseteq Z_{\mathbb{R}}(C). \tag{20}$$

2.3.1. Case I: order relations are included in the problem hypotheses

In this case, since H includes inequalities, we eliminate them to obtain H'. Suppose the proof by the Gröbner basis method succeeds. This means that $Z_{\mathbb{R}}(H') \subseteq Z_{\mathbb{R}}(C)$. Note that we assume C does not include any inequalities. Then from (18) we have $Z_{\mathbb{R}}(H) \subseteq Z_{\mathbb{R}}(C)$. Consequently, the Gröbner basis method is sound even if we eliminate all inequalities from the hypotheses of a given geometric problem.

On the other hand, suppose the geometric problem described by H and C is a valid theorem of Euclidean plane geometry, that is, assume $Z_{\mathbb{R}}(H) \subseteq Z_{\mathbb{R}}(C)$. In this case, after the inequality elimination, we may have $Z_{\mathbb{R}}(H') \not\subseteq Z_{\mathbb{R}}(C)$ because (18) holds, and as a result $Z_{\mathbb{C}}(H') \not\subseteq Z_{\mathbb{C}}(C)$. That is, H' and the negated C may not be inconsistent and consequently the Gröbner basis method may fail in proving the given problem even if it is a theorem. This means that the elimination of inequalities from the problem hypotheses can introduce another incompleteness besides the one which the Gröbner basis method originally has (i.e. (13)).

Fig. 1 shows such an example. Whereas the problem described in Fig. 1 is a valid geometric theorem, it is invalidated by the inequality elimination. In the figure, predicate `line` specifies that those points in its argument list are aligned on a straight line, and $para(x_1, x_2, x_3, x_4)$ implies that two lines specified by pairs of points x_1 and x_2, and x_3 and x_4 are parallel. The strict definitions of these predicates will be given in Section 4.

First, since predicate `line` is an order relation, it is reduced to `on` by the inequality elimination. Then, the reduced problem hypotheses result in including a geometric configuration such as shown in Fig. 2, where the conclusion `para(a,b,d,c)` does not hold. Similarly, the inequality elimination from order relation `eqang` relaxes the problem hypotheses to include a geometric configuration such as shown in Fig. 3, where again the conclusion `para(a,b,d,c)` does not hold.

```
Hypotheses:  line([a,b]).    line([b,c]).
             line([a,e,c]).  line([c,d]).
             line([a,d]).    eqang(e,a,d,e,b,c).
             line([b,e,d]).  eqseg(a,e,b,e).
Conclusion:  para(a,b,d,c).
```

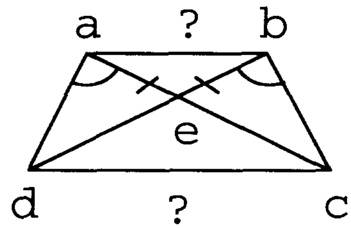

Fig. 1. Example 1: a geometric problem which includes order relations in its hypotheses.

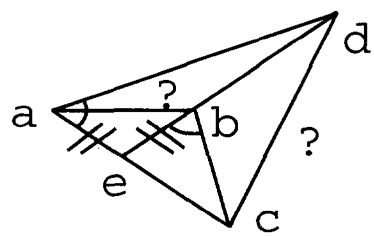

Fig. 2. A geometric configuration implied by the reduced algebraic representation of the problem hypotheses in Fig. 1, where the order of points b, e, and d is changed to d, b, and e. Then, the conclusion `para(a,b,d,c)` becomes invalid.

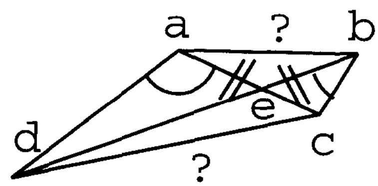

Fig. 3. Another geometric configuration implied by the reduced algebraic representation of the problem hypotheses in Fig. 1, where $\angle ead$ and $\angle ebc$ are regarded as supplementary instead of congruent. Then, the conclusion `para(a,b,d,c)` becomes invalid.

2.3.2. *Case II: order relations are included in the problem conclusion*

In this case, since C includes inequalities, we eliminate them to obtain C'. Here we assume that H does not include any inequalities. Suppose the inconsistency between H and the negated C' is proved by the Gröbner basis method and we have $Z_{\mathbb{R}}(H) \subseteq Z_{\mathbb{R}}(C')$. However, $Z_{\mathbb{R}}(H) \subseteq Z_{\mathbb{R}}(C)$ may not hold because $Z_{\mathbb{R}}(C) \subseteq Z_{\mathbb{R}}(C')$. That is, the soundness of the Gröbner basis method is invalidated by the inequality elimination from the problem conclusion C.

Fig. 4 illustrates such an example, where predicates `rangle` (right angle), `midpoint`, and `perpen` (perpendicular) are defined later. We cannot apply the Gröbner basis method to this problem, because it is not sound for this problem; the conclusion of the problem

```
Hypotheses:  line([a,d,c]).   line([c,e]).
             line([a,f,b]).   rangle(a,b,c).
             line([a,e]).     midpoint(d,a,c).
             line([c,b]).     perpen(a,b,d,e).
             line([d,f,e]).   eqseg(a,d,d,e).
Conclusion:  eqang(a,c,e,e,c,b).
```

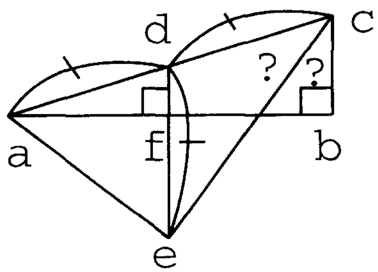

Fig. 4. Example 2: a geometric problem which includes an order relation in its conclusion.

includes order relation eqang.

In summary, with the inequality elimination, we have:

- *Expansion of incompleteness* (Case I): Some geometric theorems whose hypotheses include order relations cannot be proved.
- *Invalidation of soundness* (Case II): The proof becomes unsound for those problems whose conclusions include order relations.

3. Integrated logical and algebraic reasoning

In this section, we propose an integrated logical and algebraic reasoning method to solve the above two problems of the Gröbner basis method in handling order relations. Our basic idea is as follows. As noted before, logical reasoning can handle order relations without any problem, and moreover it can transform order relation eqang into non-order relations by using geometric theorems such as that about congruent triangles. If such a transformation is possible, then the Gröbner basis method can be applied to the transformed problem. Based on this idea, we developed a geometric theorem proving system, where both logical and algebraic reasoning modules cooperate to solve a given problem. Fig. 5 shows the organization of the system.

3.1. Scheme of the integrated reasoning

A geometric problem to be proved is described by using those predicates defined in Section 4. As shown in Figs. 1 and 4, the problems given to our system include no disjunctive or negation operators and all variables denoting points, a, b, c and so on, are regarded as being universally quantified. Note that, as discussed in Section 2.2 and as will be discussed below, this does not mean that our system cannot handle these logical operators or existential quantifiers.

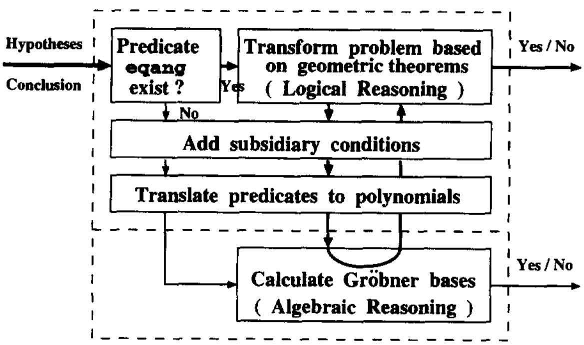

Fig. 5. Organization of the integrated logical and algebraic geometric reasoning system.

First, the system examines if the problem description includes eqang. If not, it directly activates the algebraic reasoning module based on the Gröbner basis method to prove the problem. Otherwise, the system activates the logical reasoning module.

The logical reasoning module first applies the stored geometric axioms and theorems to the problem hypotheses to derive all possible new geometric relations, which then are added to the hypotheses. If the conclusion is included in the hypotheses expanded by this forward reasoning, the problem is proved. Otherwise, if the conclusion does not include eqang, then the system activates the algebraic reasoning module to prove the conclusion under the expanded hypotheses. In this case, the inequality elimination is applied to the expanded hypotheses. If the conclusion is described by eqang, on the other hand, it starts the backward reasoning to derive sub-goals of the conclusion.

Since there are multiple applicable axioms and theorems in the backward reasoning, its process can be described by an AND–OR tree whose root node denotes the conclusion (Fig. 6). The system examines each sub-goal one by one to prove it:

(1) If the sub-goal is included in the expanded hypotheses, then it is proved.
(2) If the sub-goal is described by eqang again, then conduct the backward reasoning recursively.
(3) If the sub-goal does not include eqang, then activate the algebraic reasoning module to prove it under the expanded hypotheses.

Using these integrated reasoning methods, the system tries to form a derivation AND sub-tree included in the AND–OR tree which connects the given hypotheses to the conclusion.

While the above reasoning scheme handles only eqang, we can conduct a similar reasoning based on the other primitive order relation, between. In our system, however, we do not accept those problems whose conclusions include between, nor do we use those geometric axioms and theorems in the backward reasoning that generate sub-goals including between. This is partly because a few problems include between in their conclusions and partly because between can be transformed into non-order relations only in very special cases.

When between is included in the problem hypotheses, the system uses it as it is in the forward reasoning, while between(P,A,B) is transformed into the non-order relations

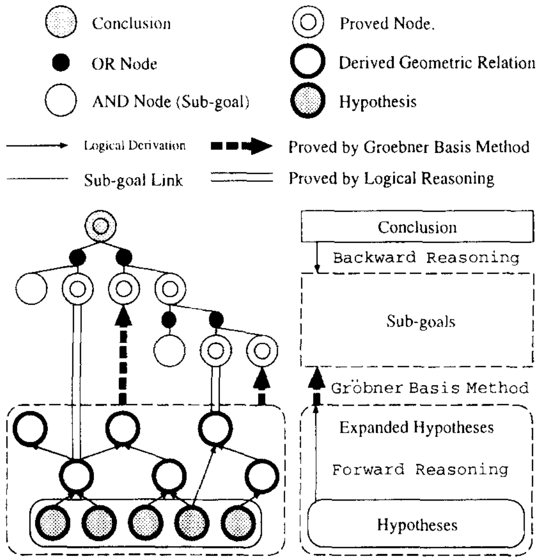

Fig. 6. AND–OR tree representing a reasoning process.

$$on(P,L) \wedge on(A,L) \wedge on(B,L) \wedge (A \neq B) \wedge (A \neq P) \wedge (P \neq B)$$

before applying the algebraic reasoning module. This is equivalent to the inequality elimination from the hypotheses, and hence the soundness of the proof is maintained. Thus, the reasoning capability of our system about `between` is very limited.

3.2. Properties of the integrated reasoning

3.2.1. Properties of the logical reasoning

In the logical reasoning module, ordinary deductive reasoning is conducted by using a set of geometric theorems $Th = \{th_1, th_2, \ldots, th_l\}$ as well as *Axioms*, where $Axioms \models th_i$ $(1 \leqslant i \leqslant l)$.

To realize efficient reasoning, the axioms and theorems used in the forward and backward reasoning processes are confined to those described by Horn clauses:

$$(\forall x)((A_1 \wedge \cdots \wedge A_r) \rightarrow B), \tag{21}$$

where A_i $(1 \leqslant i \leqslant r)$, B are positive literals. Thus, although the logical reasoning is sound, it is not complete.

3.2.2. Soundness of the integrated reasoning

The proof by the integrated reasoning can be classified into the following three types:

(1) The problem does not include `eqang` and is proved by the algebraic reasoning module.
(2) The problem includes `eqang` and is proved by the logical reasoning module.

(3) The problem includes eqang and is proved by the cooperation of the logical and algebraic reasoning modules.

The soundness of the proof in cases (1) and (2) is trivial. Note that as discussed before, the soundness is maintained even if the problem includes the other order relation `between`, because it is included only in the problem hypotheses.

As for case (3), the soundness of the proof is trivial if eqang is included only in the problem hypotheses. If the conclusion includes eqang, then the backward reasoning is conducted to generate a set of sub-goals, whose soundness is guaranteed by the incorporated axioms and theorems. When the conclusion is proved by the integrated reasoning, each sub-goal is proved by either sound logical or sound algebraic reasoning. Consequently, the soundness of the overall reasoning process is guaranteed. That is, the integrated reasoning realizes the sound proof of those geometric problems that the Gröbner basis method alone cannot perform by sound reasoning, i.e. those including eqang in their conclusions. An example of this will be given in Section 5.

3.2.3. Completeness of the integrated reasoning

Since both logical and algebraic reasoning modules are incomplete, the integrated reasoning is not complete either. However, it can prove some problems which cannot be proved by the Gröbner basis method alone. As discussed in Section 2.3.1, the Gröbner basis method fails in proving some theorems including eqang in their hypotheses. In the integrated reasoning, on the other hand, the forward reasoning expands the hypotheses by adding newly derived geometric relations before applying the Gröbner basis method. If this forward reasoning transforms eqang correctly into non-order relations, $Z_{\mathbb{C}}(Hyp \cup New) \subset Z_{\mathbb{C}}(C)$ holds and hence the Gröbner basis method can prove the problem. Here $Hyp \cup New$ denotes the algebraic representation of the expanded hypotheses after the inequality elimination. Thus, the range of the geometric problems which can be proved is expanded by the integrated reasoning. An example of this will be given in Section 5.

4. Integrated geometric theorem proving system

We developed a geometric theorem proving system based on the integrated reasoning method described above, where several additional reasoning facilities are incorporated to augment its reasoning capability.

4.1. Problem description method

A geometric problem to be proved is described as a logical formula

$$(hyp_1 \wedge hyp_2 \wedge \cdots \wedge hyp_n) \rightarrow Conc, \tag{22}$$

where $hyp_1, hyp_2, \ldots, hyp_n$, and $Conc$ are positive literals and all variables are universally quantified.

The following nine predicates are used for the problem description:

Table 4
Definitions of higher-level predicates

collinear(A,B,C)	$\Leftrightarrow$ (∃ L)(on(A,L)∧on(B,L)∧on(C,L))
online(P,A,B)	$\Leftrightarrow$ noteq(A,B)∧collinear(P,A,B)
midpoint(P,A,B)	$\Leftrightarrow$ noteq(A,B)∧collinear(P,A,B)∧eqseg(A,P,P,B)
para(A,B,C,D)	$\Leftrightarrow$ ¬(∃P)(online(P,A,B)∧online(P,C,D))
rangle(A,B,C)	$\Leftrightarrow$ noteq(A,B)∧noteq(B,C)∧ (∃P)(midpoint(B,A,P)∧eqseg(A,C,C,P))
perpen(A,B,C,D)	$\Leftrightarrow$ noteq(A,B)∧noteq(C,D)∧ ((¬noteq(A,C)∧rangle(B,A,D))∨ (noteq(A,C)∧online(A,C,D)∧rangle(B,A,C))∨ (noteq(A,C)∧¬online(A,C,D)∧online(C,A,B)∧ rangle(A,C,D))∨ (noteq(A,C)∧¬online(A,C,D)∧¬online(C,A,B)∧ (∃P)(online(P,A,B)∧online(P,C,D)∧rangle(A,P,C))))

eqseg	eqang	collinear
online	midpoint	para
rangle	perpen	line

Note that all these nine predicates take points as their arguments.

Among the four primitive predicates listed in Table 1, between is substituted by line, whose meaning will be given later. Note that since line is an order relation like between, it can be used only to describe the problem hypotheses.

We exclude on, because no information useful for reasoning is specified by describing either on(A,L) or on(A,L) ∧ on(B,L), and because the collinearity among more than three points, on(A,L) ∧ on(B,L) ∧ on(C,L) ∧ · · ·, can be described by using the predicate collinear. Note that on is used internally by the system.

4.1.1. Definition of higher-level predicates

To facilitate the problem description, it is often useful to define higher-level predicates. As pointed out in [14], however, their geometric meanings must be specified very strictly; careless loose definitions can lead to the failure of proof and sometimes introduce unsoundness into geometric reasoning. Table 4 shows the strict definitions of six higher-level predicates, which are derived by the method described in [14]. Here predicate noteq(A,B) implies that two points A and B are different and its algebraic representation is defined as follows.

$$noteq(A,B) \Leftrightarrow x_A \neq x_B \vee y_A \neq y_B. \tag{23}$$

Thus noteq is a non-order relation and is often used to specify subsidiary conditions to exclude degenerated geometric configurations. Note that several redundant noteqs are included in Table 4 to clarify non-degenerated case specifications.

As is obvious from the definitions in Table 4, these six predicates are non-order relations since they are defined in terms of the primitive non-order relations on, eqseg, and noteq. Table 5 shows their algebraic representations.

The definition of and reasoning based on para involve some complications. As shown in Tables 4 and 5, para(A,B,C,D) is originally defined as a non-order relation. In logical reasoning, however, we must regard it as an order relation to apply geometric theorems about the congruence of corresponding (alternate) angles formed by a line intersecting

Table 5
Algebraic representations of higher-level predicates

collinear(A,B,C)	$(y_B - y_A)x_C + (x_A - x_B)y_C + (x_B y_A - x_A y_B) = 0$
online(P,A,B)	$(x_A \neq x_B \vee y_A \neq y_B) \wedge (y_A - y_P)x_B + (x_P - x_A)y_B + (x_A y_P - x_P y_A) = 0$
midpoint(P,A,B)	$(x_A \neq x_B \vee y_A \neq y_B) \wedge 2x_P - x_A - x_B = 0 \wedge 2y_P - y_A - y_B = 0$
para(A,B,C,D)	$(x_A \neq x_B \vee y_A \neq y_B) \wedge (x_C \neq x_D \vee y_C \neq y_D) \wedge$ $(y_B - y_A)x_C + (x_A - x_B)y_C + (x_B y_A - x_A y_B) \neq 0 \wedge$ $(x_B - x_A)(y_D - y_C) - (x_D - x_C)(y_B - y_A) = 0$
rangle(A,B,C)	$(x_A \neq x_B \vee y_A \neq y_B) \wedge (x_B \neq x_C \vee y_B \neq y_C) \wedge$ $(x_A - x_B)(x_C - x_B) + (y_A - y_B)(y_C - y_B) = 0$
perpen(A,B,C,D)	$(x_A \neq x_B \vee y_A \neq y_B) \wedge (x_C \neq x_D \vee y_C \neq y_D) \wedge$ $(x_B - x_A)(x_D - x_C) + (y_B - y_A)(y_D - y_C) = 0$

with parallel lines. That is, to identify the corresponding (alternate) angles, we need to determine if the directions of the pair of line segments AB and CD coincide with each other or not.

To solve this problem, we regard para(A,B,C,D) as denoting the following order relation ord-para(A,B,C,D) when it is included in the problem hypotheses.

ord-para(A,B,C,D) ⇔ para(A,B,C,D) ∧
(∃ P)(between(P,A,D) ∧ between(P,B,C)),

where ord-para(A,B,C,D) means that line segments AB and CD are parallel and have the same direction. Then, the system can conduct the forward logical reasoning using geometric theorems about the corresponding (alternate) angles to derive new geometric relations.

On the other hand, if para is included in the conclusion, it is regarded as the non-order relation as defined in Tables 4 and 5. Since the algebraic representation of ord-para is reduced to that of para by the inequality elimination and ord-para appears only in the hypotheses, the soundness of the integrated reasoning is maintained. In short, the system regards those paras in the hypotheses as order relations ord-paras while paras in the conclusion are regarded as non-order relations. Users should take this into account when describing a problem.

4.2. Specification of geometric configuration

In proving geometric theorems, we usually draw a figure based on the problem description and try to find a proof strategy by analyzing the figure. Such reasoning assisted by geometric figures is called *diagrammatic reasoning* [1].

In our system, we prepare a special predicate line to describe the geometric configuration, by which the logical reasoning is facilitated. line takes a plain list of points as its argument and declares the existence of a straight line as well as the fact that the points in the list are aligned on that line. From this definition, it is obvious that line is an order relation and that between can be described by line. That is, line can appear only in the problem hypotheses to maintain the soundness of the integrated reasoning. Moreover, since line subsumes non-order relations collinear and online, we need not use the latter predicates to describe the hypotheses; `collinear` and `online` are used only to describe the conclusion.

In logical reasoning, the system tries to prove only those relations between line segments and angles which are located on and formed by those straight lines declared by `line`. That is, the system neglects as irrelevant those line segments and angles which are not included in the declared geometric configuration. With this "focusing" mechanism, the number of derived geometric relations in the forward reasoning and sub-goals in the backward reasoning is reduced considerably and as a result the efficiency of the logical reasoning is improved.

4.3. Geometric axioms and theorems

The system incorporates the following axioms and theorems for logical reasoning, where (m, n) stands for: m, the number of axioms and theorems used in the forward reasoning and n, that used in the backward reasoning.

(1) *Geometric axioms* (6, 3). Among the fifteen Hilbert axioms, only simple ones are used excluding the continuity and Archimedes axioms.

(2) *Theorems about higher-level predicates* (16, 4). These theorems specify the definitions of higher-level predicates by using primitive ones.

(3) *Fundamental geometric theorems* (30, 19). These include those theorems about congruent triangles, the congruence of corresponding and alternate angles formed by a line intersecting parallel lines, the congruence of two base angles in a equilateral triangle, and so on.

(4) *Inference rules for equivalent relations.* eqseg, eqang, and para are equivalent relations. The system applies inference rules about equivalent relations to construct equivalent classes based on each of eqseg, eqang, and para, and conducts reasoning based on the equivalent classes. In the backward reasoning to prove eqang, especially, a two stage reasoning process is conducted to generate sub-goals. Suppose the current goal is to prove $\theta_1 = \theta_2$. Then, based on the equivalent classes including θ_1 and θ_2, $\Theta_1 = \{\theta_{11}, \ldots, \theta_{1m}\} \ni \theta_1$ $\Theta_2 = \{\theta_{21}, \ldots, \theta_{2n}\} \ni \theta_2$, the system first generates a set of sub-goals: $\theta_{1i} = \theta_{2j}$ $(1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n)$. At the second stage, the geometric axioms and theorems are applied to each $\theta_{1i} = \theta_{2j}$ to derive sub-sub-goals. If a certain $\theta_{1i} = \theta_{2j}$ is proved successfully, then Θ_1 and Θ_2 are merged into one equivalent class and hence the goal $\theta_1 = \theta_2$ is proved.

In the forward reasoning all possible axioms and theorems are applied to derive new geometric relations, and the breadth-first search is conducted in the backward reasoning. Even if we use these brute force search methods, the efficiency is kept reasonable because of the focusing mechanism described before.

4.4. Transformation to algebraic representation

The logical problem description is transformed into the corresponding algebraic one based on Tables 2, 3, and 5. Note that these transformation rules are strict in the sense

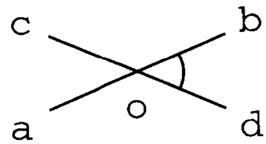

Fig. 7. Straight lines, line segments, and angles.

that appropriate subsidiary conditions to exclude degenerated geometric configurations are attached.

By defining the coordinate system properly, we can reduce the number of variables in the algebraic representation. This makes the computation by the Gröbner basis method efficient. In our system, the point which is specified to be on the largest number of lines is selected as the origin, and the one among these lines on which the largest number of points are located as the x-axis.

5. Proof examples

In this section we demonstrate the effectiveness of the integrated geometric reasoning system with several selected examples.

5.1. Notation

We use the following notations to describe the process of reasoning.

- *Internal representation of lines, line segments, and angles.* Suppose the geometric configuration shown in Fig. 7 is given. Inside the system, the straight line passing through points a, o, and b is described by the list [a,o,b], line segment ab by seg(a,b), and ∠bod by ang(o,[a,o,b],[c,o,d]).
- *Internal representation of equivalent classes.* The equivalent classes defined by eqseg, eqang, and para are described by lists of line segments, angles, and straight lines, respectively. Right angles are also grouped into a list named RANGLE. The given and derived hypotheses described by the other predicates are stored in a list named OTHERS.

5.2. Example 1

Problem. *Let* e *denote the intersection of the pair of diagonals of quadrangle* abcd. *If* ae = be *and* ∠ead = ∠ebc, *then prove* ab‖dc (Fig. 1).

As discussed before, the Gröbner basis method fails in proving this problem because order relation eqang is included in the hypotheses (see Figs. 2 and 3). In the integrated reasoning, first the forward reasoning was activated. Although eight geometric axioms and theorems were applied to derive ten new geometric relations (Fig. 8), the conclusion could not be proved. Then, since the conclusion is non-order relation para(a,b,d,c), the Gröbner basis method was applied using the expanded hypotheses and the conclusion.

```
EQSEG =[[seg(a,c),seg(b,d)]  [seg(a,d),seg(b,c)]
        [seg(a,e),seg(b,e)]  [seg(c,e),seg(d,e)]]
EQANG =[[ang(a,[a,b],[a,d]),       ang(b,[b,a],[b,c])     ]
        [ang(a,[a,b],[a,e,c]),     ang(b,[b,a],[b,e,d])   ]
        [ang(a,[a,d],[a,e,c]),     ang(b,[b,c],[b,e,d])   ]
        [ang(c,[c,b],[c,d]),       ang(d,[d,a],[d,c])     ]
        [ang(c,[c,b],[c,e,a]),     ang(d,[d,a],[d,e,b])   ]
        [ang(c,[c,d],[c,e,a]),     ang(d,[d,c],[d,e,b])   ]
        [ang(e,[a,e,c],[b,e,d]),ang(e,[c,e,a],[d,e,b])]
        [ang(e,[a,e,c],[d,e,b]),ang(e,[b,e,d],[c,e,a])]]
PARA  =[]
RANGLE=[]
OTHERS=[]
```

Fig. 8. Internal representation of the reasoning process in Example 1.

25 polynomial equations were derived from the logical descriptions, and it was shown that their Gröbner bases included 1. Thus the problem was proved successfully.

5.3. Example 2

Problem. *Let* abc *be a triangle with right angle* ∠abc *and* d *be the midpoint of the diagonal side* ac. *Let* f *be the root of the perpendicular from* d *to side* ab, *and* e *the point on the extended perpendicular such that* ad = de. *If we connect* e *with* a *respectively* c *by line segments, then prove* ∠ace = ∠ecb (Fig. 4).

Since this problem also includes eqang, the forward reasoning was conducted first to derive 12 new geometric relations (Fig. 9). Here again the logical reasoning failed, which then activated the backward reasoning because the conclusion was described by eqang. First two sub-goals were generated using the equivalent classes defined by eqang, to which four geometric axioms and theorems were applied to generate six sub-sub-goals. From a sub-goal at the first level eqang(ang(c,[c,b],[c,e]), ang(e,[e,c],[e,f,d])), para([b,c],[e,f,d]) was derived as a sub-sub-goal. Here, the directions of two lines [b,c] and [e,f,d] were shown to coincide with each other because points b and f are aligned on the same line [a,f,b] and points c and d are aligned on the same line [a,d,c]. Therefore, this sub-sub-goal can be considered as being described by non-order relation para. Consequently, the Gröbner basis method was applied using the expanded hypotheses and this sub-sub-goal. Then it was shown that the Gröbner bases of the 37 derived polynomials included 1, and hence the problem was proved successfully.

5.4. Example 3

Problem. *Given a triangle* abc, *let* d *be the point on side* bc *such that* ab = bd = dc *and* m *the point on side* ab *such that* md∥ac. *Let* e *be the midpoint of segment* bd *and* n *the intersection of line segments* ae *and* dm. *Then prove* ∠cad = ∠dae ([16] Fig. 10).

The similar forward and backward reasonings were conducted to produce the expanded hypotheses (Fig. 11) and two sub-goals based on the equivalent classes defined by eqang.

```
EQSEG =[[seg(a,d),seg(c,d),seg(d,e)]]
EQANG =[[ang(a,[a,d,c],[a,e]),  ang(e,[e,a],[e,f,d]) ]
        [ang(b,[b,c],[b,f,a]),  ang(f,[a,f,b],[d,f,e]),
         ang(f,[a,f,b],[e,f,d]),ang(f,[b,f,a],[d,f,e]),
         ang(f,[b,f,a],[e,f,d])                       ]
        [ang(c,[c,b],[c,d,a]),  ang(d,[c,d,a],[d,f,e])]
        [ang(c,[c,d,a],[c,e]),  ang(e,[e,c],[e,f,d]) ]]
PARA  =[]
RANGLE=[ang(b,[b,c],[b,f,a]),  ang(f,[a,f,b],[d,f,e]),
        ang(f,[a,f,b],[e,f,d]),ang(f,[b,f,a],[d,f,e]),
        ang(f,[b,f,a],[e,f,d])                       ]
OTHERS=[perpen([a,b,f],[b,c]),perpen([a,b,f],[d,e,f]),
        midpoint(d,a,c)                              ]
```

Fig. 9. Internal representation of the reasoning process in Example 2.

```
Hypotheses:  line([b,e,d,c]).  line([a,d]).
             line([b,m,a]).    eqseg(a,b,b,d).
             line([a,n,e]).    eqseg(b,d,d,c).
             line([a,c]).      eqseg(b,e,e,d).
             line([m,n,d]).    para(a,c,m,d).
Conclusion:  eqang(c,a,d,d,a,e).
```

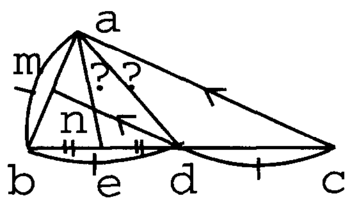

Fig. 10. Example 3 (from [16]).

```
EQSEG =[[seg(a,b),seg(b,d),seg(c,d)]
        [seg(b,e),seg(d,e)]]
EQANG =[[ang(a,[a,c],[a,d]),     ang(d,[d,a],[d,n,m])    ]
        [ang(a,[a,c],[a,m,b]),   ang(m,[a,m,b],[m,n,d])  ]
        [ang(a,[a,c],[a,n,e]),   ang(n,[a,n,e],[m,n,d]),
         ang(n,[d,n,m],[e,n,a])                          ]
        [ang(a,[a,d],[a,m,b]),   ang(d,[c,d,e,b],[d,a])  ]
        [ang(c,[c,a],[c,d,e,b]),ang(d,[c,d,e,b],[d,n,m])]
        [ang(n,[a,n,e],[d,n,m]),ang(n,[e,n,a],[m,n,d])  ]]
PARA  =[[[a,c],[m,n,d]]
        [[c,a],[d,n,m]]]
RANGLE=[]
OTHERS=[midpoint(d,b,c),midpoint(e,b,d)]
```

Fig. 11. Internal representation of the reasoning process in Example 3

The first sub-goal `eqang(ang(a,[a,d], [a,n,e]),ang(d,[d,a],[d,n,m]))` was successfully transformed to a single sub-sub-goal without any order relation `para([a,n,e],[m,n,d])` by the backward reasoning. Then, the Gröbner basis method was applied, but it was terminated by time-out. Note that since this sub-sub-goal is not a valid proposition, the Gröbner basis method cannot prove it in theory.

Next, from the second sub-goal `eqang(ang(a,[a,d], [a,n,e]), ang(d,[d,a],[d,n,m]))`, a single sub-sub-goal `eqseg(seg(a,n), seg(d,n))` was generated. This time, the Gröbner basis method worked successfully for 37 polynomial equations, and the problem was proved.

6. Conclusion

In this paper we first pointed out that the algebraic reasoning by the Gröbner basis method (and Wu's method) cannot handle order relations correctly although they are essential to geometric reasoning. To solve this problem, we proposed an integrated logical and algebraic reasoning method. In our method, the logical reasoning decomposes a given geometric problem into sub-problems as well as derives new geometric relations. Then those sub-problems without order relations are solved by the Gröbner basis method using the original and newly derived hypotheses. In other words, logical and algebraic reasoning modules cooperate with each other to solve the geometric problem. We proved the soundness of our integrated reasoning method and demonstrated its effectiveness with illustrative examples which could not be proved by the Gröbner basis method alone.

The following are future problems to be studied.

(1) The current system cannot prove those problems whose conclusions (and subgoals) include order relations `between`. Moreover, reasoning based on `ord-para` and `para` in the current system is not sophisticated. To handle these order relations, we have to analyze their fundamental conceptual structures extensively and develop a more capable logical reasoning method.

(2) The current system cannot handle algebraic relations between angles: e.g. the algebraic constraint that the sum of three interior angles of a triangle is equal to 180°. To handle these relations, we have to introduce another algebraic reasoning mechanism.

(3) Many geometric problems include complex objects with internal structures (e.g. polygons) and propositions about their quantitative properties (e.g. area sizes). To handle these geometric concepts, we have to introduce a higher-level descriptive method to represent regions and planes by sets of points.

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