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$\mathcal{U}_q(sl(2))$ Satisfies a Bernstein Duality

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It is well known that in some cases the functor $\mathrm{Ext}^{\mu}_{R}(-, R)$ defines a duality between module categories. In earlier papers we studied when this duality can be represented by a bimodule and have characterized when this happens. In this paper, using some computational methods of noncommutative Gröbner bases in the construction of projective resolutions of irreducible finite-dimensional representations, we show new examples of algebras satisfying this property.

Introduction

Let R be a K-algebra over a field K. In the particular case of an enveloping algebra of a finite-dimensional solvable Lie algebra a local duality was studied by G. Barou and M. P. Malliavin in Barou and Malliavin (1985). Later this duality was extended to a wide class of algebras by the authors (see, Gómez *et al.*, 1996; Jara and Jódar, 2000). Algebras in this class satisfy certain properties: Auslander–Gorenstein condition, they are Cohen–Macaulay, $\mathrm{idim}(R) = \mathrm{GKdim}(R)$ and in addition they have the strong second layer condition. We refer to Gómez *et al.* (1996) and Jara and Jódar (2000) as basic papers on this theory where appear many examples of algebras satisfying all these properties.

Local dualities defined by R^0, the underlying R-bimodule of the dual coalgebra R^0, are characterized as those such that, for any cofinite prime ideal P of R, the K-dimension of R/P and its image by the duality are equal. See Proposition (1.17) in Jara and Jódar (2000). Later, applying this study to the Bernstein duality, i.e. the duality defined by $\mathrm{Ext}^{\mu}_{R}(-, R)$, being $\mu = \mathrm{idim}(R) = \mathrm{GKdim}(R)$, we show that in many examples R^0 defines the Bernstein duality. See Section 2 in Jara and Jódar (2000).

The aim of this work is to extend the number of examples in which the Bernstein duality may be defined by R^0. In this case we study the algebra $\mathcal{U}_q(sl(2))$, being q a *root of unity* (in the case in which q is not a root of unity the algebra $\mathcal{U}_q(sl(2))$ does not satisfy the strong second layer condition). Therefore first we classify the finite dimensional irreducible representations M of $\mathcal{U}_q(sl(2))$, after that we study what homological properties satisfies $\mathcal{U}_q(sl(2))$ to finally study the dimension of $\mathrm{Ext}^{\mu}_{\mathcal{U}_q(sl(2))}(M, \mathcal{U}_q(sl(2)))$.

The paper is divided into three sections. The first two sections include the necessary background. In Section 1 we show some properties of $\mathcal{U}_q(sl(2))$ and deal with the finite dimensional irreducible representations of $\mathcal{U}_q(sl(2))$; our main references are Kassel's and Jantzen's books, where we address the readers to get complete proofs of the results

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in this section. In section two we study homological properties of $\mathcal{U}_q(sl(2))$; it satisfies the Auslander–Gorenstein condition, it is Cohen–Macaulay and injective and Gelfand–Kirillov dimensions are well known. In addition $\mathcal{U}_q(sl(2))$ satisfies the strong second layer condition.

The most relevant contribution of the paper appears in Section 3. In this section we show an example in which noncommutative Gröbner bases techniques are applied to solve some problems in noncommutative algebras.

Our goal in this section is to compute the vector space dimension of $\mathrm{Ext}^{\mu}_R(R/P,R)$, for any cofinite prime ideal P of $R=\mathcal{U}_q(sl(2))$, as we did in Jara and Jódar (2000) with $R=\mathbb{C}_q[X_1,\dots,X_n]$ the quantum polynomial ring. Otherwise, in the case of $R=\mathcal{U}_q(sl(2))$ we follow a different approach to the problem because a description of finite dimensional irreducible representations is available. Therefore a simple way to get the dimension of $\mathrm{Ext}^{\mu}_R(R/P,R)$ is to compute the dimension of $\mathrm{Ext}^{\mu}_R(V,R)$, for any irreducible finite dimensional representation V of R. We devote this section to perform this computation. A final word with respect to the used techniques: it is well known that $\mathcal{U}_q(sl(2))$ is the quotient, by a regular element, of a noncommutative polynomial algebra S. Hence we may do the computation with noncommutative Gröbner bases in the noncommutative polynomial algebra and after that we translate the results on $\mathrm{Ext}_S(-,S)$ to $\mathcal{U}_q(sl(2))$ via Rees' theorem.

Finally, we would like to thank one of the referees for pointing out an alternative approach by showing the equality $\mathrm{Ext}^{\mu}_R(K,R)\simeq K$ using some Hopf algebra results.

1. Irreducible Finite Dimensional Representations of $\mathcal{U}_q(sl(2))$

Definitions and Notation

Let $q\in\mathbb{C}$ a dth primitive root of unity nonzero element such that $q^2\neq 1$. We define

$$e=\begin{cases} d & \text{if } d \text{ is odd and}\\ d/2 & \text{if } d \text{ is even.}\end{cases}$$

For any integer number n we define

$$[n]=\frac{q^n-q^{-n}}{q-q^{-1}}=q^{n-1}+q^{n-3}+\cdots+q^{-n+3}+q^{-n+1}.$$

Therefore we have

$$[n]=0 \qquad \text{if and only if} \qquad n\equiv 0 \mod e.$$

Also we may define the factorial and binomial coefficients. Let $0\le k\le n$ be an integer number, we define

$$[0]!=1,$$
$$[k]!=[1][2]\cdots[k]$$

and

$$\begin{bmatrix} n\\ k\end{bmatrix}=\frac{[n]!}{[k]![n-k]!}.$$

We define the $\mathbb{C}$-algebra $\mathcal{U}_q(sl(2))$ as the $\mathbb{C}$-algebra generated by variables E, F, K,

K^{-1} satisfying the relationships

$$\begin{array}{ll} KK^{-1} = K^{-1}K & = 1, \\ KEK^{-1} & = q^2 E, \\ KFK^{-1} & = q^{-2} F, \\ [E,F] & = \frac{K-K^{-1}}{q-q^{-1}}. \end{array}$$

Before computing the irreducible finite dimensional representations of $\mathcal{U}_q(sl(2))$ we point out some general results on the commutativity of elements in $\mathcal{U}_q(sl(2))$.

Lemma 1.1. (Kassel, 1995) *Let $r \geq 0$ and $s \in \mathbb{Z}$ integer numbers, then the following statements hold.*

(1) $E^r K^s = q^{-2rs} K^s E^r, \qquad F^r K^s = q^{2rs} K^s F^r$
(2) $[E, F^r] = [r] F^{r-1} \frac{q^{-(r-1)}K - q^{r-1}K^{-1}}{q-q^{-1}} = [r] \frac{q^{r-1}K - q^{-(r-1)}K^{-1}}{q-q^{-1}} F^{r-1}$
(3) $[E^r, F] = [r] \frac{q^{-(r-1)}K - q^{r-1}K^{-1}}{q-q^{-1}} E^{r-1} = [r] E^{r-1} \frac{q^{r-1}K - q^{-(r-1)}K^{-1}}{q-q^{-1}}.$

Proposition 1.1. (Kassel, 1995) *(1) $\{E^i F^j K^l\}_{i,j \in \mathbb{N}; l \in \mathbb{Z}}$ is a basis of $\mathcal{U}_q(sl(2))$ as vector space over $\mathbb{C}$.*
(2) The graded algebra $Gr(\mathcal{U}_q(sl(2))$ is associative with generators $\overline{E}$, $\overline{F}$, $\overline{K}$, $\overline{K^{-1}}$ satisfying the conditions

$$\begin{array}{ll} \overline{K}\,\overline{K^{-1}} & = \overline{K^{-1}}\,\overline{K} = 1, \\ \overline{K}\,\overline{E}\,\overline{K^{-1}} & = q^2\,\overline{E}, \\ \overline{K}\,\overline{F}\,\overline{K^{-1}} & = q^{-2}\,\overline{F}, \\ \overline{E}\,\overline{F} & = \overline{F}\,\overline{E} \end{array}$$

(3) $\mathcal{U}_q(sl(2))$ is a noetherian $\mathbb{C}$-algebra without non-zero zero divisors.

In this section we classify finite dimensional irreducible representations of $\mathcal{U}_q(sl(2))$ showing that there is a upper bound on its dimension and that there are two different classes of irreducible representations: the first one consists of irreducible representations of dimension $<e$ (they are similar to irreducible representations of $\mathcal{U}_q(sl(2))$ if q is not a root of unity), and the second one that consists of irreducible representations of dimension equal to e, which are new and completely different of representations of $\mathcal{U}_q(sl(2))$ if q is not a root of unity.

We give now the theorems about the classification of finite dimensional irreducible representations of $\mathcal{U}_q(sl(2))$ and the necessary notions to apply this study in the third section. The reader can consult proofs and related topics either in Jantzen (1996) or Kassel (1995) books.

irreducible representations of dimension $<e$

Let V be a $\mathcal{U}_q(sl(2))$-module. For any complex number λ we define $V^\lambda = \{x \in V \colon Kx = \lambda x\}$. If $V^\lambda \neq 0$ we say V^λ is a *weight space* and λ is a *weight* of V; elements in V^λ are called *weight vectors*.

Let V be a $\mathcal{U}_q(sl(2))$-module and λ be a complex number. An element $x \in V$ such that $Ex = 0$ and $Kx = \lambda x$ is called a *highest weight vector* and λ is called a *highest weight* of V.

A $\mathcal{U}_q(sl(2))$-module is a *weight module* if it is generated by the weight subspaces of V and it is called a *highest weight module of weight* λ if it is generated by a highest weight vector of weight λ.

The first result we need is the following one

Proposition 1.2. *Let V be a nonzero $\mathcal{U}_q(sl(2))$-module of dimension $<e$, then V contains a highest weight vector.*

The second result is technical and we need it to apply in third section.

Proposition 1.3. *Let v be a highest weight vector of weight λ and let $v_0 = v$ and $v_p = F^p v$ for any $p > 0$, then we have the following relationships:*

$$\begin{aligned} Kv_p &= \lambda q^{-2p} v_p; \\ Ev_p &= [p]\frac{\lambda q^{-(p-1)} - \lambda^{-1} q^{p-1}}{q - q^{-1}}; \\ Fv_p &= v_{p+1}. \end{aligned}$$

Now we may state the main theorem

Theorem 1.1. *(1) Let V be a $\mathcal{U}_q(sl(2))$-module of dimension $<e$ generated by a highest weight vector v of weight λ, then the following assertions hold:*

- *(a) $\lambda = \varepsilon q^n$, for $\varepsilon = \pm 1$ and n satisfying $\dim(V) = n + 1$;*
- *(b) If we define $v_p = F^p v$, then $v_p = 0$ if $p > n$ and the set $\{v = v_0, v_1, \ldots, v_n\}$ is a $\mathbb{C}$-basis of V;*
- *(c) The operator K acting on V is diagonal and it has $n+1$ different eigenvalues $\{\varepsilon q^n, \varepsilon q^{n-2}, \ldots, \varepsilon q^{-n+2}, \varepsilon q^{-n}\}$;*
- *(d) Any other highest weight vector in V is an scalar multiple of v and it has weight λ;*
- *(e) V is a simple $\mathcal{U}_q(sl(2))$-module.*

(2) Any simple $\mathcal{U}_q(sl(2))$-module of dimension $<e$ is generated by a highest weight vector and two simple $\mathcal{U}_q(sl(2))$-modules generated by highest weight vector with the same weight are isomorphic.

As a consequence, for any integral number n ($0 \leq n \leq e-2$) there is, up to isomorphism, only one simple $\mathcal{U}_q(sl(2))$-module of dimension $n+1$; it is generated by a highest weight vector of weight $\lambda = \varepsilon q^n$. If we write $V_{\varepsilon,n}$ for this simple module, using relationships given in Proposition 1.3, a $\mathbb{C}$-basis of V is $\{v_0, \ldots, v_n\}$ and the action of $\mathcal{U}_q(sl(2))$ is given by the formulae

$$\begin{array}{ll} Kv_p = \lambda q^{-2p} v_p & \text{if } 0 \leq p \leq n; \\ Ev_{p+1} = [p+1]\frac{\lambda q^{-p} - \lambda^{-1} q^p}{q - q^{-1}} v_p & \text{if } 0 \leq p \leq n-1; \\ Ev_0 = 0; & \\ Fv_p = v_{p+1} & \text{if } 0 \leq p \leq n-1; \\ Fv_n = 0, & \end{array}$$

where $\lambda = \varepsilon q^n$ and $\varepsilon = \pm 1$.

IRREDUCIBLE REPRESENTATIONS OF DIMENSION $>e$

In this subsection we show that e is a upper bound for the dimension of irreducible finite dimensional representations of $\mathcal{U}_q(sl(2))$. This result is a direct consequence of two facts: (1) the center of $\mathcal{U}_q(sl(2))$ is too big in $\mathcal{U}_q(sl(2))$, in fact, it is generated by E^e, F^e, K^e, K^{-e} and the quantum Casimir element C_q, see next section, and (2) any central element acts as the multiplication by an scalar on any finite dimensional irreducible representation. Combining these two facts one has.

Theorem 1.2. *There are not simple $\mathcal{U}_q(sl(2))$-modules of dimension $>e$.*

IRREDUCIBLE REPRESENTATIONS OF DIMENSION e

To classify all irreducible representations of dimension e first we show a good description of any simple $\mathcal{U}_q(sl(2))$-module of dimension e.

Proposition 1.4. *Let V be a simple $\mathcal{U}_q(sl(2))$-module of dimension e, then there is a basis $\{v_0, \ldots, v_{e-1}\}$ and elements a, $b \in \mathbb{C}$ and $\lambda \in \mathbb{C} \setminus \{0\}$ satisfying*

$$\begin{array}{ll} Kv_p = \lambda q^{-2p} v_p & \text{if } p = 0, \ldots, e-1; \\ Fv_p = v_{p+1} & \text{if } p = 0, \ldots, e-2; \\ Fv_{e-1} = bv_0; & \\ Ev_0 = av_{e-1}; & \\ Ev_{p+1} = \big(\frac{\lambda q^{-p} - \lambda^{-1} q^p}{q - q^{-1}} + ab\big) v_p & \text{if } p = 0, \ldots, e-2. \end{array}$$

In order to get a complete catalog of simple modules we only need to find conditions on a, b and λ to get simple $\mathcal{U}_q(sl(2))$-modules. To simplify we will denote by $V(\lambda, a, b)$ the module described in the above Proposition, then we have the theorem

Theorem 1.3. *Any simple $\mathcal{U}_q(sl(2))$-module of dimension e is of one of the following types:*

(1) $V(\lambda, a, b)$ where $a \in \mathbb{C}$ and λ, $b \in \mathbb{C} \setminus \{0\}$;
(2) $V(\lambda, a, 0)$ where $a \in \mathbb{C}$ and $\lambda \in \mathbb{C} \setminus \{0\}$, $\lambda \neq \pm q^n$ for any $n \in \{0, \ldots, e-2\}$.

2. Homological Properties of $\mathcal{U}_q(sl(2))$

We will relate in this section homological properties of $\mathcal{U}_q(sl(2))$. Some of them may be known, but, by completeness, we include the proof of all of them.

STRONG SECOND LAYER CONDITION

In order to introduce the strong second layer condition let us remember Jategaonkar's Main Lemma. First we start with a noetherian ring and with the definition of an *affiliate series* of a right R-module M, it is a sequence $0 = M_0 \subseteq M_1 \subsetneqq \cdots \subsetneqq M_n = M$ of submodules together with an ordered set of prime ideals $\{P_1, \ldots, P_n\}$ such that each P_i is a maximal annihilator prime of M_i/M_{i-1} and $M_i/M_{i-1} = \mathrm{Ann}_{M_i/M_{i-1}}(P_i)$. The Jategaonkar's Main Lemma studies the behaviour of affiliated series of length two.

Theorem 2.1. (Goodearl and Warfield, 1989) *Let R be a noetherian ring and let M be a right R-module with an affiliated series $0 \subsetneqq U \subsetneqq M$ and affiliated prime ideals $\{P, Q\}$, such that U is essential in M. Let M' be a submodule of M, properly containing U, such that the ideal $I = \mathrm{Ann}_R(M')$ is maximal among annihilators of submodules of M properly containing U. Then exactly one of the following two alternatives occurs:*

(1) $Q \subsetneqq P$ and $M'Q = 0$. In this case, M' and M'/U are faithful torsion R/Q-modules;
(2) $Q \rightsquigarrow P$ and $J = Q \cap P/I$ is a linking bimodule between Q and P. In this case, if U is torsionfree as right R/P-module, then M'/U is torsion free as a right R/Q-module.

A noetherian ring R satisfies the *(right) strong second layer condition* if, given the hypothesis in the above theorem, conclusion (1) never occurs. In a similar way we define *(left) strong second layer condition* and *strong second layer condition* which is left and right strong second layer conditions. We refer to Goodearl and Warfield (1989) or Bell (1988) for non-defined terms in this subsection.

With this background we may now establish the result:

Proposition 2.1. *$\mathcal{U}_q(sl(2))$ satisfies the strong second layer condition.*

Proof. First of all we need to compute the center of $\mathcal{U}_q(sl(2))$. It is generated by E^e, F^e, K^e, K^{-e} and the quantum Casimir element $C_q = EF + \frac{Kq^{-1}+K^{-1}q}{(q-q^{-1})^2}$. The proof of this result appears in De Concini–Kac's paper (de Concini and Kac, 1990). After that $\mathcal{U}_q(sl(2))$ is finitely generated as left and right module over its center, and Letzter's theorem (see, Letzter, 1990), gives us that $\mathcal{U}_q(sl(2))$ satisfies the strong second layer condition. □

Auslander–regular condition

Let R be a noetherian ring. An R-module M satisfies the *Auslander condition* if for any $n \geq 0$ and any submodule $N \subseteq \mathrm{Ext}^n_R(M, R)$ we have $j_R(N) \geq n$, being $j_R(N)$ the *grade* of N, which is defined

$$j_R(N) = \inf\{i\colon \mathrm{Ext}^i_R(N, R) \neq 0\} \in \mathbb{N} \cup \{\infty\}.$$

The ring R satisfies the *Auslander–Gorenstein condition* (resp. *Auslander–regular condition*) if any finitely generated R-module satisfies the Auslander condition and R has finite left and right injective dimension (resp. finite global dimension).

Our main goal in this subsection is to prove the following result.

Lemma 2.1. *$\mathcal{U}_q(sl(2))$ satisfies the Auslander–regular condition.*

Proof. We will use that $\mathcal{U}_q(sl(2))$ is an iterated Ore extension, indeed

$$\mathcal{U}_q(sl(2)) = \mathbb{C}[K, K^{-1}][F; \sigma_1][E; \sigma_2, \delta]$$

where

$$\sigma_1\colon \mathbb{C}[K, K^{-1}] \longrightarrow \mathbb{C}[K, K^{-1}]$$

is an automorphism defined by $\sigma_1(K) = q^2K$,

$$\sigma_2\colon \mathbb{C}[K, K^{-1}][F; \sigma_1] \longrightarrow \mathbb{C}[K, K^{-1}][F; \sigma_1]$$

is an automorphism defined by $\sigma_2(F^iK^j) = q^{-2j}F^iK^j$ for any $i \in \mathbb{N}$ and any $j \in \mathbb{Z}$, and finally

$$\delta\colon \mathbb{C}[K, K^{-1}][F; \sigma_1] \longrightarrow \mathbb{C}[K, K^{-1}][F; \sigma_1]$$

is an σ_2-derivation is defined by $\delta(F) = \frac{K-K^{-1}}{q-q^{-1}}$ and $\delta(K) = 0$. Now we only need to apply (Li and van Oystaeyen, 1996, Theorem III.3.4.6) which asserts that if R is an Auslander-regular ring, σ an automorphism and δ an σ-derivation, then $R[X; \sigma, \delta]$ and $R[X, X^{-1}; \sigma]$ are Auslander regular. □

GLOBAL DIMENSION

In order to compute the global dimension of $\mathcal{U}_q(sl(2))$, as it is an iterated Ore extension, we may use those results which relate global dimension in Ore extensions. Thus it is well known that the global dimension of $\mathbb{C}[K, K^{-1}][F, \sigma_1]$ is 2 (see, McConnell and Robson, 1987, Theorem 7.5.3). And the same theorem gives us that $\mathrm{gldim}(\mathcal{U}_q(sl(2)))$ is either 2 or 3. Thus we have a simple method to discriminate between these two values: it consists in looking for an R-module M such that $\mathrm{Ext}^3_R(M, N) \neq 0$; we will find many of them in the last section of this paper. Thus we have

Lemma 2.2. $\mathrm{gldim}(\mathcal{U}_q(sl(2))) = 3$.

Gelfand–Kirillov dimension

In the particular case we are studying $\mathcal{U}_q(sl(2))$ is an algebra extension of its center $C := \mathbb{C}[E^e, F^e, K^e, K^{-e}, C_q]$ finitely generated as C-module, hence $\mathrm{GKdim}(\mathcal{U}_q(sl(2))) = \mathrm{GKdim}(C)$. Therefore the problem of computing the Gelfand–Kirillov dimension is reduced to the commutative case and the solution is easy: $\mathrm{GKdim}(\mathcal{U}_q(sl(2))) = 3$.

Let us now consider the general case, i.e. q is not necessarily a root of unity. Then we consider the filtration $\{R_n\}_n$ whose generating subspace has $\{E, F, K, K^{-1}\}$ as basis. We know by PBW theorem that the set of monomials $\{F^\alpha E^\beta K^\gamma\colon \alpha, \beta \in \mathbb{N}, \gamma \in \mathbb{Z}\}$ is a $\mathbb{C}$-basis of $\mathcal{U}_q(sl(2))$. The number of such monomials in R_n can be counted as α, $\beta \leq n$ and $|\gamma| \leq n$. Then we have $\mathrm{GKdim}(\mathcal{U}_q(sl(2))) = 3$.

Lemma 2.3. $\mathrm{GKdim}(\mathcal{U}_q(sl(2))) = 3$.

Cohen–Macaulay condition

The algebra $\mathcal{U}_q(sl(2))$ can be described as a quotient of the algebra S generated by elements E, F, K and H satisfying the relationships

$$\begin{array}{llll} HK & = KH & & \\ KE & = q^2EK & \quad & HE = q^{-2}EH \\ KF & = q^{-2}FK & \quad & HF = q^2FH \\ [E, F] & = \frac{K-H}{q-q^{-1}} & & \end{array}$$

by the ideal $I = (HK - 1)$. Recall that $HK - 1$ is a regular central element of S. Indeed it is central as commutes with E and F. On the other hand it is easy to prove that the set $\{F^{\alpha_1}E^{\alpha_2}K^{\alpha_3}H^{\alpha_4}\colon \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{N}\}$ is a $\mathbb{C}$-basis of S by a simple PBW argument, hence given an element $s = \sum_\alpha \lambda_\alpha F^{\alpha_1} E^{\alpha_2} K^{\alpha_3} H^{\alpha_4} \in S$ if $(HK-1)s = 0$ then we have the identity

$$\sum_\alpha \lambda_\alpha F^{\alpha_1} E^{\alpha_2} K^{\alpha_3+1} H^{\alpha_4+1} = \sum_\alpha \lambda_\alpha F^{\alpha_1} E^{\alpha_2} K^{\alpha_3} H^{\alpha_4}$$

and as a consequence $\lambda_{\alpha_1\alpha_2\alpha_3+1\alpha_4+1} = \lambda_{\alpha_1\alpha_2\alpha_3\alpha_4}$, for any α, and we have all coefficients are equal zero, hence $s = 0$ and $(HK-1)$ is regular.

It is also easy, from the above PBW arguments, to prove that $\mathrm{GKdim}(S) = 4$. On the other hand, as $\mathrm{gldim}(\mathcal{U}_q(sl(2))) = 3$ and $HK - 1$ is a regular central element of S such that $\mathcal{U}_q(sl(2)) = S/(HK-1)$, then $\mathrm{gldim}(S) \geq 4$. Now as S is an iterated Ore extension as $\mathcal{U}_q(sl(2))$ was, the same arguments show $\mathrm{gldim}(S)$ is either 3 or 4. Therefore we obtain $\mathrm{gldim}(S) = 4$. It is also easy to prove that S is Auslander-regular using iterated Ore extension arguments.

Now we are interesting in showing that S satisfies the *Cohen–Macaulay condition*, i.e. $\mathrm{GKdim}(S) \in \mathbb{N}$ and for every finitely generated S-module M we have $j_S(M) + \mathrm{GKdim}(M) = \mathrm{GKdim}(S)$.

To do that we consider Levasseur and Stafford (1993, Lemma) and apply it twice to the chain of Ore extensions with graduation of the base ring as it is indicated

$$\begin{aligned}
&A := \mathbb{C}[K, H];\\
&\{A_n\}_n;\\
&A_n = \mathbb{C}(K^{\alpha_3} H^{\alpha_4}\colon \alpha_3 + \alpha_4 = n);\\
\\
&B := \mathbb{C}[K, H][E; \sigma_1];\\
&\sigma_1(K^{\alpha_3} H^{\alpha_4}) = q^{-2\alpha_3+2\alpha_4} K^{\alpha_3} H^{\alpha_4};\\
&\{B_n\}_n;\\
&B_n = \mathbb{C}(E^{\alpha_2} K^{\alpha_3} H^{\alpha_4}\colon \alpha_2 + \alpha_3 + \alpha_4 = n);\\
\\
&S = \mathbb{C}[K, H][E; \sigma_1][F; \sigma_2, \delta];\\
&\sigma_2(E^{\alpha_2} K^{\alpha_3} H^{\alpha_4}) = q^{2\alpha_3-2\alpha_4} E^{\alpha_2} K^{\alpha_3} H^{\alpha_4};\\
&\delta(E) = \tfrac{H-K}{q-q^{-1}}, \qquad \delta(K) = \delta(H) = 0
\end{aligned}$$

Thus S is Cohen–Macaulay.

Lemma 2.4. *$\mathcal{U}_q(sl(2))$ is Cohen–Macaulay.*

Proof. We may also apply the third part in Levasseur and Stafford (1993, Lemma) to obtain that the quotient $S/(HK-1) = \mathcal{U}_q(sl(2))$ is Cohen–Macaulay. □

3. Bernstein Duality in $\mathcal{U}_q(sl(2))$

Our main aim is to compare $\mathbb{C}$-dimensions of $\mathrm{Ext}^3_R(R/P, R)$ and R/P for any cofinite prime ideal P of $R = \mathcal{U}_q(sl(2))$ and see that they are equal, being $\mathrm{GKdim}(\mathcal{U}_q(sl(2)))$ $= \mathrm{gldim}(\mathcal{U}_q(sl(2))) = 3$. To do that it is enough study the same property for any finite dimensional simple $\mathcal{U}_q(sl(2))$-module M and prove that $\dim_{\mathbb{C}}(M) = \dim_{\mathbb{C}}(\mathrm{Ext}^3_{\mathcal{U}_q(sl(2))}$ $(M, \mathcal{U}_q(sl(2))))$ as there is a bijection between finite dimensional simple $\mathcal{U}_q(sl(2))$-modules and cofinite prime ideals of $\mathcal{U}_q(sl(2))$ by setting the image of M equals to $\mathrm{Ann}(M)$.

To perform this study we only need to consider any finite dimensional simple $\mathcal{U}_q(sl(2))$-module M and compute $\mathrm{Ext}^3_{\mathcal{U}_q(sl(2))}(M, \mathcal{U}_q(sl(2)))$. We do that using the description of these modules given in Section 1 and some computational techniques of non-commutative Gröbner bases developed in Jara (1999) (see also, Bueso *et al.*, 1998).

Let us consider the algebra S as defined before Lemma 2.4. We rewrite the generators by setting

$$X_1 := E; \qquad X_2 := F; \qquad X_3 := K; \qquad X_4 := H$$

and consider the lexicographical order with $X_1 < X_2 < X_3 < X_4$. Then the relations of S appear as

$$\begin{aligned} X_2X_1 &= X_1X_2 - \frac{1}{q-q^{-1}}X_3 + \frac{1}{q-q^{-1}}X_4 \\ X_3X_1 &= q^2X_1X_3 \\ X_4X_1 &= q^{-2}X_1X_4 \\ X_3X_2 &= q^{-2}X_2X_3 \\ X_4X_2 &= q^2X_2X_4 \\ X_4X_3 &= X_3X_4 \end{aligned}$$

and $\mathcal{U}_q(sl(2)) \cong S/(X_3X_4 - 1)$. Let us call $f := X_3X_4 - 1$.

Let M be a finite dimensional simple $\mathcal{U}_q(sl(2))$-module, we may consider it as S-module by the obvious change ring. As $fM = 0$ and f is a central regular element of S, by Rees' theorem we have

$$\operatorname{Ext}^n_S(M,S) \cong \operatorname{Ext}^{n-1}_{S/Sf}(M,S/Sf) = \operatorname{Ext}^{n-1}_{\mathcal{U}_q(sl(2))}(M,\mathcal{U}_q(sl(2))) \qquad n \geq 2.$$

Thus in our case we obtain

$$Ext^3_{\mathcal{U}_q(sl(2))}(M,\mathcal{U}_q(sl(2))) \cong \operatorname{Ext}^4_S(M,S).$$

Then the problem is compute the $\mathbb{C}$-dimension of $\operatorname{Ext}^4_S(M,S)$.

SIMPLE MODULES OF DIMENSION $<e$

Let us consider $M = V_{\varepsilon,n}$ a simple $\mathcal{U}_q(sl(2))$-module of dimension $n+1 < e$. In this case M is isomorphic to a quotient of $\mathcal{U}_q(sl(2))$, in fact M is isomorphic to $\mathcal{U}_q(sl(2))/\mathcal{U}_q(sl(2))$ $(E, F^{n+1}, K-\lambda)$. Hence M is isomorphic to

$$S/S(X_1, X_2^{n+1}, X_3 - \lambda, X_3X_4 - 1)$$

as S-module. We have $S(X_3 - \lambda, X_3X_4 - 1) = S(X_3 - \lambda, X_4 - \lambda^{-1})$ as the relationships hold

$$\begin{aligned} X_4 - \lambda^{-1} &= -\lambda^{-1}X_4(X_3 - \lambda) + \lambda^{-1}(X_3X_4 - 1); \qquad \text{and} \\ X_3X_4 - 1 &= \lambda^{-1}(X_3 - \lambda) + \lambda(X_4 - \lambda^{-1}) + (X_3 - \lambda)(X_4 - \lambda^{-1}). \end{aligned}$$

Then M is a S-module isomorphic to $S/S(X_1, X_2^{n+1}, X_3 - \lambda, X_4 - \lambda^{-1})$.

To compute $\operatorname{Ext}^4_S(M,S)$ we consider the Gröbner bases theory on the ring S as it appears in Jara (1999). A Gröbner basis if $I = S(X_1, X_2^{n+1}, X_3 - \lambda, X_4 - \lambda^{-1})$ is

$$\mathbb{G} = \{G_1 := X_1, G_2 := X_2^{n+1}, G_3 := X_3 - \lambda, G_4 := X_4 - \lambda^{-1}\}.$$

Let us define a free presentation φ_1: $S^4 \to I$ of I by setting $\varphi_1(e_i) = G_i$, $i = 1, \ldots, 4$. In order to compute the kernel of φ we need to compute the semisyzygies $S(G_i, G_j)$, $i < j$, and divide them with respect to the Gröbner basis $\mathbb{G}$.

$$S(G_1, G_2) = X_2^{n+1}G_1 - X_1G_2 = -[n+1]\frac{q^{-n}}{q-q^{-1}}X_2^nG_3 + [n+1]\frac{q^n}{q-q^{-1}}X_2^nG_4;$$
$$S(G_1, G_3) = q^{-2}X_3G_1 - X_1G_3 = \lambda G_1;$$
$$S(G_1, G_4) = q^2X_4G_1 - X_1G_4 = \lambda^{-1}G_1;$$
$$S(G_2, G_3) = q^{2(n+1)}X_3G_2 - X_2^{n+1}G_3 = \lambda G_2;$$
$$S(G_2, G_4) = q^{-2(n+1)}X_4G_2 - X_2^{n+1}G_4 = \lambda^{-1}G_2;$$
$$S(G_3, G_4) = X_4G_3 - X_3G_4 = \lambda^{-1}G_3 - \lambda G_4.$$

If $S(G_i, G_j) = C_{ij}G_i - C_{ji}G_j = \sum_h Q_{ijh}G_h$, where C_{ij}, C_{ji} are coefficients in S and define $s_{ij} = C_{ij}e_i - C_{ji}e_j - \sum_h Q_{ijh}e_h$, then it is well known that the set $\{s_{ij} \colon 1 \leq i < j \leq 4\}$ is a system of generators of $\mathrm{Ker}(\varphi_1)$; in fact it is a Gröbner basis with respect to a particular monomial order. See Jara (1999) for details.

Let us define

$$H_1 := s_{12} = \left(X_2^{n+1}, -X_1, [n+1]\frac{q^{-n}}{q-q^{-1}}X_2^n, -[n+1]\frac{q^n}{q-q^{-1}}X_2^n\right);$$
$$H_2 := s_{13} = (q^{-2}X_3 - \lambda, 0, -X_1, 0);$$
$$H_3 := s_{14} = (q^2X_4 - \lambda^{-1}, 0, 0, -X_1);$$
$$H_4 := s_{23} = (0, q^{2(n+1)}X_3 - \lambda, -X_2^{n+1}, 0);$$
$$H_5 := s_{24} = (0, q^{-2(n+1)}X_4 - \lambda^{-1}, 0, -X_2^{n+1});$$
$$H_6 := s_{34} = (0, 0, X_4 - \lambda^{-1}, -X_3 + \lambda).$$

With this notation $\mathbb{H} = \{H_1, H_2, H_3, H_4, H_5, H_6\}$ is a Gröbner basis of $\mathrm{Ker}(\varphi_1)$ and we may build a free presentation $\varphi_2 \colon S^6 \to \mathrm{Ker}(\varphi_1)$ of $\mathrm{Ker}(\varphi_1)$ by setting $\varphi_2(e_i) = H_i$, $i = 1, \ldots, 6$. To compute the kernel of φ_2 first we need to compute the minimum common multiple X_{ij} of H_i and H_j, $i < j$. Second we consider those which are non-zero. In that case

$$X_{12} = X_2^{n+1}X_3; \qquad X_{23} = X_3X_4$$
$$X_{13} = X_2^{n+1}X_4; \qquad X_{45} = X_3X_4.$$

As a consequence $\mathrm{Ker}(\varphi_2)$ has four generators. They can be described as

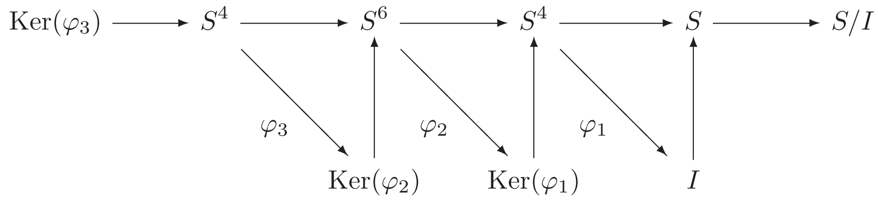
$$I_1 := s_{12} = \left(q^{2(n+1)}X_3 - \lambda q^2, -q^2X_2^{n+1}, 0, q^2X_1, 0, -q^2[n+1]\frac{q^n}{q-q^{-1}}X_2^n\right);$$
$$I_2 := s_{13} = \left(q^{-2(n+1)}X_4 - \lambda^{-1}q^{-2}, 0, -q^{-2}X_2^{n+1}, 0, q^{-2}X_1, \right.$$
$$\left. -q^{-2}[n+1]\frac{q^{-n}}{q-q^{-1}}X_2^n\right);$$
$$I_3 := s_{23} = (0, q^2X_4 - \lambda^{-1}, -q^{-2}X_3 + \lambda, 0, 0, X_1);$$
$$I_4 := s_{45} = (0, 0, 0, q^{-2(n+1)}X_4 - \lambda^{-1}, -q^{2(n+1)}X_3 + \lambda, X_2^{n+1}).$$

Again we obtain that $\mathbb{I} = \{I_1, I_2, I_3, I_4\}$ is a Gröbner basis of $\mathrm{Ker}(\varphi_2)$ with respect to a particular monomial order. Hence we may continue the process and define a free presentation $\varphi_3: S^4 \to \mathrm{Ker}(\varphi_2)$ of $\mathrm{Ker}(\varphi_2)$ by setting $\varphi_3(e_i) = I_i$, $i = 1, \ldots, 4$. The kernel of φ_3 can be computed with the same technique. First we compute the non-zero minimum common multiples X_{ij} of I_i and I_j, $i < j$. The only one is $X_{12} = X_3X_4$. Hence the only generator of $\mathrm{Ker}(\varphi_3)$ is

$$J := s_{12} = (q^{-2(n+1)}X_4 - \lambda^{-1}q^{-2}, -q^{2(n+1)}X_3 + \lambda q^2, X_2^{n+1}, -X_1)$$

and $\mathbb{J} = \{J\}$ is a Gröbner basis of $\mathrm{Ker}(\varphi_3)$.

Putting together all the information, we obtain a free resolution of S/I building as follows:



To compute $\mathrm{Ext}^4_S(S/I, S)$ we use the isomorphism

$$\mathrm{Ext}^4_S(S/I, S) \cong \mathrm{Ext}^1_S(\mathrm{Ker}(\varphi_2), S)$$

and the free presentation

$$0 \longrightarrow \mathrm{Ker}(\varphi_3) \xrightarrow{\nu} S^4 \xrightarrow{\varphi_3} \mathrm{Ker}(\varphi_2) \longrightarrow 0.$$

Then we have a long exact sequence

$$\cdots \longrightarrow \mathrm{Hom}_S(S^4, S) \xrightarrow{\nu^*} \mathrm{Hom}_S(\mathrm{Ker}(\varphi_3), S) \longrightarrow \mathrm{Ext}^1_S(\mathrm{Ker}(\varphi_2), S) \longrightarrow 0 \cdots$$

and we obtain

$$\begin{aligned} \mathrm{Ext}^1_S(\mathrm{Ker}(\varphi_2), S) &\cong \frac{\mathrm{Hom}_S(\mathrm{Ker}(\varphi_3), S)}{\mathrm{Im}(\nu^*)} \\ &\cong \frac{S}{(q^{-2(n+1)}X_4 - \lambda^{-1}q^{-2}, -q^{2(n+1)}X_3 + \lambda q^2, X_2^{n+1}, -X_1)S}. \end{aligned}$$

As a consequence the $\mathbb{C}$-dimension is $n + 1$.

SIMPLE MODULES OF DIMENSION e

If M is a finite dimensional simple $\mathcal{U}_q(sl(2))$-module of dimension e, then either $M = V(\lambda, a, b)$ with $a \in \mathbb{C}$ and b, $\lambda \in \mathbb{C} \setminus \{0\}$ or $M = V(\lambda, a, 0)$ with $a \in \mathbb{C}$ and $\lambda \in \mathbb{C} \setminus \{0\}$, $\lambda \neq \pm q^n$ for any $n \in \{0, 1, \ldots, e-2\}$.

In the first case M is isomorphic to $\mathcal{U}_q(sl(2))/\mathcal{U}_q(sl(2))(E - aF^{e-1}, F^e - b, K - \lambda)$. Hence M is isomorphic to the quotient $S/S(X_1 - aX_2^{e-1}, X_2^e - b, X_3 - \lambda, X_4 - \lambda^{-1})$. Let us call $I = S(X_1 - aX_2^{e-1}, X_2^e - b, X_3 - \lambda, X_4 - \lambda^{-1})$. After developing similar calculations that in the previous example we obtain an isomorphism

$$\mathrm{Ext}^4_S(S/I, S) \cong \frac{S}{(X_4 - \lambda^{-1}q^{-2}, -X_3 + \lambda q^2, X_2^e - b, -X_1 + aX_2^{e-1})S}.$$

Hence its $\mathbb{C}$-dimension is e.

In the second case $M \cong \mathcal{U}_q(sl(2))/\mathcal{U}_q(sl(2))(E - aF^{e-1}, F^e, K - \lambda)$ and M is isomorphic to $S/S(X_1 - aX_2^{e-1}, X_2^e, X_3 - \lambda, X_4 - \lambda^{-1})$ as S-module. If we call $I = S(X_1 -$

$aX_2^{e-1}, X_2^e, X_3-\lambda, X_4-\lambda^{-1})$ and developing similar calculations than in the former case we obtain an isomorphism

$$\mathrm{Ext}^4_S(S/I,S) \cong \frac{S}{(X_4-\lambda^{-1}q^{-2}, -X_3+\lambda q^2, X_2^e, -X_1+aX_2^{e-1})S}.$$

Hence its $\mathbb{C}$-dimension is e.

We have proved the following theorem.

THEOREM 3.1. *Let M be a finite dimensional simple $\mathcal{U}_q(sl(2))$-module, then*

$$\dim_{\mathbb{C}}(M) = \dim_{\mathbb{C}}(Ext^3_{\mathcal{U}_q(sl(2))}(M, \mathcal{U}_q(sl(2)))).$$

As a consequence we have

COROLLARY 3.1. *Let P be a cofinite prime ideal of $\mathcal{U}_q(sl(2))$, then*

$$\dim_{\mathbb{C}}(\mathcal{U}_q(sl(2))/P) = \dim_{\mathbb{C}}(Ext^3_{\mathcal{U}_q(sl(2))}(\mathcal{U}_q(sl(2))/P, \mathcal{U}_q(sl(2)))).$$

REMARK. We have in addition that a minimal injective resolution of $\mathcal{U}_q(sl(2))$ has last point $E_3 \cong \mathcal{U}_q(sl(2))^\circ$ and it is the direct sum of injective hulls of $\mathcal{U}_q(sl(2))/P$, being P a cofinite prime ideal of $\mathcal{U}_q(sl(2))$, i.e.

$$\mathcal{U}_q(sl(2))^\circ \cong \oplus\{E(\mathcal{U}_q(sl(2))/P)\colon P \subseteq \mathcal{U}_q(sl(2)) \text{ cofinite and prime}\}.$$

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