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# Involution approach to investigating polynomial systems

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**Abstract**

An involution approach to solving systems of polynomial equations is proposed. A notion of involutive polynomial system is introduced and an algorithm for its construction is presented. The relation between involutive systems and Gröbner bases is clarified, and some advantages of the involution approach are discussed.

*Keywords:* Involutive polynomial systems; Computer algebra; Gröbner bases; Zero-dimensional ideals

## 1. Introduction

In this paper we propose a new algorithmic approach to solving algebraic systems based on the concepts and results of the formal theory of partial differential equations, see [1,2] and references therein. The modern formal theory gives a foundation of the constructive methods of PDE analysis going back to the pioneering works of Riquier [3] and Janet [4]. Using these methods, it is possible in a finite number of steps (differentiations and eliminations of derivatives) either to establish an incompatibility of a given system of PDEs or to transform it into the involutive form that gives complete information on the formal solutions.

Applying the results of [1,2] to the systems of algebraic equations we use a well-known formal analogy between polynomials and linear homogeneous PDEs with constant coefficients and with one unknown function. That is, unknown function is interpreted as unity, differentiation – as multiplication by variable; addition, subtraction and multiplication by coefficients are left unchanged; e.g., instead of $u_{xxx} + 2u_{yy} = 0, u_{xxyy} - 3u = 0$ one writes $x^3 + 2y^2 = 0, x^2y^2 - 3 = 0$. The analogy is also valid for the solution spaces [2]. For example, if the general solution of the PDE-system depends on $N$ arbitrary constants and does not contain arbitrary functions then the corresponding system of polynomials generates a zero-dimensional ideal with $N$ roots.

We give a formulation of Janet's method in the modern way restricting ourselves with algebraic systems. We introduce a notion of Janet normal form and formulate the involution conditions in terms of it. Then we describe an algorithm for transformation of a given polynomial system to be the involutive form and

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clarify the relation between the involutive system and the Gröbner basis [5]. Some possible advantages of the involution approach are also discussed.

## 2. Basic definitions

Throughout this paper we use the notations:

| | |
|---|---|
| $K$ | arbitrary zero characteristic field |
| $a, b$ | elements of $K$ |
| $K[x_1 \dots, x_n]$ | polynomial ring over $K$ |
| $f, g, h, p$ | polynomials from $K[x_1, \dots, x_n]$ |
| $F, G, H, P$ | finite subsets in $K[x_1, \dots, x_n]$ |
| $u, v, w, s$ | terms in polynomials (without coefficients from $K$) |
| $\deg(u)$ | total degree of $u$ |
| $cf(f, u)$ | coefficient of $u$ in $f$ |
| $Ideal(F)$ | ideal generated by $F$ |

Let $<_{\mathrm{T}}$ be the admissible term ordering and let variables be ordered as $x_1 <_{\mathrm{T}} x_2 <_{\mathrm{T}} \cdots <_{\mathrm{T}} x_n$. We denote by $lt(f)$ the leading term in $f$ w.r.t. $<_{\mathrm{T}}$; $lc(f)$ the leading coefficient in $f$, i.e. $cf(f, lt(f))$; and $red(f) = f - lc(f) \cdot lt(f)$.

**Definition 1.** (Pommaret [1]). Variable $x_i$ is *multiplicative* for the term $u$ if its index $i$ is not greater than the index of the lowest variable in $u$. Otherwise, $x_i$ is *non-multiplicative* for $u$ (symbolically, $x_i \in Nonmult(u)$).

**Definition 2.** *Class of a term* is the index of its lowest variable. *Class of a polynomial* is the class of its leading term.

Denote $u \cdot v$ by $u \times v$ if all variables in $v$ are multiplicative for $u$ of if $v = 1$. Write also $g \cdot u = g \times u$ if $lt(g) \cdot u = lt(g) \times u$.

**Definition 3.** Term $u$ is called a *Janet divisor* for the term $w$ if there exists a term $v$ such that $w = u \times v$ (symbolically $u|_{\mathrm{J}} w$).

The following properties of Janet divisors are obvious:
(1) if $u|_{\mathrm{J}} v$ and $v|_{\mathrm{J}} w$ then $u|_{\mathrm{J}} w$ *(transitivity)*;
(2) if $u|_{\mathrm{J}} w$ and $v|_{\mathrm{J}} w$ then $u|_{\mathrm{J}} v$ or $v|_{\mathrm{J}} u$;
(3) if $\neg(u|_{\mathrm{J}} v)$ then $\forall w, s, \neg(u \times w|_{\mathrm{J}} v \times s)$.

**Definition 4.** Polynomial $f$ *is reduced* to $h$ modulo $G$ *in the sense of Janet* if there exists $g \in G$ and $u$ such that $lt(g) \cdot u \equiv lt(g) \times u$, $a \equiv cf(f, lt(g) \times u) \neq 0$ and $h = f - a \cdot g \times u$. Polynomial $f$ is given in the *Janet normal form* modulo $G$ if for each term in $f$ there are no Janet divisors in $\{lt(g)|g \in G\}$. Polynomial $h$ is a *Janet reduced form* of $f$ modulo $G$ (symbolically $h = NF_{\mathrm{J}}(f, G)$) if there exists a chain of Janet reductions from $f$ to $h$ and $h$ is given in the Janet normal form modulo $G$.

In contrast to a Janet normal form, we denote by $NF(f, G)$ usual normal form of $f$ modulo $G$. An algorithm for computing $NF_{\mathrm{J}}$ may be obtained from one for $NF$ [5] replacing usual division of terms by a Janet division.

**Example 1.** *Let* $G = \{xy\}$, $f = x^2y + xy^2$, $x >_T y$. *Then* $NF_J(f, G) = x^2y \neq NF(f, G) = 0$.

**Definition 5.** $G$ is *autoreduced* (in the sense of Janet) if

$$\forall g, g' \in G, g \neq g', \quad \neg(lt(g)|_J lt(g')).$$

**Proposition 1.** *If* $G$ *is autoreduced then for any term* $u$ *there exists no more than one Janet divisor in* $\{lt(g) \mid g \in G\}$.

**Proof.** This is an immediate consequence of Definition 5 and property (2) of Janet divisors. □

Denote by *Autoreduce* $(F)$ a function that for given $F$ computes $G$ which is autoreduced and $Ideal(F) = Ideal(G)$. An algorithm *Autoreduce* may be obtained from the well-known algorithm *Reduce All* [5] replacing usual $NF$ by $NF_J$.

Denote by $M(G)$ a set of finite sums

$$M(G) = \left\{ \sum_i g_i \times \sum_j a_{ij} u_{ij} \;\middle|\; g_i \in G \right\}.$$

It is obvious that $h = NF_J(f, G)$ implies $(f - h) \in M(G)$.

**Theorem 1.** *If* $G$ *is autoreduced and* $f \in M(G)$ *then* $NF_J(f, G) = 0$ *for any sequence of Janet reductions.*

**Proof.** First let us prove that for $lt(f)$ there exists a Janet divisor in $\{lt(g) \mid g \in G\}$ and it is unique. By definition of $M(G)$,

$$f = \sum_i \left[ lt(g_i) \times \sum_j a_{ij} u_{ij} + red(g_i) \cdot \sum_j a_{ij} u_{ij} \right], \tag{1}$$

where the like terms are not assumed to be collected.

Find a maximal term (w.r.t. $<_T$) in the r.h.s. of (1). Evidently, it is the greatest of the maxima over each square bracket in (1). From $\forall j, lt(g_i) \times u_{ij} >_T lt(red(g_i) \cdot u_{ij})$ it follows that the maximum over each bracket is $\max_j\{lt(g_i) \times u_{ij}\} = lt(g_i) \times v_i$ where $v_i = \max_j u_{ij}$ and the maximal term in the r.h.s. of (1) is $w \equiv \max_i\{lt(g_i) \times v_i\}$. Since $G$ is autoreduced, we have $\forall i \neq k, \neg(lt(g_i)|_J lt(g_k))$. Hence, by property (2) of Janet divisors, $\forall i \neq k, lt(g_i) \times v_i \neq lt(g_k) \times v_k$. It means that there exists a unique maximum $w \equiv lt(g_m) \times v_m$ in $\{lt(g_i) \times v_i\}$. Since all other terms in the r.h.s. of (1) are less than $w$, we conclude that $lt(f) = w \equiv lt(g_m) \times v_m$. We have proven that $lt(f)$ has a unique Janet divisor $lt(g_m)$. Our next step is to prove that $NF_J(f, G) = 0$ for any sequence of Janet reductions. Assume that there exists a chain of Janet reductions such that $NF_J(f, G) = h \neq 0$ and force a contradiction. Since $h$ is in Janet normal form modulo $G$, term $lt(h)$ has no Janet divisors in $\{lt(g) \mid g \in G\}$. On the other hand, from $f \in M(G)$ and $h = NF_J(f, G)$ we have $h \in M(G)$ and as proved above there exists $g \in G$ such that $lt(g)|_J lt(h)$. We come to a contradiction. Hence, $NF_J(f, G) = 0$. □

**Theorem 2.** (Uniqueness of a Janet normal form). *If $G$ is autoreduced and $h_1$, $h_2$ are Janet normal forms of $f$ modulo $G$ then $h_1 = h_2$.*

**Proof.** Since $h_1 \equiv NF_J(h_1, G)$, $h_2 \equiv NF_J(h_2, G)$, for each term of $h_1 - h_2$ there are no Janet divisors in $\{lt(g) \mid g \in G\}$. Therefore

$$NF_J(h_1 - h_2, G) \equiv h_1 - h_2 \tag{2}$$

for any sequence of Janet reductions. On the other hand, $h_1 = NF_J(f, G)$ and $h_2 = NF_J(f, G)$, hence $h_1 - h_2 \in M(G)$. Then, by Theorem 1, $NF_J(h_1 - h_2, G) = 0$ for any sequence of Janet reductions. Comparing with (2) we obtain $h_1 = h_2$. □

**Theorem 3.** *If $G$ is autoreduced, then*

$$\forall f_1, f_2, \quad NF_J(f_1 + f_2, G) = NF_J(f_1, G) + NF_J(f_2, G).$$

**Proof.** Let

$$h_1 = NF_J(f_1, G), \qquad h_2 = NF_J(f_2, G), \qquad h_3 = NF_J(f_1 + f_2, G). \tag{3}$$

Since for each term in $h_1$, $h_2$ and $p$ there are no Janet divisors in $\{lt(g) \mid g \in G\}$, we have

$$NF_J(h_1 + h_1 - h_3, G) = h_1 + h_2 - h_3. \tag{4}$$

One the other hand, from (3) it follows that $h_1 + h_2 - h_3 \in M(G)$. Hence, by Theorem 1, we obtain that $NF_J(h_1 + h_2 - h_3, G) = 0$. Comparing with (4) we have $h_3 = h_1 + h_2$, that is $NF_J(f_1 + f_2, G) = NF_J(f_1, G) + NF_J(f_2, G)$. □

## 3. Involutive systems

**Definition 6.** *Prolongation* of polynomial $g$ by variable $x$ is a product $g \cdot x$. If $x \in Nonmult(lt(g))$ then the prolongation is called non-multiplicative, otherwise multiplicative.

**Definition 7.** (Pommaret [1]). $G$ is an *involutive system* if it is autoreduced and all non-multiplicative prolongations of its elements are the linear combinations of multiplicative prolongations of its element, i.e.

$$\forall g \in G, \ \forall x \in Nonmult(lt(g)), \quad NF_J(g \cdot x, G) = 0. \tag{5}$$

Note that involution conditions (5) are non-trivial because any non-multiplicative prolongation $g \cdot x$ is not reduced to zero in the sense of Janet by means of polynomial $g$.

The following algorithm for a given $F$ computes an equivalent involutive system $G$.

**Algorithm 1** ($G = \text{Invsys}(F)$).
$G := \emptyset$;
**while** $F \neq \emptyset$ **do**
    $G := \text{Autoreduce}(G \cup F)$
    $F := \emptyset$;

**for each** $g$ **in** $G$ **do**
**for each** $x_i$ **in** Nonmult($lt(g)$) **do**
  $f := NF_J(g \cdot x_i, G)$;
  **if** $f \neq 0$ **then**
    $F := F \cup \{f\}$;

In addition, for the output system $G$ the integers $\alpha_i^q$, called *Cartan characters* of $G$ [1], are computed

$$\alpha_i^q = C_{q+n-i}^q - \beta_i^q, \quad q = \max\{\deg lt(g) \mid g \in G\} \qquad i = 1, 2, \ldots, n,$$

where $C_{q+n-i}^q$ is equal to the number of all terms of the classes $\geqslant i$ and degree $q$, and $\beta_i^q$ is a number of all such terms which have Janet divisors in the set $\{lt(g) \mid g \in G\}$.

**Example 2.** Let $<_T$ be the degree reverse laxicographical ordering, $z <_T y <_T x$ and

$$F = \{x^3 + y^2 + z - 3, y^3 + z^2 + x - 3, z^3 + x^2 + y - 3\}.$$

An involutive system $G$ equivalent to $F$ is

$$\begin{aligned} G = \{ & x^2y^2z^3 - 3x^2y^2 - xy^2z - x^2z^2 + xyz^2 + x^2y + 3xy^2 + 3x^2 - 3xy + y^2 + z - 3, \\ & x^2yz^3 + x^2y^2 - 3x^2y - xyz + xz^2 + x^2 + 3xy - 3x, \\ & xy^2z^3 - 3xy^2 - y^2z - xz^2 + yz^2 - x^2 + xy + 3y^2 + 3x - 3y, \\ & x^2y^3 + x^2z^2 - 3x^2 - y^2 - z + 3, x^2z^3 + x^2y - xy^2 - 3x^2 - xz + 3x, \\ & xyz^3 + xy^2 - 3xy - yz + z^2 + x + 3y - 3, y^2z^3 + x^2y^2 - 3y^2 - z^2 - x + 3, \\ & xy^3 + xz^2 + x^2 - 3x, xz^3 + xy - y^2 - 3x - z + 3, yz^3 + x^2y + y^2 - 3y, \\ & x^3 + y^2 + z - 3, y^3 + z^2 + x - 3, z^3 + x^2 + y - 3\}. \end{aligned}$$

Cartan characters of $G$ are $\alpha_1^7 = \alpha_2^7 = \alpha_3^7 = 0$.

Note that (5) is a termination condition for Algorithm 1. Hence, the output system $G$ is involutive. On the other hand, there exist polynomial systems which cannot be transformed to involutive form at all (in the sense of condition (5)). It means that generally Algorithm 1 is non-terminating process. For example, for any single element set $F = \{f(x_1, x_2, \ldots, x_n)\}$ where $x_1 <_T \cdots <_T x_n$, $u \equiv lt(f)$, $\partial u/\partial x_1 \neq 0$, $\partial u/\partial x_n \neq 0$, $n \neq 1$, Algorithm 1 generates an infinite sequence of prolongations of the form $f \cdot x_n^k$, $k = 1, 2, \ldots$, which cannot be reduced to zero in the sense of Janet.

Thus there exist "irregular" polynomial systems for which Algorithm 1 does not terminate, contrary to the "regular" ones.

**Theorem 4.** *(Pommaret [2], Zharkov [6]). If dimension of $Ideal(F)$ is zero, then Algorithm 1 terminates.*

By Theorem 4, Algorithm 1 may not terminate only for the positive-dimension ideals. In these cases sometimes it is sufficient to reorder variables so that the system becomes regular. However, there exist the systems which are irregular for any order of variables. Note that reordering of variables is a particular case of invertible linear transformation of the vector space $\{(x_1, \ldots, x_n)\}$. The following result forms the foundation of the involution approach.

**Theorem 5.** *(Pommaret [1]). For any given $F = \{f \mid f \in K[x_1, \ldots, x_n]\}$ there exists an invertible linear transformation*

$$
\begin{aligned}
&x = A \cdot y, \\
&x \equiv (x_1, \ldots, x_n), \qquad y \equiv (y_1, \ldots, y_n), \qquad y_1 < \cdots < y_n, \\
&A = \|a_{ij}\|, \quad a_{ij} \in K, \quad i, j = 1, \ldots, n, \quad \det(A) \neq 0,
\end{aligned} \tag{6}
$$

*such that $F$ is regular w.r.t. $y_1, \ldots, y_n$.*

By Theorem 5, any polynomial system can be transformed to involutive form by a suitable transformation (6). It may be proved that "almost all" transformations of the form (6) lead to the desirable result.

To understand the importance of involution approach to solving polynomial equations it is necessary to clarify the relation between involutive systems and Gröbner bases. It turns out that any involutive system is nothing else but a special form of non-reduced Gröbner basis. To show this we first prove the following theorem.

**Theorem 6.** *Let $G$ be involutive. Then*

$$\forall f \in Ideal(G), \quad NF_J(f, G) = 0.$$

**Proof.** Let $g$ be any polynomial from $G$ and $u$ be an arbitrary term. Represent $g \cdot u$ as $v \cdot g \times w$, where $v \cdot w = u$, all variables in $v$ are non-multiplicative and all variables in $w$ are multiplicative for $g$. Fix some variable $x$ in $v$ and write $g \cdot u = v_1 \cdot (xg) \times w$ where $v_1 = v/x$. Because of (5),

$$x \cdot g = g_1 \times s_1 + \sum_{k,l} a_{kl} g_k \times s_{kl}, \tag{7}$$

where $g_i \in G$, and $g_1$ is such that $lt(g_1) \times s_1 = x \cdot lt(g)$. Since $G$ is autoreduced, $g_1$ is unique. Evidently, $\max\{lt(gk) \times s_{kl}\} <_T lt(g_1) \times s_1$. Substituting (7) into $g \cdot u \equiv v_1 \cdot (xg) \times w$ we have

$$g \cdot u = v_1 \cdot g_1 \times w_1 + \sum_{k,l} a_{kl} g_k \cdot u_{kl},$$

where $w_1 = s_1 \cdot w$ and, by admissibility of the term ordering, $\max\{lt(g_k) \cdot u_{kl}\} <_T lt(g) \cdot u$. Repeating the same considerations for $g_1$, taking into account that $\deg(v_1) = \deg(v) - 1$ and acting recursively we obtain in a finite number of steps

$$g \cdot u = g_1' \times w_1' + \sum_{k,l} a_{kl}' g_k' \cdot u_{kl}', \tag{8}$$

where $g_i' \in G$, $lt(g_1') \times w_1' = lt(g) \cdot u$ and

$$\max\{lt(g_k') \cdot u_{kl}'\} <_T lt(g) \cdot u. \tag{9}$$

Consider any $f \in Ideal(G)$ and represent it in the form

$$f = \sum_{i,j} a_{ij} g_i \times w_{ij} + \sum_{k,l} a_{kl} g_k \cdot u_{kl}.$$

where $g_i, g_k \in G$ and $g_k \cdot u_{kl} \neq g_k \times u_{kl}$. Representing each polynomial $g_k \cdot u_{kl}$ by Eq. (8), we obtain

$$f = \sum_{i,j} a'_{ij} g'_i \times w'_{ij} + \sum_{k,l} a'_{kl} g'_k \cdot u'_{kl},$$

where $\max\{lt(g'_k)u'_{kl}\} <_{\mathrm{T}} \max\{lt(gk)u_{kl}\}$. Repeating this process and taking into account that the admissible term ordering is noetherian, we obtain in a finite number of steps that $f = \sum_{i,j} \tilde{a}_{ij} \tilde{g}_i \times \tilde{w}_{ij}$, that is $f \in M(G)$. Consequently, by Theorem 1, $NF_{\mathrm{J}}(f, G) = 0$. □

**Corollary 1.** *Any involutive system G is a Gröbner basis in the given term ordering.*

**Proof.** According to [5], it is sufficient to prove that for some fixed algorithm of $NF$ the following relations hold:

$$\forall g, g' \in G, \quad NF(Spol(g, g'), G) = 0, \tag{10}$$

where

$$Spol(g, g') \equiv \frac{lcm(lt(g), lt(g'))}{lt(g)} \cdot g - \frac{lc(g)}{lc(g')} \cdot \frac{lcm(lt(g), lt(g'))}{lt(g')} \cdot g',$$

$lcm(u, v) \equiv$ least common multiple of $u$, $v$.

Choose an algorithm of $NF$ so that $NF(f, G) \equiv NF'(NF_{\mathrm{J}}(f, G), G)$ where $NF'$ is computed using arbitrary fixed normal form algorithm. Then, since $Spol(g, g') \in Ideal(G)$ and because of Theorem 6, we have $NF_{\mathrm{J}}(Spol(g, g'), G) = 0$ that implies (10). □

In spite of the fact that involutive systems sometimes appear more complicated than reduced Gröbner bases they enable to obtain information about ideals in a more natural way. The following result may be deduced from [2].

**Theorem 7.** *(Pommaret [2]). Let $<_{\mathrm{T}}$ be the total-degree ordering and let G be an involutive system. Dimension of Ideal(G) is equal to the number of non-zero characters $\alpha_i^q$ of G. If dimension of Ideal(G) is zero, then the number of roots is*

$$D = C_{n+q-1}^{q-1} - \sum_{k=1}^{q-1} \beta_1^k, \qquad q = \max\{\deg g \mid g \in G\}$$

E.g., for $G$ in Example 2 dimension of $Ideal(G)$ is zero and $D = 27$.

Let $F$ be a system with finite number of roots. To find the roots of $F$ it is sufficient to construct the Gröbner basis of $Ideal(F)$ in the lexicographical ordering. It turns out that involutive system $G$ equivalent to $F$ and constructed in the total-degree ordering may be interpreted as a system of linear algebraic equations over field $K(x_1)$ w.r.t. all the terms contained in lexicographical Gröbner basis of $Ideal(F)$ considered in $K(x_1)[x_2, \ldots, x_n]$. The compatibility condition of this linear system gives an equation for $x_1$. As a rule, to obtain such a system it is sufficient to consider not the whole involutive system, but only its elements of class 1. For instance, involutive system $G$ in Example 2 contains nine polynomials of class 1 which form a linear algebraic system over $Q(z)$ w.r.t. the terms $x^2y^2, x^2y, xy^2, x^2, xy, y^2, x, y$ considered as independent unknowns. The compatibility condition gives

$$z^{27} - 27z^{24} + 317z^{21} - 18z^{19} - 2067z^{18} - 50z^{17} + 279z^{16} + 8156z^{15} + 645z^{14} - 1674z^{13}$$
$$- 20359z^{12} - 3044z^{11} + 4645z^{10} + 33644z^{9} + 6288z^{8} - 6388z^{7} - 36936z^{6} - 5925z^{5}$$
$$+ 4957z^{4} + 23187z^{3} + 4063z^{2} - 4342z - 5352 = 0.$$

Solving the linear system w.r.t. the terms $x$, $y$ and eliminating other terms, we obtain two equations of the form $x + p_1(z) = 0$, $y + p_2(z) = 0$, $\deg(lt(p_1)) = \deg(lt(p_2)) = 26$, which give a reduced lexicographical Gröbner basis together with equation in $z$.

Thus, involutive system being non-reduced Gröbner basis in some admissible term ordering is an implicit form of reduced lexicographical Gröbner basis which can be easily constructed by means of linear algebra, compare with [7]. The theoretical explanation of this fact will be given elsewhere [6].

Note that the notions of a Janet normal form, involutive system and Theorems 1–6 are valid for any admissible term ordering. However, our definition of the Cartan characters and Theorem 7 make sense only for the total degree orderings.

## 4. Examples

An improved version of Algorithm 1 [6] is implemented in the form of REDUCE package INVSYS. We present the results of comparison of INVSYS with standard REDUCE package GROEBNER [8,9] at several examples of zero-dimensional ideals taken from paper [7]. Note that Examples 4 and 5 distinguish from each other in only one term and this leads to drastic distinction in computing time.

**Example 3.**

$$x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2 + x_1x_2x_3 + x_1x_2 + x_1x_3 + x_2x_3 = 0,$$
$$x_1^2x_2^2x_3 + x_1x_2^2x_3^2 + x_1^2x_2x_3 + x_1x_2x_3 + x_2x_3 + x_1 + x_3 = 0,$$
$$x_1^2x_2^2x_3^2 + x_1^2x_2^2x_3 + x_1x_2^2x_3 + x_1x_2x_3 + x_1x_3 + x_3 + 1 = 0.$$

**Example 4.**

$$x + x_2 + x_3 + x_4 + x_5 = 0,$$
$$x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1 = 0,$$
$$x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_1 + x_5x_1x_2 = 0,$$
$$x_1x_2x_3x_4 + x_2x_3x_4x_5 + x_3x_4x_5x_1 + x_4x_5x_1x_2 + x_5x_1x_2x_3 = 0,$$
$$x_1x_2x_3x_4x_5 - 1 = 0.$$

**Example 5.**

$$x_1 + x_2 + x_3 + x_4 + x_5 = 0,$$
$$x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1 = 0,$$
$$x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_1 + x_5x_1x_2 = 0,$$
$$\underline{x_2}x_3x_4 + x_2x_3x_4x_5 + x_3x_4x_5x_1 + x_4x_5x_1x_2 + x_5x_1x_2x_3 = 0,$$
$$x_1x_2x_3x_4x_5 - 1 = 0.$$

Table 1
Results of comparison for different variable orderings

| Example | Variable ordering | $T_1$ | $T_2$ |
|---|---|---|---|
| 3 | $x_1 > x_2 > x_3$ | 16″ | 33″ |
| 4 | $x_1 > x_2 > x_3 > x_4 > x_5$ | 11″ | 8″ |
| 4 | $x_1 > x_2 > x_5 > x_3 > x_4$ | 9″ | 7″ |
| 5 | $x_1 > x_2 > x_3 > x_4 > x_5$ | 2.5″ | 5.7′ |
| 5 | $x_1 > x_2 > x_5 > x_3 > x_4$ | 32′ | 51′ |
| 5 | $x_4 > x_1 > x_5 > x_2 > x_3$ | 1.5′ | 20′ |
| 6 | $x_1 > x_2 > x_6 > x_3 > x_4 > x_5$ | 2 h | > 40 h |
| 6 | $x_5 > x_4 > x_3 > x_6 > x_2 > x_1$ | 1 h | 22 h |

**Example 6.**

$$\begin{aligned}
&x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 0,\\
&x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_6 + x_6x_1 = 0,\\
&x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_6 + x_5x_6x_1 + x_6x_1x_2 = 0,\\
&x_1x_2x_3x_4 + x_2x_3x_4x_5 + x_3x_4x_5x_6 + x_4x_5x_6x_1 + x_5x_6x_1x_2 + x_6x_1x_2x_3 = 0,\\
&x_1x_2x_3x_4x_5 + x_2x_3x_4x_5x_6 + x_3x_4x_5x_6x_1 + x_4x_5x_6x_1x_2 + x_5x_6x_1x_2x_3 + x_6x_1x_2x_3x_4 = 0,\\
&x_1x_2x_3x_4x_5x_6 - 1 = 0.
\end{aligned}$$

All computations using INVSYS and GROEBNER have been performed on an 25 Mhz MS-DOS based AT/386 computer with 8 Mb RAM. The results of comparison for different variable orderings are given in Table 1. We denote by $T_1$ the time for computing involutive system using INVSYS and by $T_2$ the time for computing Gröbner basis in the degree reverse lexicographical ordering using GROEBNER.

The results of comparison enable to hope that the involution approach is sufficiently powerful tool for solving polynomial systems. The complexity of the appropriate algorithm is an open question.

We want to emphasize a great importance of the theory developed in [1,2] for the computer algebra that becomes clear just now.

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