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SAGBI Bases Under Composition

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Polynomial composition is the operation of replacing the variables in a polynomial with other polynomials. In this paper we give a sufficient and necessary condition on a set Θ of polynomials to assure that the set $F \circ \Theta$ of composed polynomials is a SAGBI basis whenever F is.

1. Introduction

Our interest in the subject of this paper is inspired by Hong (1998), where Hoon Hong addresses the problem of the behavior of Gröbner bases under composition of polynomials. More precisely, let Θ be a set of polynomials, as many as the variables in our polynomial ring. The question then is under which conditions on these polynomials it is true that for an arbitrary Gröbner basis G (with respect to some term ordering), the composed set $G \circ \Theta$ is also a Gröbner basis (with respect to the same ordering). If this is the case, then we say that the composition commutes with Gröbner bases computation. The main result in Hong (1998) is that this happens if and only if the composition is "compatible" with the ordering and the nondivisibility (see Section 2.2 and Section 4 respectively, for the terminology).

An algorithmic approach for studying subalgebras of polynomial rings, analogous to Buchberger's Gröbner bases theory for ideals, has been introduced by Robbiano and Sweedler (1990), and independently by Kapur and Madlener (1989); the bases hereby constructed are called SAGBI bases (**S**ubalgebra **A**nalog to **G**röbner **B**ases for **I**deals). Since many of the basic concepts of Gröbner bases transfer to the subalgebra case, it is natural to ask under which conditions composition by a set Θ commutes with SAGBI bases computation. The main contribution of this paper is to show that this is the case if and only if the composition is compatible with the ordering, i.e. we need only one of the conditions necessary for Gröbner bases. The reason that only this condition is sufficient in the subalgebra case is that it implies the "SAGBI counterpart", called nonequality below, of the nondivisibility. It follows that commutation with SAGBI bases computation is a weaker condition than commutation with Gröbner bases computation.

The natural application of the results in this paper is the same as for Gröbner bases in Hong (1998): composed objects often occur in real-life mathematical models, and given a set F of polynomials in which the variables are defined in terms of other variables, it should be more efficient to compute a SAGBI basis of F before carrying out the composition. (Note, however, that in contrast to Gröbner bases, SAGBI bases computation may

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not terminate). We also mention that polynomial decomposition is a widely studied area. Since commutation with Gröbner bases computation implies commutation with SAGBI bases computation, all the compatible compositions in Section 6 of Hong (1998) clearly apply to the SAGBI case.

Since the theory of SAGBI bases transfers very smoothly to noncommutative polynomial rings (see Nordbeck, 1998), we will end this paper with a brief treatment of noncommutative SAGBI bases under composition. We will see that we get exactly the same results in the noncommutative case.

We mention that composition of Gröbner bases has been studied further by Hong (1996), by Gutierrez and San Miguel (1998) and by the author of this paper in Nordbeck (2001). The first of these papers is devoted to the case when the composed Gröbner bases may be under a possibly different ordering, the second concerns reduced Gröbner bases and the last noncommutative Gröbner bases.

2. Basic Definitions and Notation

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite alphabet, and let $K[X]$ denote the polynomial algebra over the arbitrary field K. By a *term* in $K[X]$ we mean an element of the form $x_1^{e_1} \cdots x_n^{e_n}$ with $e_1, \ldots, e_n \in \mathbb{N} = \{0, 1, 2, \ldots\}$; we denote the set of all terms by T. Note that $\mathbf{1} = x_1^0 \cdots x_n^0 \in T$.

If H is a subset of $K[X]$ (not necessarily finite), then the subalgebra of $K[X]$ generated by H is usually denoted $K[H]$. This notion is natural since the elements of $K[H]$ are precisely the polynomials in the set of formal variables H, viewed as elements of $K[X]$. The elements of $K \subset K[H]$ correspond to the constant polynomials.

We will reserve the word *monomial* for the “terms in H” (in contrast to X). Thus, *by a monomial we will always mean a finite product of elements from H* (or some other subset of $K[X]$ that we are working with; see Remark 4 below). To emphasize this fact, such a product will usually be written $m(H)$. We will also allow the “empty” monomial $\mathbf{1}$. Note that our monomials are not (in general) terms viewed as elements of $K[X]$.

It is worth mentioning that in Robbiano and Sweedler’s paper (1990), the role of monomials is played by exponent functions; for a monomial $h_1^{\alpha_1} h_2^{\alpha_2} \cdots h_l^{\alpha_l}$, $\alpha_i \in \mathbb{N}$, we then speak of the exponent vector $\alpha = (\alpha_1, \ldots, \alpha_l)$. Since this approach is not possible in the non-commutative case, we speak of monomials in this paper.

In what follows we will always assume that T is given a *term ordering* (see, for example, Adams and Loustaunau, 1994). We can then associate to every non-zero polynomial $f \in K[X]$ its *leading term*, denoted $\widehat{f}$ ($\in T$). We call the coefficient of $\widehat{f}$ the *leading coefficient* of f. We also define, for a subset $F \subset K[X]$, $\widehat{F} = \{\widehat{f} \mid f \in F\}$.

2.1. SAGBI Bases

Here we gather the theory concerning SAGBI bases that we will need. For a more complete exposition we refer to Robbiano and Sweedler (1990).

Definition 1. Let S be a subalgebra of $K[X]$. A subset $H \subset S$ is called a *SAGBI basis* for S if for every non-zero $s \in S$, there exists a monomial m such that $\widehat{s} = \widehat{m(H)}$.

Remark 1. Since term orderings are preserved by multiplication, $m(H) = \prod_i h_i$ ($h_i \in H$) implies $\widehat{m(H)} = \prod_i \widehat{h_i}$, i.e.

$$\widehat{m(H)} = m(\widehat{H}). \tag{1}$$

Thus an equivalent formulation of the previous definition is that H is a SAGBI basis if the leading term of every non-zero element in S can be written as a product of leading terms of elements in H.

We can show that if H is a SAGBI basis for S, then H generates S, i.e. $S = K[H]$. We may (and will) therefore simply say that H is a SAGBI basis, meaning that H is a SAGBI basis for $K[H]$.

Definition 2. We say that two monomials m, m' form a *critical pair* (m, m') of H if $\widehat{m(H)} = \widehat{m'(H)}$. If $c \in K$ is such that $m(H)$ and $cm'(H)$ have the same leading coefficient, then we define the *T-polynomial*† of (m, m') as $T(m, m') = m(H) - cm'(H)$.

The constant is chosen such that the leading words cancel in $T(m, m')$, and thus we have $\widehat{T(m, m')} < \widehat{m(H)} = \widehat{m'(H)}$.

We will also use the characterization of SAGBI bases in the following theorem. The proof is essentially the same as the proof of, for example, Theorem 2.8 in Robbiano and Sweedler (1990).

Theorem 1. *A subset $H \subset K[X]$ is a SAGBI basis if and only if the T-polynomial of every critical pair (m, m') of H either is equal to zero, or can be written*

$$T(m, m') = \sum_{i=1}^{t} c_i m_i(H), \qquad \widehat{m_i(H)} < \widehat{m(H)} = \widehat{m'(H)} \;\; \forall i, \tag{2}$$

where the m_i are monomials and the $c_i \in K$.

Remark 2. In particular, it follows that every subset $H \subset K[X]$ consisting only of terms (or terms times coefficients) is a SAGBI basis; every T-polynomial is then clearly equal to zero.

2.2. composition of polynomials

We now define the process of composition of polynomials.

Definition 3. Let $\Theta = \{\theta_1, \ldots, \theta_n\}$ be a subset of $K[X]$ ($= K[x_1, \ldots, x_n]$), and let $f \in K[X]$. We define the *composition* of f by Θ, written $f \circ \Theta$, as the polynomial obtained from f by replacing each occurrence of the x_i with θ_i. We also define, for a subset $F \subset K[X]$, $F \circ \Theta = \{f \circ \Theta | f \in F\}$.

Remark 3. We will, in addition, assume that $\theta_i \notin K$ for all i. This will be used later to ensure that $t \neq \mathbf{1}$ implies $t \circ \widehat{\Theta} \neq \mathbf{1}$ for every term $t \in T$. We will make some more comments on this matter in Remark 6 below.

†The Gröbner bases term is *S-polynomial*. We here use the T since a critical pair in Robbiano and Sweedler (1990) is called a *tête-a-tête*.

Remark 4. With the notation in Definition 3, it seems tempting to write our monomials $m(H)$ above as $m \circ H$. But we retain our notation $m(H)$ to distinguish our "compositions" by H from our compositions by Θ. We leave to the reader to convince her/himself that the two forms of compositions are associative in the following sense:

$$m(H) \circ \Theta = m(H \circ \Theta). \tag{3}$$

The notion $m(H \circ \Theta)$ clearly makes sense since we have a natural correspondence between the sets $H = \{h_1, h_2, \dots\}$ and $H \circ \Theta = \{h_1 \circ \Theta, h_2 \circ \Theta, \dots\}$.

We clearly have, for $f, g \in K[X]$,

$$(fg) \circ \Theta = f \circ \Theta g \circ \Theta, \tag{4}$$

$$(f + g) \circ \Theta = f \circ \Theta + g \circ \Theta. \tag{5}$$

Since our ordering is preserved by multiplication we also have, in analogy with (1), for every term $t \in T$,

$$\widehat{t \circ \Theta} = t \circ \widehat{\Theta}. \tag{6}$$

Definition 4. We say that composition by Θ *commutes with SAGBI bases computation* if for every SAGBI basis H, also $H \circ \Theta$ is a SAGBI basis (under the same ordering as H).

As mentioned before, our main task in this paper will be to decide under which conditions on Θ, composition by Θ commutes with SAGBI bases computation.

Remark 5. In Hong's paper, the counterpart of Definition 4 requires that if G is a Gröbner basis for the ideal generated by a set of polynomials F, then $G \circ \Theta$ is a Gröbner basis for the ideal generated by $F \circ \Theta$. A direct translation to our subalgebra language would of course be: if H is a SAGBI basis for the subalgebra generated by a set of polynomials F, then $H \circ \Theta$ is a SAGBI basis for the subalgebra generated by $F \circ \Theta$. That this implies the statement in Definition 4 is clear (take $F = H$). The two formulations are in fact equivalent since it is easy to prove (using (4) and (5) above) that $K[H] = K[F]$ implies $K[H \circ \Theta] = K[F \circ \Theta]$.

Definition 5. We say that composition by Θ is *compatible with the given ordering* if for all terms $t, t' \in T$ we have

$$t > t' \Longrightarrow t \circ \widehat{\Theta} > t' \circ \widehat{\Theta}. \tag{7}$$

Let $f \in K[X]$ be written as a linear combination of terms in decreasing order: $f = c_1 t_1 + \cdots + c_s t_s$, $t_1 > \cdots > t_s$. If composition by Θ is compatible with the ordering, then we have $t_1 \circ \widehat{\Theta} > \cdots > t_s \circ \widehat{\Theta}$, so using (5) and (6) we get

$$\widehat{f \circ \Theta} = \widehat{f} \circ \widehat{\Theta}. \tag{8}$$

Definition 6. We say that composition by Θ is *compatible with nonequality* if for all terms $t, t' \in T$ we have

$$t \neq t' \Longrightarrow t \circ \widehat{\Theta} \neq t' \circ \widehat{\Theta}. \tag{9}$$

The previous two definitions contain the two sufficient conditions needed for commutation of SAGBI bases computation. The following trivial lemma shows that the second condition in fact follows from the first one.

LEMMA 1. *If composition by Θ is compatible with the ordering, then composition by Θ is compatible with nonequality.*

Lemma 1 holds because term orderings are total.

3. Main Theorem

We now state our main result.

THEOREM 2. *Composition by Θ commutes with SAGBI bases computation if and only if the composition is compatible with the ordering.*

We will prove this theorem in the following two sections.

3.1. PROOF OF SUFFICIENCY

In this section, we prove the sufficiency of the compatibility condition (in Theorem 2) for commutativity. The key to this sufficiency is the following lemma.

LEMMA 2. *Assume that composition by Θ is compatible with the ordering. If (m, m') is a critical pair of $H \circ \Theta$, then (m, m') is also a critical pair of H.*

PROOF. Assume that (m, m') is a critical pair of $H \circ \Theta$. We know from (3) that $m(H \circ \Theta) = m(H) \circ \Theta$ and $m'(H \circ \Theta) = m'(H) \circ \Theta$, so $\widehat{m(H \circ \Theta)} = \widehat{m'(H \circ \Theta)}$ implies $\widehat{m(H)} \circ \widehat{\Theta} = \widehat{m'(H)} \circ \widehat{\Theta}$ by (8). Since our composition is compatible also with nonequality (by Lemma 1), we now get $\widehat{m(H)} = \widehat{m'(H)}$, i.e. (m, m') is a critical pair of H. $\Box$

PROPOSITION 1. *If composition by Θ is compatible with the ordering, then composition by Θ commutes with SAGBI bases computation.*

PROOF. For an arbitrary SAGBI basis H, we need to show that $H \circ \Theta$ is also a SAGBI basis. We will use Theorem 1, so let (m, m') be an arbitrary critical pair of $H \circ \Theta$. From the previous lemma, we know that (m, m') is also a critical pair of H. Since H is a SAGBI basis, by Theorem 1 we can then write

$$m(H) - cm'(H) = \sum_i c_i m_i(H) \quad \text{(or zero)}, \qquad \widehat{m_i(H)} < \widehat{m(H)} = \widehat{m'(H)} \quad \forall i. \tag{10}$$

Composing the equation containing the T-polynomial by Θ we get, using (3), (4) and (5),

$$m(H \circ \Theta) - cm'(H \circ \Theta) = \sum_i c_i m_i(H \circ \Theta) \quad \text{(or zero)}. \tag{11}$$

Further, composing the inequality in (10) by $\widehat{\Theta}$ we get, now using (6) and (7),

$$m_i\widehat{(H \circ \Theta)} < m\widehat{(H \circ \Theta)} = m'\widehat{(H \circ \Theta)} \ \forall i. \tag{12}$$

Since the leading words in the left-hand side of (11) then clearly cancel, the constant c must be the same as in the definition of the T-polynomial of (m, m') with respect to $H \circ \Theta$. Thus (11) and (12) together give a representation as in Theorem 1, and since the critical pair (m, m') of $H \circ \Theta$ was arbitrary, we conclude that $H \circ \Theta$ is a SAGBI basis. □

3.2. proof of necessity

We now prove the necessity of the compatibility condition (in Theorem 2) for commutativity. We show that the commutation of SAGBI bases computation implies compatibility with nonequality, and then use this result in the proof of the compatibility with the ordering. We start with a lemma.

Lemma 3. *Let $u, v \in T$ be two terms with $u \neq v$ but $u \circ \widehat{\Theta} = v \circ \widehat{\Theta}$. Then for every $w < u$, $H = \{u - w, v\}$ is a SAGBI basis.*

Proof. We first note that both u and v must be different from $\mathbf{1}$. For if say $v = \mathbf{1}$ then $u \neq \mathbf{1}$, and since the θ_i are nonconstant (recall Remark 3), $v \circ \widehat{\Theta} = u \circ \widehat{\Theta} \neq \mathbf{1}$, a contradiction.

We prove the stronger statement that H has no non-trivial critical pairs, i.e. if $\widehat{m(H)} = \widehat{m'(H)}$, then $m = m'$. It is then clear that H is a SAGBI basis; every T-polynomial must be identically equal to zero. Take therefore an arbitrary critical pair (m, m') of H; since $\widehat{H} = \{u, v\}$ we have $\widehat{m(H)} (= m(\widehat{H})) = u^k v^l$ and $\widehat{m'(H)} = u^s v^t$, and $u^k v^l = u^s v^t$. Assuming this critical pair to be non-trivial, we have $k \neq s$ and $l \neq t$, more precisely $k > s$ and $l < t$ or vice versa. Cancelling common powers of u and v we thus end up with $u^i = v^j$, $i, j > 0$. Composing this last equality by $\widehat{\Theta}$ we get, using (4), $(u \circ \widehat{\Theta})^i = (v \circ \widehat{\Theta})^j$. Since $u \circ \widehat{\Theta} = v \circ \widehat{\Theta} \neq \mathbf{1}$ we then clearly have $i = j$. From $u^i = v^j$ it now finally follows that $u = v$, a contradiction. Thus (m, m') is trivial, and H is a SAGBI basis. □

Proposition 2. *If composition by Θ commutes with SAGBI bases computation, then composition by Θ is compatible with nonequality.*

Proof. Let composition by Θ commute with SAGBI bases computation, and assume, in contrast, that there exist two terms $u, v \in T$ with $u \neq v$ but $u \circ \widehat{\Theta} = v \circ \widehat{\Theta}$; as in the previous proof we have $u, v \neq \mathbf{1}$. We know from Remark 2 that $H = \{u, v\}$ is a SAGBI basis, so $H \circ \Theta = \{u \circ \Theta, v \circ \Theta\}$ is also a SAGBI basis. If $f = u \circ \Theta - v \circ \Theta \in K[H \circ \Theta]$ is not equal to $\mathbf{1}$ or zero, then we are finished; since $\widehat{f} < u \circ \widehat{\Theta} = v \circ \widehat{\Theta}$, $\widehat{f}$ can then not be written as a product from $\widehat{H \circ \Theta} = \{u \circ \widehat{\Theta}, v \circ \widehat{\Theta}\}$, so $H \circ \Theta$ cannot be a SAGBI basis (recall Remark 1).

Now, let $u' = ux_i, v' = vx_i$ for some (arbitrary) $x_i \in X$. It is clear that for example $u' \circ \Theta = (u \circ \Theta)\theta_i$, and that

$$u' \neq v' \qquad \text{and} \qquad \widehat{u' \circ \Theta} = \widehat{v' \circ \Theta}. \tag{13}$$

If $f = u \circ \Theta - v \circ \Theta = \mathbf{1}$, then we use $H' = \{u', v'\}$ which is a SAGBI basis by Remark 2.

We then have

$$f' = u' \circ \Theta - v' \circ \Theta = (u \circ \Theta)\theta_i - (v \circ \Theta)\theta_i = f\theta_i = \theta_i \in K[H' \circ \Theta],$$

and as above it follows that $H = \{u' \circ \Theta, v' \circ \Theta\}$ cannot be a SAGBI basis ($\widehat{\theta_i} < \widehat{u' \circ \Theta} = \widehat{v' \circ \Theta}$). (We note that we cannot use $f = \mathbf{1}$ as a counterexample directly; $m = \mathbf{1}$ is a permitted monomial in Definition 1.)

It remains to exclude the case $f = u \circ \Theta - v \circ \Theta = 0$ (e.g. if $\Theta = \widehat{\Theta}$). The set we now use is $H' = \{u' + x_i, v'\}$. Since we have (13) and $x_i < u' = ux_i$, H' is a SAGBI basis by Lemma 3. As in the previous paragraph, we now get a contradiction from $f' = u' \circ \Theta - v' \circ \Theta = \theta_i \in K[H' \circ \Theta]$. □

Remark 6. We have above used our assumption in Remark 3; $\theta_i \notin K$ for all i. Hong's paper on Gröbner bases uses instead the fact that in the definition of a Gröbner basis G, it is common to assume that zero (i.e. the zero polynomial) is not allowed as a member of G. Assume that we follow this line and do not allow zero in SAGBI bases. If $\theta_i = c \in K$ for some i, then, since $H = \{x_i - c\}$ is obviously a SAGBI basis, we conclude that $H \circ \Theta = \{x_i \circ \Theta - c\} = \{\theta_i - c\} = \{0\}$ contradicts the commutation of SAGBI bases computation. This shows that it would have worked equally well to assume that SAGBI bases may not contain the zero polynomial.

We can now complete the proof of our main theorem.

Proposition 3. *If composition by Θ commutes with SAGBI bases computation, then composition by Θ is compatible with the ordering.*

Proof. Assume that composition by Θ commutes with SAGBI bases computation. If $u, v \in T$ are two terms with $u > v$, then we have to show that $u \circ \widehat{\Theta} > v \circ \widehat{\Theta}$. Since in particular $u \neq v$, we know from the previous proposition that we cannot have $u \circ \widehat{\Theta} = v \circ \widehat{\Theta}$, so we need only exclude the case $u \circ \widehat{\Theta} < v \circ \widehat{\Theta}$.

We first claim that $H = \{u - v, v\}$ is a SAGBI basis. In fact, we know from Remark 2 that $H' = \{u, v\}$ is a SAGBI basis so, having Remark 1 in mind, our claim follows since $K[H] = K[H']$ and $\widehat{H} = \widehat{H'}$. We conclude that $H \circ \Theta = \{u \circ \Theta - v \circ \Theta, v \circ \Theta\}$ must be also a SAGBI basis.

Assume now that $u \circ \widehat{\Theta} < v \circ \widehat{\Theta}$. We then have $\widehat{H \circ \Theta} = \{v \circ \widehat{\Theta}\}$, and $u \circ \Theta = (u \circ \Theta - v \circ \Theta) + v \circ \Theta \in K[H \circ \Theta]$. But (as in the previous proof) since $u \circ \widehat{\Theta} < v \circ \widehat{\Theta}$, $u \circ \widehat{\Theta} \neq \mathbf{1}$ cannot be written as a power of $v \circ \widehat{\Theta}$, so $H \circ \Theta$ cannot be a SAGBI basis. Thus our assumption that $u \circ \widehat{\Theta} < v \circ \widehat{\Theta}$ was false, so composition by Θ is compatible with the ordering. □

4. Comparison with the Ideal Case

The main theorem in Hong's paper (1998) is the following:

Theorem 3. (Theorem 3.1. in Hong, 1998) *Composition by Θ commutes with Gröbner bases computation if and only if both of the following conditions hold:*

1. *Composition by Θ is compatible with the ordering and*

2. for all terms $t, t' \in T$, if t does not divide t', then $t \circ \widehat{\Theta}$ does not divide $t' \circ \widehat{\Theta}$.

The second condition, in Hong (1998) called compatibility with nondivisibility, is used to guarantee that the least common multiples coincide in the sense that

$$\mathrm{lcm}(t, t') \circ \widehat{\Theta} = \mathrm{lcm}(t \circ \widehat{\Theta}, t' \circ \widehat{\Theta}) \tag{14}$$

for all terms t, t'. For a subset $H \subset K[X]$, this leads to a correspondence between the "critical pairs" of H and $H \circ \Theta$ essential for Gröbner bases theory. We have seen in Lemma 2 that compatibility with nonequality guarantees the crucial correspondence of the critical pairs in our subalgebra setting. (In particular, (14) implies that there are no (non-trivial) "Gröbner critical pairs" arising from the set Θ, and compatibility with nonequality implies the absence of "SAGBI critical pairs" of Θ.)

Since the first condition in Theorem 3 was sufficient for our SAGBI bases, we clearly have the following:

Corollary 1. *If composition by Θ commutes with Gröbner bases computation, then composition by Θ commutes with SAGBI bases computation.*

It is also easy to see that condition 2 in Theorem 3 alone implies its SAGBI counterpart, i.e. compatibility with nonequality: if $t \neq t'$, then it is clear that either t does not divide t' or vice versa (or both), let's say t does not divide t'. If condition 2 is fulfilled, then this implies that $t \circ \Theta$ does not divide $t' \circ \Theta$; in particular we must have $t \circ \Theta \neq t' \circ \Theta$.

To show that the converse of Corollary 1 is false, we need to find a set Θ such that composition by Θ is compatible with some ordering, but condition 2 in Hong's theorem is false. To find sufficient and necessary conditions for a given composition to be compatible with a given ordering is stated as an open problem in Hong (1998). However, this problem is now solved by Hong and Weispfenning (1999). The reader can easily check that composition by $\Theta = \{x_1x_2, x_2\} \subset K[x_1, x_2]$ is compatible with the pure lexicographic ordering with $x_1 > x_2$. Since we have $x_2 \nmid x_1$ but $x_2 \circ \widehat{\Theta} \mid x_1 \circ \widehat{\Theta}$ (i.e. $x_2 \mid x_1x_2$), we see that condition 2 above is not valid. Thus composition by this set Θ commutes with SAGBI bases computation but not with Gröbner bases computation.

5. Composition of Noncommutative SAGBI Bases

We now turn to the noncommutative polynomial algebra; the free associative algebra $K\langle X\rangle$. The elements of $K\langle X\rangle$ are all polynomials in the noncommuting variables $X = \{x_1, \ldots, x_n\}$, and the "terms" in $K\langle X\rangle$ are now *words*. (For the definition of an ordering in the noncommutative case, we refer to Nordbeck, 2001.)

The subalgebra $K\langle H\rangle$ of $K\langle X\rangle$ generated by a set of noncommutative polynomials $H \subset K\langle X\rangle$ consists of all polynomials in the formal noncommuting variables H. We may of course also speak of monomials, which are now "words in H".

As mentioned in the introduction, we can define noncommutative SAGBI bases; this has been done in Nordbeck (1998). While commutative and noncommutative Gröbner bases differ at some essential points, the theories of commutative and noncommutative SAGBI bases are very much alike. Section 2.1, where we collected some facts about SAGBI bases, can be translated to the noncommutative case word by word, as long as we remember that there is no commutativity (replace the brackets [..] by $\langle..\rangle$). Also

Section 2.2 concerning compositions transfers directly to the noncommutative case; all definitions, remarks and conclusions are exactly the same.

We will now see that our main theorem has the same wording in the noncommutative setting.

THEOREM 4. *Composition by* Θ *commutes with noncommutative SAGBI bases computation if and only if the composition is compatible with the ordering.*

The reader can check that the proofs of Lemma 2 and Proposition 1 still work when H and the monomials are noncommutative, and thus we can prove the sufficiency of the compatibility condition in Theorem 4.

In the proof of the necessity, we make a short remark on the proof of Lemma 3. First we need of course to replace "terms" by "words" in the statement of the proposition; we assume that this is done for the whole of Section 3.2. Recall that in order to prove Lemma 3, we need to show that $\widehat{m(H)} = \widehat{m'(H)}$ implies $m = m'$. Since m, m' are now noncommutative, we cannot use the same technique as in the original proof, but the argument can be replaced by the following:

We know that $\widehat{m(H)}, \widehat{m'(H)}$ are "words" in u, v. In the equality $\widehat{m(H)} = \widehat{m'(H)}$, we begin cancelling common factors of u and v from the left. If we end up with the equality $\mathbf{1} = \mathbf{1}$, then it is clear that we had $m = m'$. But if we assume that $m \neq m'$, then we end up with an equality $uw = vw'$ for some words w, w' (we cannot end up with $\mathbf{1}$ at only one side). Since $u \neq v$, we then must have that u is a proper left subword of v or vice versa. In either case, it is easy to see we cannot have $u \circ \widehat{\Theta} = v \circ \widehat{\Theta}$, using that the θ_i are nonconstant. This contradiction shows that we cannot have $m \neq m'$, so Lemma 3 is valid also in our noncommutative setting.

It is now straightforward to check that the proofs of Propositions 2 and 3 go through, and thus we can show the necessity in Theorem 4, thereby finishing the proof of Theorem 4.

Finally we mention that if we extract the main theorem on composition of noncommutative Gröbner bases from Nordbeck (2001), then we directly see (not surprisingly) that also in the noncommutative case, commutation with SAGBI bases computation is a weaker condition than commutation with Gröbner bases computation.

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