GRÖBNER BASES AND REGULARITY OF REES ALGEBRAS

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Introduction

Let $B = k[x_1, \dots, x_n]$ be a polynomial ring over a field k and $A = B/J$ a quotient ring of B by a homogeneous ideal J. Let $\mathfrak{m}$ denote the maximal graded ideal of A. Then the Rees algebra $R = A[\mathfrak{m}t]$ may be considered a standard graded k-algebra and has a presentation $B[y_1, \dots, y_n]/I_J$. For instance, if $A = k[x_1, \dots, x_n]$, then

$$R \cong k[x_1, \dots, x_n, y_1, \dots, y_n]/(x_iy_j - x_jy_i|\ i, j = 1, \dots, n).$$

In this paper we want to compare the ideals J and I_J as well as their homological properties.

The generators of I_J can be easily described as follows. For any homogeneous form $f = \sum_{1\le i_1\le \cdots \le i_d \le n} a_{i_1\cdots i_d} x_{i_1}\cdots x_{i_d} \in B$ of degree d we set

$$f^{(k)} := \sum_{1\le i_1\le \cdots \le i_d \le n} a_{i_1\cdots i_d} x_{i_1}\cdots x_{i_{d-k}} y_{i_{d-k+1}}\cdots y_{i_d}$$

for $k = 0, \dots, d$. For any subset $L \subset B$ of homogeneous polynomials in S we set

$$L' := \{f^{(k)}|\ f \in L, k = 0, \dots, \deg f\},$$

and let

$$H := \{x_iy_j - x_jy_i|\ 1 \le i < j \le n\}.$$

If L is a minimal system of generators of I, then $L' \cup H$ is a minimal system of generators of I_J (Proposition 1.1). We will show that if L is Gröbner basis of J for the reverse lexicographic order induced by $x_1 > \cdots > x_n > y_1 > \cdots > y_n$, then $L' \cup H$ is Gröbner basis of I_J (Theorem 1.3). As a consequence, if J has a quadratic Gröbner basis, then so does I_J.

The main concern of this paper is however the regularity which is a measure for the complexity of the resolution of a standard graded algebra (see [EiG], [BM]). Recall that the Castelnuovo-Mumford regularity of A is defined by

$$\operatorname{reg}(A) := \max\{b_i - i|\ i > 0\},$$

where b_i denotes the largest degree of a generator of the ith syzygy module of A. The regularity and related invariants of a graded k-algebra (for example, the extremal Betti numbers introduced in [BCP]) can be expressed in terms of the cohomological invariants $a_i = \max\{a|\ H^i(A)_a \ne 0\}$, where $H^i(A)$ denotes the ith local cohomology of A with support $\mathfrak{m}$ (see Section 2 for more details). For instance, $\operatorname{reg}(R) = \max\{a_i + i|\ i \ge 0\}$. In particular, we will also study the invariant

$$a^*(A) := \max\{a_i|\ i \ge 0\} = \max\{b_i|\ i \ge 0\} - n,$$

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which is another kind of regularity for A [Sh], [T2], [T3]. Our results are based on the observation that the local cohomology of R can be estimated in terms of the local cohomology of A.

If R is the Rees algebra of an arbitrary homogeneous ideal I of A generated by forms of the same degree, then R is still a standard k-algebra. In this case, we have the following estimations:

$$\begin{aligned} a^*(A) - s &\le a^*(R) \le \max\{a^*(A), a^*(G)\}, \\ \operatorname{reg}(A) &\le \operatorname{reg}(R) \le \max\{\operatorname{reg}(A)+1, \operatorname{reg}(G)\}, \end{aligned}$$

where s is the minimal number of generators of I and G denotes the associated graded ring of I (Theorem 3.4 and Theorem 3.5). These bounds are sharp. In particular, if R is the Rees algebra of the graded maximal ideal of A, then

$$\begin{aligned} a^*(A) - n &\le a^*(R) \le a^*(A), \\ \operatorname{reg}(A) &\le \operatorname{reg}(R) \le \operatorname{reg}(A)+1. \end{aligned}$$

It is shown in Theorem 5.3 and Theorem 5.7 that $a^*(R) = a^*(A)$ if and only if $a^*(A) \neq -1$ and that $\operatorname{reg}(R) = \operatorname{reg}(A)+1$ if and only if there is an integer i such that $\operatorname{reg}(A) = a_i + i$ and $a_i \le -2$. The proofs follow from the fact that the bigraded components of the local cohomology of R can be expressed completely by the graded components of the local cohomology of A (see Theorem 4.2). In particular, we can show that $\operatorname{reg}(R) = \operatorname{reg}(A)+1$ if $\operatorname{reg}(A) = b_i - i$ and $b_i \le n-2$ for some index i at which A has an extremal Betti number (Corollary 5.9). However, an example shows that this condition is only sufficient. As applications, we compare the regularity of the Rees algebra of the ring $B/\operatorname{in}(I)$, where $\operatorname{in}(I)$ denotes the initial ideal of I, with that of R and we estimate this regularity for the generic initial ideal $\operatorname{Gin}(J)$ with respect to the reverse lexicographic term order.

We will also compute the projective dimension of I_J. In Proposition 4.3 we give a precise formula for the depth of R in terms of invariants of A. In fact,

$$\operatorname{depth} R = \max\{i |\ H^j_{\mathfrak{m}}(A)_a = 0 \text{ for } a \neq -1, j < i-1, \text{ and } a_{i-1} < 0\}.$$

This formula is better than Huckaba and Marley's estimation for the depth of the Rees algebra of an arbitrary ideal in a local ring [HM]. Inspired by a construction of Goto [G] we give examples showing that for arbitrary positive numbers $2 \le r < d$ there exists a standard graded k-algebra A of dimension d with $\operatorname{depth} A = r$ and $\operatorname{depth} R = d+1$. In these examples R is Cohen-Macaulay, since $\dim R = d+1$. Though the difference between the depth of A and of R may be large, this is not the case for the Rees ring R^* of a polynomial ring extension $A[z]$ of A. Here we have that $\operatorname{depth} R^* = \operatorname{depth} A + 1$ if $a_s \ge 0$ and $\operatorname{depth} A[z] = \operatorname{depth} A + 2$ if $a_s < 0$ (Corollary 4.4).

We would like to mention that if R is the Rees algebra of a homogeneous ideal generated by forms of different degree, R is not a standard k-algebra. Since R is a standard graded algebra over A, one can still define the Castelnuovo-Mumford regularity and the a^*-invariant of R with respect to this grading. These invariants have been studied recently by several authors (see e.g. [JK], [Sh], [T1], [T2]).

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1. Gröbner basis of Rees algebras

Let A be a standard graded k-algebra with graded maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$. Then $A = B/J$ where $B = k[x_1, \dots, x_n]$ is a polynomial ring, and $J \subset B$ a graded ideal. The Rees algebra $R = A[\mathfrak{m}t]$ may be considered as a bigraded module over the bigraded polynomial ring $S = k[x_1, \dots, x_n, y_1, \dots, y_n]$ (where $\deg x_i = (1, 0)$ and $\deg y_i = (1, 1)$ for all i) via the bigraded epimorphism $\phi : S \to R$ with $\phi(x_i) = x_i$ and $\phi(y_i) = x_i t$ for $i = 1, \dots, n$. Let I_J denote the kernel of this epimorphism.

We are interested in the generators and the Gröbner basis of I_J. In order to describe I_J we introduce the following notations.

Let $f = \sum_{1 \le i_1 \le \cdots \le i_d \le n} a_{i_1 \cdots i_d} x_{i_1} \cdots x_{i_d} \in B$ be homogeneous of degree d. For $k = 0, \dots, d$ we set

$$f^{(k)} = \sum_{1 \le i_1 \le \cdots \le i_d \le n} a_{i_1 \cdots i_d} x_{i_1} \cdots x_{i_{d-k}} y_{i_{d-k+1}} \cdots y_{i_d}.$$

Notice that $f^{(k)}$ is bihomogeneous of degree (d, k). For any subset $L \subset B$ of homogeneous polynomials in S we set

$$L' := \{f^{(k)} |\ f \in L, k = 0, \dots, \deg f\}.$$

We further let

$$H := \{x_i y_j - x_j y_i |\ 1 \le i < j \le n\}.$$

With these notations we have

Proposition 1.1. *Let L be a (minimal) system of generators of J, then $L' \cup H$ is a (minimal) system of generators of I_J.*

Proof. Let $P = k[x_1, \dots, x_n, x_1 t, \dots, x_n t] \subset k[x_1, \dots, x_n, t]$, and $\phi_1 : S \to P$, $\phi_2 : P \to R$ be the k-algebra homomorphisms given by $\phi_1(x_i) = x_i$, $\phi_1(y_i) = x_i t$, and $\phi_2(x_i) = \bar{x}_i$, $\phi_2(x_i t) = \bar{x}_i t$ for $i = 1, \dots, n$. We have $\phi = \phi_2 \circ \phi_1$, and since ϕ is bigraded, the ideal I_J is bigraded. We clearly have $L' \cup H \subset I_J$. Let $f \in I_J$ be bigraded with $\deg f = (a, b)$. Then $\phi_1(f) = f(x, xt) = f(x, x)t^b$, and so $0 = \phi(f) = f(\bar{x}, \bar{x})t^b$, that is, $f(\bar{x}, \bar{x}) = 0$. Therefore, there exist homogeneous elements $g_i \in B$ and $f_i \in J$ such that $f(x, x) = \sum_{i=1}^m g_i f_i$. Let $b_i = \min\{\deg f_i, b\}$. Then

$$\phi_1(f) = f(x, x)t^b = \sum_{i=1}^m (g_i t^{b-b_i})(f_i t^{b_i}) = \phi_1(\sum_{i=1}^m g_i^{(b-b_i)} f_i^{(b_i)}),$$

and so $f \in L' \cup H$, since $\operatorname{Ker} \phi_1$ is generated by H.

Now let L be a minimal system of generators of J. We first show that $\phi_1(L')$ is a minimal system of generators of the ideal $L_J = \phi_1(I_J)$ in P. Indeed, $\phi_1(L') =$

$\{f_it^b|\ f_i \in L, \quad b = 0, \dots, \deg f_i\}$. Suppose this is not a minimal system of generators of L_J. Then there exists an equation

$$f_it^b = \sum_j \sum_k (f_jt^{b_{jk}})(g_{jk}t^{c_{jk}}),$$

where $b_{jk} \le \deg f_j$, $b_{jk} + c_{jk} = b$ and $f_jt^{b_{jk}} \ne f_it^b$ for all j and k, and where all summands are bihomogeneous of degree (d, b) with $d = \deg f_i$. Notice that the right hand sum contains no summand of the form $(f_it^{b_{ik}})(g_{ik}t^{c_{ik}})$. In fact, otherwise we would have $\deg g_{ik}t^{c_{ik}} = (0, b - b_{ik})$, and so $b_{ik} = b$ which is impossible. It follows that $f_i = \sum_{j\ne i}(\sum_k g_{jk})f_j$, a contradiction.

Now suppose that $L' \cup H$ is not a minimal system of generators of I_J. If one of the $f_i^{(k)}$ is a linear combination of the other elements of $L' \cup H$, then $\phi_1(L')$ is not a minimal system of generators of L_J, a contradiction. Next suppose one of the elements of H, say, $x_1y_2 - x_2y_1$ is a linear combination of the other elements of $L' \cup H$. Only the elements of bidegree $(2, 1)$ can be involved in such a linear combination. In other words,

$$x_1y_2 - x_2y_1 = \sum \lambda_f f^{(1)} + h \quad \text{with} \quad \lambda \in k.$$

Here the sum is taken over all $f \in L$ with $\deg f = 2$, and h is a k-linear combination of the polynomials $x_iy_j - x_jy_i$ different from $x_1y_2 - x_2y_1$. Since the monomial x_2y_1 does not appear in any polynomial on the right hand side of the equation, we get a contradiction. $\square$

We will now compute a Gröbner basis of I_J. For the proof we will use the following Gröbner basis criterion.

Lemma 1.2. *Let $Q = k[x_1, \dots, x_r]$ be the polynomial ring, $I \subset Q$ a graded ideal and L a finite subset of homogeneous elements of I. Given a term order $<$, there exists a unique monomial k-basis C of $Q/(\mathrm{in}(L))$ (which we call a "standard basis" with respect to $<$ and L). This k-basis C is a system of generators for the k-vector space Q/I, and L is a Gröbner basis of I with respect to $<$, if and only if C is a k-basis of Q/I.*

Theorem 1.3. *Let $<$ be the reverse lexicographic order induced by $x_1 > \dots > x_n > y_1 > \dots > y_n$. If L is a Gröbner basis of J with respect to the term order $<$ restricted to B, then $L' \cup H$ is a Gröbner basis of I_J with respect to $<$.*

Proof. Let C be a standard basis of B with respect to $<$ and L, and set

$$C' := \{u^{(k)}|\ u \in C, \quad k = 0, \dots, \deg u\}.$$

We will show that

(i) C' is a standard basis with respect to $<$ and $L' \cup H$, and
(ii) C' is a k-basis of R.

Let v be a monomial of T which does not belong to the ideal $(\mathrm{in}(L') \cup \mathrm{in}(H))$. Since $v \notin (\mathrm{in}(H))$, it follows that $v = u^{(k)}$ for some monomial $u \in S$. Suppose that $u \notin C$. Then $u \in \mathrm{in}(L)$, and since $\mathrm{in}(L') = \mathrm{in}(L)'$ it follows that $u^{(k)} \in (\mathrm{in}(L'))$, a contradiction. Thus $u \in C$, and hence $v \in C'$.

Conversely, if $v \in C'$, then $v = u^{(k)}$ for some $u \in C$. Monomials of the form $u^{(k)}$ cannot be multiples of monomials of $\mathrm{in}(H)$. Suppose $u^{(k)}$ is a multiple of a monomial $w \in \mathrm{in}(L)'$. Then $w = v^{(l)}$ for some $v \in \mathrm{in}(L)$ and some l, and $v^{(l)}$ divides $u^{(k)}$. It follows that v divides u, a contradiction. This proves (i).

Let $C_i = \{u \in C|\ \deg u = i\}$, and similarly $C'_i = \{u \in C'|\ \deg u = i\}$. Since C is a k-basis of A, it follows that $|C_i| = \dim_k A_i$, and since $C'_i = \{u^{(k)}|\ u \in C, k = 0, \dots, i\}$, it follows that $|C'_i| = (i+1)|C_i| = (i+1)\dim_k A_i$. It is easy to show that $\dim_k R_i = (i+1)\dim_k A_i$, $|C'_i| = \dim_k R_i$ for all i. This shows that the elements of C' are k-linearly independent, and proves (ii). Hence the desired conclusion follows from Lemma 1.2. □

Corollary 1.4. *If J has a quadratic Gröbner basis, then so does I_J.*

We would like to remark that if L is a reduced Gröbner basis, then $L' \cup H$ need not be reduced.

Example. Let $A = k[x_1, x_2, x_3]/(x_1x_2 - x_3^2)$. Then $L = \{x_1x_2 - x_3^2\}$ is a reduced Gröbner basis of J, but $L' \cup H$ is not reduced, since $x_1y_2 = \mathrm{in}(x_1y_2 - x_3y_3)$ appears in $x_1y_2 - x_2y_1$.

2. Regularity and local cohomology of graded algebras

The aim of this section is to prepare some facts on the relationships between the regularities and local cohomology modules of a graded module.

Let $B = k[x_1, \dots, x_n]$ be a polynomial ring over a field k. Let E be a finitely graded module over B. Let $\mathbb{F} : 0 \to F_r \to \cdots \to F_1 \to F_0 \to E \to 0$ be a minimal free resolution of E. For all integer i we denote by $b_i(E)$ the largest degree of the generators of F_i, where $b_i(E) := -\infty$ for $i < 0$ or $i > r$. The Castelnuovo-Mumford regularity of E [EiG] is defined by

$$\mathrm{reg}(E) := \max\{b_i(E) - i|\ i \geq 0\}.$$

This notion is refined by D. Bayer, H. Charalambous, and S. Popescu [BCP] as follows. For any integer j let

$$j\text{-}\mathrm{reg}(E) := \max\{b_i(E) - i|\ i \geq j\}.$$

Similarly, we can define the invariants

$$\begin{aligned} b^*(E) &:= \max\{b_i(E)|\ i \geq 0\}, \\ b_j^*(E) &:= \max\{b_i(E)|\ i \geq j\}. \end{aligned}$$

It is known that these invariants can be also characterized by means of the graded local cohomology modules of E.

Let $A = B/J$ be any graded quotient ring of B. Let E now be a finitely generated module over A. Let $\mathfrak{m}$ denote the maximal graded ideal of A. For any integer i we denote by $H^i_{\mathfrak{m}}(E)$ the ith local cohomology module of E. Since $H^i_{\mathfrak{m}}(E)$ is a graded artinian A-module, $H^i_{\mathfrak{m}}(E)_a = 0$ for a large enough. Therefore we can consider the largest non-vanishing degree $a_i(E) := \max\{a|\ H^i_{\mathfrak{m}}(E)_a \neq 0\}$, where $a_i(E) = -\infty$ if

$H^i_{\mathfrak{m}}(E) = 0$. Note that $H^i_{\mathfrak{m}}(E) = 0$ for $i < 0$ and $i > d := \dim E$ and that $a_d(E)$ is the a-invariant of E [GW]. For any integer j we define

$$\begin{aligned} a^*_j(E) &:= \max\{a_i(E)|\ i \le j\}, \\ \operatorname{reg}_j(E) &:= \max\{a_i(E) + i|\ i \le j\}. \end{aligned}$$

In particular, we set

$$a^*(E) := \max\{a_i(E)|\ i \ge 0\}.$$

These cohomological invariants do not depend on the presentation of A. See [Sh], [T1], [T2], [T3] for more information on these invariants.

Theorem 2.1. [T3, Theorem 3.1] *For any integer j we have*
(i) $b^*_j(E) = a^*_{n-j}(E) + n$,
(ii) $j\text{-}\operatorname{reg}(E) = \operatorname{reg}_{n-j}(E)$.

Theorem 2.1 has the following immediate consequence.

Corollary 2.2. (i) $b^*(E) = \max\{a_i(E)|\ i \ge 0\} + n = a^*(E) + n$,
(ii) $\operatorname{reg}(E) = \max\{a_i(E) + i|\ i \ge 0\}$ [EG, Theorem 1.2].

¿From Theorem 2.1 we also obtain the following relationship between the invariants $b_j(E)$ and $a_{n-j}(E)$. Following [BCP] we say that E has an *extremal Betti number* at j if $b_j(E) - j > b_i(E) - i$ for all $i > j$ or, equivalently, $j\text{-}\operatorname{reg}(E) > (j+1)\text{-}\operatorname{reg}(E)$.

Corollary 2.3. *Assume that E has an extremal Betti number at j. Then $b_j(E) = a_{n-j}(E) + n$.*

Proof. By the assumption, $j\text{-}\operatorname{reg}(E) = b_j(E) - j$. By Theorem 2.1,

$$\operatorname{reg}_{n-j}(E) = j\text{-}\operatorname{reg}(E) > (j+1)\text{-}\operatorname{reg}(E) = \operatorname{reg}_{n-j-1}(E).$$

Therefore, $\operatorname{reg}_{n-j}(E) = a_{n-j}(E) + n - j$. ¿From this it follows that $b_j(E) = a_{n-j}(E) + n$. ∎

For later applications we also prepare some facts on the regularity of polynomial extensions and quotient modules.

Lemma 2.4. *Let $A[z]$ be a polynomial ring over A in one variable. Let $\mathfrak{n}$ denote the maximal graded ideal of $A[z]$. Put $E[z] = E \otimes A[z]$ and $E[z^{-1}] = E \otimes A[z^{-1}]$. For every integer $i \ge 1$ we have*

$$H^i_{\mathfrak{n}}(E[z]) = H^{i-1}_{\mathfrak{m}}(E)(1)[z^{-1}].$$

Proof. By local duality (see e.g. [BH, Theorem 3.6.19]) we know that

$$\begin{aligned} H^i_{\mathfrak{m}}(E) &= \operatorname{Ext}^{n-i}_B(E, B(-n))^\vee, \\ H^i_{\mathfrak{n}}(E[z]) &= \operatorname{Ext}^{n+1-i}_{B[z]}(E[z], B[z](-n-1))^\vee, \end{aligned}$$

where $^\vee$ denotes the Matlis duality. Since $B \longrightarrow B[z]$ is a flat extension, we have

$$\operatorname{Ext}^{n+1-i}_{B[z]}(E[z], B[z]) = \operatorname{Ext}^{n+1-i}_B(E, B)[z].$$

From this it follows that

$$\begin{aligned} H^i_{\mathfrak{n}}(E[z]) &= \operatorname{Ext}_B^{n+1-i}(E, B(-n-1))^\vee[z^{-1}] \\ &= \operatorname{Ext}_B^{n+1-i}(E, B(-n))^\vee(1)[z^{-1}] \;=\; H^{i-1}_{\mathfrak{m}}(E)(1)[z^{-1}]. \end{aligned}$$

□

Proposition 2.5. *With the above notation we have*
(i) $a_i(E[z]) = a_{i-1}(E) - 1$ *for all* $i \geq 0$,
(ii) $a^*(E[z]) = a(E) - 1$,
(iii) $\operatorname{reg}(E[z]) = \operatorname{reg}(E)$.

Proof. By Lemma 2.4 we have

$$H^i_{\mathfrak{n}}(E[z])_n = \bigoplus_{a \geq n-1} H^{i-1}_{\mathfrak{m}}(E)_a.$$

Hence (i) is immediate. The formulas (ii) and (iii) are consequences of (i). □

Proposition 2.6. *Assume that* $\operatorname{depth} E > 0$ *and* $f \in A$ *is a regular form of degree* c *for* E. *Then*
(i) $a^*(E/fE) = a^*(E) + c$,
(ii) $\operatorname{reg}(E/fE) = \operatorname{reg}(E) + c - 1$.

Proof. From the exact sequence $0 \longrightarrow E(-c) \xrightarrow{f} E \longrightarrow E/fE \longrightarrow 0$ we obtain the following exact sequence of local cohomology modules:

$$H^i_{\mathfrak{m}}(E)_a \longrightarrow H^i_{\mathfrak{m}}(E/fE)_a \longrightarrow H^{i+1}_{\mathfrak{m}}(E)_{a-c} \longrightarrow H^{i+1}_{\mathfrak{m}}(E)_a\ .$$

From this it immediately follows that $a_i(E/fE) \leq \max\{a_i(E), a_{i+1}(E) + c\}$. For $a \geq a_i(E/fE)$, the map $H^{i+1}_{\mathfrak{m}}(E)_{a-c} \longrightarrow H^{i+1}_{\mathfrak{m}}(E)_a$ is injective. Since $H^{i+1}_{\mathfrak{m}}(E)_a = 0$ for all large a, this injective map yields $H^{i+1}_{\mathfrak{m}}(E)_{a-c} = 0$. Therefore, we get $a_{i+1}(E) + c \leq a_i(E)$. Taking the maxima over i of the inequalities

$$a_{i+1}(E) + c \leq a_i(E/fE) \leq \max\{a_i(E), a_{i+1}(E) + c\}$$

we will get (i). For (ii) we only need to take the maxima over i of the inequalities

$$a_{i+1}(E) + i + c \leq a_i(E/fE) + i \leq \max\{a_i(E) + i, a_{i+1}(E) + i + c\}.$$

□

3. Rees algebras of ideals generated by forms of the same degree

Let A be a standard graded algebra over a field. Let $I = (f_1, \ldots, f_s)$ be a homogeneous ideal in A such that $f_1, \ldots, f_m$ have the same degree. Then the Rees algebra $R = A[It]$ can be considered as a standard $\mathbb{N}$-graded algebra over k. Let M denote the maximal graded ideal of R.

Let $\mathfrak{m} = (\bar{x}_1, \dots, \bar{x}_n)$ be the maximal homogeneous ideal of A. We can refine the $\mathbb{N}$-graded structure of R by a bigrading with

$$\begin{aligned} \operatorname{bideg} \bar{x}_i &= (1,0),\ i = 1, \dots, r, \\ \operatorname{bideg} f_j t &= (1,1),\ j = 1, \dots, s. \end{aligned}$$

It is easy to verify that if $z \in R$ is a bihomogeneous element with $\operatorname{bideg} z = (a,b)$, then $\deg z = a$. There is also the natural bigrading $\operatorname{bideg} \bar{x}_i = (1,0)$ and $\operatorname{bideg} f_j t = (0,1)$. But we shall see that the first bigrading is more suitable for our investigation.

Let E be any bigraded R-module. Then E has the natural $\mathbb{Z}$-graded structure $E_a = \bigoplus_{b \in \mathbb{Z}} E_{(a,b)}$. In particular, $H^i_M(E)$ is both a bigraded R-module and a $\mathbb{Z}$-graded R-module with

$$H^i_M(E)_a = \bigoplus_{b \in \mathbb{Z}} H^i_M(E)_{(a,b)}.$$

Let R_+ denote the ideal of R generated by the elements $f_j t$. Let $G = \oplus_{n \geq 0} I^n/I^{n+1}$ be the associated graded ring of I. To estimate $a_i(R)$ we consider the following two short exact sequences of bigraded R-modules:

$$\begin{gathered} 0 \longrightarrow R_+ \longrightarrow R \longrightarrow A \longrightarrow 0, \\ 0 \longrightarrow R_+(0,1) \longrightarrow R \longrightarrow G \longrightarrow 0. \end{gathered}$$

¿From the above short exact sequences we obtain the following long exact sequences of bigraded local cohomology modules:

$$\cdots \longrightarrow H^{i-1}_M(A)_{(a,b)} \longrightarrow H^i_M(R_+)_{(a,b)} \longrightarrow H^i_M(R)_{(a,b)} \longrightarrow H^i_M(A)_{(a,b)} \longrightarrow \cdots \quad (1)$$

$$\cdots \longrightarrow H^{i-1}_M(G)_{(a,b)} \longrightarrow H^i_M(R_+)_{(a,b+1)} \longrightarrow H^i_M(R)_{(a,b)} \longrightarrow H^i_M(G)_{(a,b)} \longrightarrow \cdots \quad (2)$$

These sequences allow us to study the vanishing of the bigraded local cohomology modules of R by means of those of A and G.

Lemma 3.1. *For a fixed integer a assume that there is an integer b_0 such that*
(i) $H^{i-1}_M(A)_{(a,b)} = 0$ *for* $b \geq b_0$,
(ii) $H^i_M(G)_{(a,b)} = 0$ *for* $b > b_0$.
Then $H^i_M(R)_{(a,b)} = 0$ *for* $b \geq b_0$.

Proof. The assumptions (i) and (ii) implies that for $b \geq b_0$,
(i') $H^i_M(R_+)_{(a,b)} \longrightarrow H^i_M(R)_{(a,b)}$ is injective,
(ii') $H^i_M(R_+)_{(a,b+1)} \longrightarrow H^i_M(R)_{(a,b)}$ is surjective.
Since $H^i_M(R)$ is an artinian R-module, $H^i_M(R)_{(a,b)} = 0$ for b large enough. Once we have $H^i_M(R)_{(a,b+1)} = 0$ for some integer $b \geq b_0$, we can use (i') and (ii') to deduce first that $H^i_M(R_+)_{(a,b+1)} = 0$ and then that $H^i_M(R)_{(a,b)} = 0$. This can be continued until $b = b_0$. $\square$

Proposition 3.2. $a_i(R) \leq \max\{a_{i-1}(A), a_i(G)\}$.

Proof. Fix an arbitrary integer $a > \max\{a_{i-1}(A), a_i(G)\}$. Since $a > a_{i-1}(A)$, $H_M^{i-1}(A)_a = 0$. Therefore,

$$H_M^{i-1}(A)_{(a,b)} = 0 \text{ for all } b.$$

Since $a > a_i(G)$, $H_M^i(G)_a = 0$. Therefore,

$$H_M^i(G)_{(a,b)} = 0 \text{ for all } b.$$

By Lemma 3.1 we obtain $H_M^i(R)_{(a,b)} = 0$ for all b. That implies $H_M^i(R)_a = 0$ or, equivalently, $a_i(R) \le \max\{a_{i-1}(A), a_i(G)\}$. □

On the other hand, there is the following relation between the maximal shifts of the terms of the minimal free resolutions of A and R.

Proposition 3.3. *$b_i(R) \ge b_i(A)$.*

Proof. We consider a minimal free resolution $\mathbb{F} : 0 \longrightarrow F_l \longrightarrow \cdots \longrightarrow F_0 \longrightarrow R$ of R as a bigraded module over the polynomial ring $S := k[X_1, \ldots, X_n, Y_1, \ldots, Y_s]$. Let $\mathbb{F}^*$ denote the exact sequence:

$$0 \longrightarrow \bigoplus_{a\in\mathbb{Z}} (F_l)_{(a,0)} \longrightarrow \cdots \longrightarrow \bigoplus_{a\in\mathbb{Z}} (F_0)_{(a,0)} \longrightarrow \bigoplus_{a\in\mathbb{Z}} R_{(a,0)} = A$$

It is clear that $\mathbb{F}^*$ is a free resolution of A as a graded module over the polynomial ring $B = k[X_1, \ldots, X_n]$.

To estimate the shifts of the twisted free modules of $\mathbb{F}^*$ we consider a twisted free S-module $S(-c, -d)$. Then $(S(-c, -d))_{(a,0)} = S_{(a-c,-d)}$ is a direct sum of $\binom{s-d-1}{s-1}$ copies of $B_{a-c+d} = B(-(c-d))_a$, where we set $\binom{s-d-1}{s-1} = 0$ if $d > 0$. Therefore, $\oplus_a(S(-c, -d))_{(a,0)}$ is the direct sum of $\binom{s-d-1}{s-1}$ copies of $B(-(c-d))$. If $S(-c, -d)$ runs over all twisted free modules of F_i, then $b_i(R) = \max\{c\} \ge \max\{c - d\}$. Since $\max\{c-d\}$ is the maximum shift of the ith term of $\mathbb{F}^*$, we have $\max\{c-d\} \ge b_i(A)$. So we obtain $b_i(R) \ge b_i(A)$. □

¿From the above propositions we can easily derive upper and lower bounds for $a^*(R)$ and $\text{reg}(R)$ in terms of A and G.

Theorem 3.4. *Let s denote the minimal number generators of I. Then*

$$a^*(A) - s \le a^*(R) \le \max\{a^*(A), a^*(G)\}.$$

Proof. By definition we have $a^*(E) = \max\{a_i(E)|\ i \ge 0\}$ for any finitely generated graded R-module E. Therefore, from Proposition 3.2 we immediately obtain the upper bound $a^*(R) \le \max\{a^*(A), a^*(G)\}$. On the other hand, by Corollary 2.2 we have

$$\begin{aligned} a^*(A) &= \max\{b_i(A)|\ i \ge 0\} - n, \\ a^*(R) &= \max\{b_i(R)|\ i \ge 0\} - n - s. \end{aligned}$$

Therefore, from Proposition 3.3 we can immediately deduce the lower bound $a^*(A) - s \le a^*(R)$. □

Theorem 3.5. $\text{reg}(A) \le \text{reg}(R) \le \max\{\text{reg}(A) + 1, \text{reg}(G)\}$

Proof. By Corollary 2.2 we have $\operatorname{reg}(E) = \max\{a_i(E) + i|\ i \geq 0\}$ for any finitely generated graded R-module E. By Proposition 3.2, $a_i(R) + i \leq \max\{a_{i-1}(A) + i, a_i(G) + i\}$. Hence we get the upper bound $\operatorname{reg}(R) \leq \max\{\operatorname{reg}(A) + 1, \operatorname{reg}(G)\}$. On the other hand, using Proposition 3.3 we obtain the lower bound

$$\begin{aligned} \operatorname{reg}(A) &= \max\{b_i(A) + i|\ i \geq 0\} \\ &\leq \max\{b_i(R) + i|\ i \geq 0\} = \operatorname{reg}(R). \end{aligned}$$

□

Corollary 3.6. *Let I be an ideal generated by a regular sequence of s forms of degree c. Then*
(i) $a^*(A) - s \leq a^*(R) \leq a^*(A) + s(c-1)$,
(ii) $\operatorname{reg}(A) \leq \operatorname{reg}(R) \leq \max\{\operatorname{reg}(A) + 1, \operatorname{reg}(A) + s(c-1)\}$.

Proof. We have $G \cong (A/I)[z_1, \ldots, z_s]$, where $z_1, \ldots, z_s$ are indeterminates. By Proposition 2.5, this implies $a^*(G) = a^*(A/I) - s$ and $\operatorname{reg}(G) = \operatorname{reg}(A/I)$. On the other hand, Proposition 2.6 gives $a^*(A/I) = a^*(A) + sc$ and $\operatorname{reg}(A/I) = \operatorname{reg}(A) + s(c-1)$. Therefore, $a^*(G) = a^*(A) + s(c-1)$ and $\operatorname{reg}(G) = \operatorname{reg}(A) + s(c-1)$. Hence the conclusion follows from Theorem 3.4 and Theorem 3.5. □

The following example shows that the above upper and lower bounds for $a^*(R)$ and $\operatorname{reg}(R)$ are sharp.

Example. Let $A = k[x_1, \ldots, x_n]$ and $I = (x_1, \ldots, x_n)$. By Proposition 2.5 we have $a^*(A) = -n$ and $\operatorname{reg}(A) = 0$. In the next sections we shall see that

$$\begin{aligned} a^*(R) &= \begin{cases} -2 & \text{for} \quad n = 1, \\ -n & \text{for} \quad n > 1. \end{cases} \\ \operatorname{reg}(R) &= \begin{cases} 0 & \text{for} \quad n = 1, \\ 1 & \text{for} \quad n > 1. \end{cases} \end{aligned}$$

4. Local cohomology of Rees algebras of maximal graded ideals

Let A be a standard graded algebra over a field k. Let $\mathfrak{m}$ be the maximal graded ideal of A. From now on, R will denote the Rees algebra $A[\mathfrak{m}t]$.

We first note that R has a bigraded automorphism ψ induced by the map $\psi(x) = xt$ and $\psi(xt) = x$ for any element $x \in \mathfrak{m}$. It is clear that if $f \in R$ is a bihomogeneous element with $\operatorname{bideg} f = (a, b)$, then $\operatorname{bideg} \psi(f) = (a, a-b)$. In particular, ψ induces an isomorphism between A and G as R-modules.

Since A concentrates only in degree of the form $(a, 0)$, we have

$$H^i_M(A)_{(a,b)} = \begin{cases} 0 & \text{for} \quad b \neq 0, \\ H^i_{\mathfrak{m}}(A)_a & \text{for} \quad b = 0. \end{cases}$$

¿From this it follows that

$$H^i_M(G)_{(a,b)} = \begin{cases} 0 & \text{for} \quad b \neq a, \\ H^i_{\mathfrak{m}}(A)_a & \text{for} \quad b = a. \end{cases}$$

Therefore, using (1) and (2) we obtain

$$H^i_M(R_+)_{(a,b)} \cong H^i_M(R)_{(a,b)} \text{ for } b \neq 0, \tag{3}$$

$$H^i_M(R_+)_{(a,b+1)} \cong H^i_M(R)_{(a,b)} \text{ for } b \neq a. \tag{4}$$

The following lemma gives a complete description of the bigraded local cohomology modules of R in terms of those of A.

Lemma 4.1. *For any integer a we have*

$$H^i_M(R)_{(a,b)} = \begin{cases} 0 & \text{if } \; b \geq \max\{0, a+1\}, \\ H^i_{\mathfrak{m}}(A)_a & \text{if } \; 0 \leq b < \max\{0, a+1\}, \\ H^{i-1}_{\mathfrak{m}}(A)_a & \text{if } \; \min\{0, a+1\} \leq b < 0, \\ 0 & \text{if } \; b < \min\{0, a+1\}. \end{cases}$$

Proof. By (3) and (4), $H^{i-1}_M(A)_{(a,b)} = 0$ for $b > 0$ and $H^i_M(G)_{(a,b)} = 0$ for $b \geq a+1$. Therefore, using Lemma 3.1 we get $H^i_M(R)_{(a,b)} = 0$ for $b \geq \max\{0, a+1\}$. By the above automorphism of R, this implies $H^i_M(R)_{(a,b)} = 0$ for $a - b \geq \max\{0, a+1\}$ or, equivalently, for $b < \min\{0, a+1\}$.

To prove that $H^i_M(R)_{(a,b)} = H^i_{\mathfrak{m}}(A)_a$ for $0 \leq b < \max\{0, a+1\}$ we may assume that $a+1 > 0$. By (2) we have the exact sequence

$$H^i_M(R_+)_{(a,a+1)} \longrightarrow H^i_M(R)_{(a,a)} \longrightarrow H^i_M(G)_{(a,a)} \longrightarrow H^{i+1}_M(R_+)_{(a,a+1)}.$$

By (5) we have $H^j_M(R_+)_{(a,a+1)} \cong H^j_M(R)_{(a,a+1)} = 0$ for all j. Therefore,

$$H^i_M(R)_{(a,a)} \cong H^i_M(G)_{(a,a)} = H^i_{\mathfrak{m}}(A)_a.$$

Using (5) and (6) we obtain $H^i_M(R)_{(a,b)} \cong H^i_M(R)_{(a,b+1)}$ for $0 \leq b < a$. Thus, $H^i_M(R)_{(a,b)} \cong H^i_{\mathfrak{m}}(A)_a$ for $0 \leq b < a+1 = \max\{0, a+1\}$.

To prove that $H^i_M(R)_{(a,b)} = H^{i-1}_{\mathfrak{m}}(A)_a$ for $\min\{0, a+1\} \leq b < 0$ we may assume that $a+1 < 0$. By (1) we have the exact sequence

$$H^{i-1}_M(R)_{(a,0)} \longrightarrow H^{i-1}_M(A)_{(a,0)} \longrightarrow H^i_M(R_+)_{(a,0)} \longrightarrow H^i_M(R)_{(a,0)}.$$

Since $a+1 < 0$, we have $H^j_M(R)_{(a,0)} = 0$ for all j. Therefore,

$$H^i_M(R_+)_{(a,0)} \cong H^{i-1}_M(A)_{(a,0)} = H^{i-1}_{\mathfrak{m}}(A)_a.$$

By (2) we have the exact sequence

$$H^{i-1}_M(G)_{(a,-1)} \longrightarrow H^i_M(R_+)_{(a,0)} \longrightarrow H^i_M(R)_{(a,-1)} \longrightarrow H^i_M(G)_{(a,-1)}.$$

By (4) we have $H^j_M(G)_{(a,-1)} = 0$ for all j. Therefore,

$$H^i_M(R)_{(a,-1)} \cong H^i_M(R_+)_{(a,0)} \cong H^{i-1}_{\mathfrak{m}}(A)_a.$$

Using (5) and (6) we obtain $H^i_M(R)_{(a,b)} \cong H^i_M(R)_{(a,b+1)}$ for $a+1 \leq b < -1$. Thus, $H^i_M(R)_{(a,b)} \cong H^{i-1}_{\mathfrak{m}}(A)_a$ for $\min\{0, a+1\} = a+1 \leq b \leq -1$. $\square$

The above lemma on the bigraded local cohomology modules of R can be formulated for the $\mathbb{N}$-graded structure as follows.

Theorem 4.2.
$$H^i_M(R)_a = \begin{cases} \bigoplus_{a+1\text{ copies}} H^i_{\mathfrak{m}}(A)_a & \text{if} \quad a \geq 0, \\ 0 & \text{if} \quad a = -1, \\ \bigoplus_{-(a+1)\text{ copies}} H^{i-1}_{\mathfrak{m}}(A)_a & \text{if} \quad a \leq -2. \end{cases}$$

Proof. The statement follows from the formula
$$H^i_M(R)_a = \bigoplus_{b\in\mathbb{Z}} H^i_M(R)_{(a,b)}$$
and Lemma 4.1. Indeed, if $a \geq 0$, $\max\{0, a+1\} = a+1$ and $\min\{0, a+1\} = 0$. Therefore,
$$H^i_M(R)_{(a,b)} = \begin{cases} 0 & \text{for} \quad b \geq a+1, \\ H^i_{\mathfrak{m}}(A)_a & \text{for} \quad 0 \leq b \leq a, \\ 0 & \text{for} \quad b < 0. \end{cases}$$
If $a = -1$, $\max\{0, a+1\} = \min\{0, a+1\} = 0$. Hence $H^i_M(R)_{(-1,b)} = 0$ for all b. If $a \leq -2$, we have
$$H^i_M(R)_{(a,b)} = \begin{cases} 0 & \text{for} \quad b \geq 0, \\ H^{i-1}_{\mathfrak{m}}(A)_a & \text{for} \quad a+1 \leq b < 0, \\ 0 & \text{for} \quad b \leq a. \end{cases}$$
□

In the following we will denote $a_j(A)$ by a_j for all j. An immediate consequence of Theorem 4.2 is the following formula for the depth of the Rees algebra (see [HM] for the depth of the Rees algebra of an arbitrary ideal).

Proposition 4.3.
$$\operatorname{depth} R = \max\{i|\ H^j_{\mathfrak{m}}(A)_a = 0 \text{ for } a \neq -1, j < i-1, \text{ and } a_{i-1} < 0\}.$$
In particular, $\operatorname{depth} R \geq \operatorname{depth} A$.

Proof. We have $H^i_M(R) = 0$ if and only if $H^{i-1}_{\mathfrak{m}}(A)_a = 0$ for $a \leq -2$ and $H^i_{\mathfrak{m}}(A)_a = 0$ for $a \geq 0$. Putting this in the formula
$$\operatorname{depth} R = \max\{i|\ H^j_M(R) = 0 \text{ for } j < i\}.$$
we obtain the conclusion. □

If $\operatorname{depth} A = 0$, we must have $\operatorname{depth} R = 0$. If $\operatorname{depth} A = 1$, we can not have $H^1_{\mathfrak{m}}(A)_a = 0$ for $a \neq -1$. For, we have $H^1_{\mathfrak{m}}(A)_{-1} \cong H^0_{\mathfrak{m}}(A/xA)_0 = 0$, where x is a regular linear form of A. Therefore, Proposition 4.3 gives $\operatorname{depth} R = 1$ if $\operatorname{depth} A = 1$. If $\operatorname{depth} A \geq 2$, $\operatorname{depth} R$ can be arbitrarily large than $\operatorname{depth} A$.

Example. Let $2 \leq r < d$ be arbitrary positive numbers. We will construct a graded algebra A with $\operatorname{depth} A = r$ and $\operatorname{depth} R = d+1$ (i.e. R is a Cohen-Macaulay ring).

Let $T = k[x_1, \ldots, x_d]$ and $\mathfrak{n}$ the maximal graded ideal of T. Let E be the rth syzygy module of k over T. Then $H^i_{\mathfrak{n}}(E) = 0$ for $i \neq r, d$,
$$H^r_{\mathfrak{n}}(E)_a = \begin{cases} 0 & \text{for} \quad a \neq 0, \\ k & \text{for} \quad a = 0, \end{cases}$$
and $H^d_{\mathfrak{n}}(E) = H^d_{\mathfrak{n}}(T)$.

Let C be the idealization of the graded T-module $E(r-1)$ (see e.g. [N, p.2]). Since E is generated by elements of degree r, $E(r-1)$ is generated by elements of degree 1. Hence C is a standard graded algebra over k. By the construction of the idealization we have a natural exact sequence of the form $0 \to E(r-1) \to C \to T \longrightarrow 0$, where all homomorphisms have degree 0. Let $\mathfrak{m}_C$ denote the maximal graded ideal of C. Then $H^i_{\mathfrak{m}_C}(C) = H^i_{\mathfrak{n}}(E(r-1))$ for $i \neq d$ and there is the exact sequence $H^d_{\mathfrak{n}}(E(r-1)) \longrightarrow H^d_{\mathfrak{m}_C}(C) \longrightarrow H^d_{\mathfrak{n}}(T) \longrightarrow 0$. From this it follows that $H^i_{\mathfrak{m}_C}(C) = 0$ for $i \neq r, d$,

$$H^r_{\mathfrak{m}_C}(C)_a = \begin{cases} 0 & \text{for} \quad a \neq 1-r, \\ k & \text{for} \quad a = 1-r, \end{cases}$$

and $a_d(C) = a_d(T) = -d$.

Now let A be the $(r-1)$th Veronese subring of C. Then $H^i_{\mathfrak{m}}(A)_a = H^i_{\mathfrak{m}_C}(C)_{a(r-1)}$ [GW, Theorem 3.1.1]. Hence $H^i_{\mathfrak{m}}(A) = 0$ for $i \neq r, d$,

$$H^r_{\mathfrak{m}}(A)_a = \begin{cases} 0 & \text{for} \quad a \neq -1, \\ k & \text{for} \quad a = -1, \end{cases}$$

and $a_d \leq -1$. Therefore, $\operatorname{depth} A = r$ and $\operatorname{depth} R = d+1$ by Proposition 4.3.

Remark. The above example is inspired by Goto's construction of Buchsbaum local rings of minimal multiplicity with local cohomology modules of given lengths [G, Example (4.11)(2)]. Evans and Griffith [EvG] have constructed graded domains A whose local cohomology modules $H^i_{\mathfrak{m}}(A)$, $i < d$, are isomorphic to given graded modules of finite length with a shifting. Since the shift could not be computed explicitly, we can not use their construction for our purpose.

Despite the eventually big difference between the depths of a given ring and its Rees algebra, the depth of the Rees algebra of a polynomial extension $A[z]$ of A is rather rigid.

Corollary 4.4. *Let R^* denote the Rees algebra of a polynomial ring $A[z]$ over A in one variable. Put $s = \operatorname{depth} A$. Then*

$$\operatorname{depth} R^* = \begin{cases} s+1 & \text{if} \quad a_s \geq 0, \\ s+2 & \text{if} \quad a_s < 0. \end{cases}$$

Proof. By Proposition 2.4, $H^i_{\mathfrak{n}}(A[z]) = 0$ if $H^{i-1}_{\mathfrak{m}}(A) = 0$ and $H^i_{\mathfrak{n}}(A[z])$ is a module of infinite length with $a_i(A[z]) = a_{i-1} - 1$ if $H^{i-1}_{\mathfrak{m}}(A) \neq 0$. As a consequence, $H^i_{\mathfrak{n}}(A[z])_a = 0$ for $a \neq -1$ if and only if $H^{i-1}_{\mathfrak{m}}(A) = 0$. Therefore,

$$\begin{aligned} \operatorname{depth} R^* &= \max\{i \mid H^j_{\mathfrak{n}}(A[z])_a = 0 \text{ for } a \neq -1, j < i-1, \text{ and } a_{i-1} < 0\} \\ &= \max\{i \mid H^j_{\mathfrak{m}}(A) = 0 \text{ for } j < i-2, \text{ and } a_{i-2} < 0\} \end{aligned}$$

Since $s = \max\{i \mid H^j_{\mathfrak{m}}(A) = 0 \text{ for } j < i\}$, we get

$$\max\{i \mid H^j_{\mathfrak{m}}(A) = 0 \text{ for } j < i-2, \text{ and } a_{i-2} < 0\} = \begin{cases} s+1 & \text{if} \quad a_s \geq 0, \\ s+2 & \text{if} \quad a_s < 0. \end{cases}$$

$\square$

The following criterion for the Cohen-Macaulayness of R can be also derived from a more general criterion for the Cohen-Macaulayness of the Rees algebra of an arbitrary ideal of Trung and Ikeda [TI].

Corollary 4.5. *R is a Cohen-Macaulay ring if and only if $H^j_{\mathfrak{m}}(A)_a = 0$ for $a \neq -1$, $j < d$, and $a_d < 0$.*

Proof. This follows from Proposition 4.3 (the case depth $R = d+1$). □

Another consequence of Theorem 4.2 is the following formula for $a_i(R)$. This formula is crucial for the estimation of $a^*(R)$ and $\operatorname{reg}(R)$ in the next section.

Proposition 4.6.

$$a_i(R) = \begin{cases} a_i & \text{if} \quad a_i \geq 0, \\ \max\{a \mid a \leq -2 \text{ and } H^{i-1}_{\mathfrak{m}}(A)_a \neq 0\} & \text{if} \quad a_i < 0. \end{cases}$$

In particular, $a_i(R) = a_{i-1}$ if $a_{i-1} \leq -2$ and $a_i < 0$.

Proof. Note that $H^j_{\mathfrak{m}}(A)_a = 0$ for $a > a_j$ and $H^i_{\mathfrak{m}}(A)_{a_j} \neq 0$ if $a_j \neq -\infty$. If $a_i \geq 0$, using Theorem 4.2 we get $H^i_M(R)_a = 0$ for $a > a_i$ and $H^i_M(R)_{a_i} \neq 0$, hence $a_i(R) = a_i$. If $a_i < 0$, we get $H^i_M(A)_a = 0$ for $a \geq -1$. For $a \leq -2$, $H^i_M(R)_a \neq 0$ if and only if $H^{i-1}_{\mathfrak{m}}(A)_a \neq 0$. Therefore, $a_i(R) = \max\{a \mid a \leq -2 \text{ and } H^{i-1}_{\mathfrak{m}}(A)_a \neq 0\}$, which is exactly a_{i-1} if $a_{i-1} \leq -2$. □

From Proposition 4.6 we immediately obtain the following bounds for $a_i(R)$.

Corollary 4.7. (i) $a_i(R) \leq \max\{a_{i-1}, a_i\}$ *for $i \leq d$,*
(ii) $a_{d+1}(R) \leq \min\{-2, a_d\}$.

Example. Let $A = k[x_1, \ldots, x_n]$. We know that $a_i = -\infty$ for $i \neq n$ and $a_n = -n$ with $H^n_{\mathfrak{m}}(A)_a \neq 0$ for $a \leq -n$. Therefore, $a_i(R) = -\infty$ for $i \neq n+1$ and

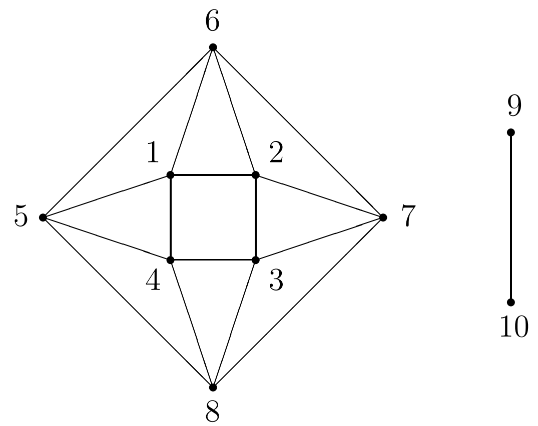
$$a_{n+1}(R) = \begin{cases} -2 & \text{if} \quad n = 1, \\ -n & \text{if} \quad n > 1. \end{cases}$$

One may expect that $a_i(R) = -2$ if $a_{i-1} > -2$ and $a_i < 0$. But the following example shows that is not always the case.

Example. Let Δ be the simplicial complex on ten vertices $\{1, \ldots, 10\}$ with the maximal faces

$$\{1,2,6\}, \{2,6,7\}, \{2,3,7\}, \{3,7,8\}, \{3,4,8\}, \{4,5,8\}, \{1,4,5\}, \{9,10\}.$$

Note that Δ is topologically the disjoint union of a circle and a point.



Let $T = k[x_1, \ldots, x_{10}]$ and $\mathfrak{n}$ the maximal graded ideal of T. Let I be the monomial ideal of Δ in T:

$$\begin{aligned} I \;=\; & (x_2x_{10}, x_2x_9, x_2x_8, x_2x_5, x_2x_4, x_3x_{10}, x_3x_9, x_3x_5, \\ & x_1x_3, x_5x_{10}, x_4x_{10}, x_1x_{10}, x_5x_9, x_4x_9, x_1x_9, x_1x_8, x_8x_9, x_8x_{10}, \\ & x_1x_7, x_4x_7, x_5x_7, x_7x_9, x_7x_{10}, x_3x_6, x_4x_6, x_5x_6, x_6x_8, x_6x_9, x_6x_{10}). \end{aligned}$$

By Hochster's formula for the local cohomology modules of the Stanley-Reisner ring $k[\Delta] = T/I$ (see e.g. [BH, Theorem 5.8]) we have $H^0_{\mathfrak{n}}(k[\Delta]) = 0$,

$$H^1_{\mathfrak{n}}(k[\Delta])_a = \begin{cases} 0 & \text{for} \quad a \neq 0, \\ k & \text{for} \quad a = 0, \end{cases}$$

$$H^2_{\mathfrak{n}}(k[\Delta])_a = \begin{cases} 0 & \text{for} \quad a > 0 \text{ and } a = -1, \\ k & \text{for} \quad a = 0 \text{ and } a \leq -2, \end{cases}$$

and $H^3_{\mathfrak{n}}(k[\Delta])_a = 0$ for $a > -2$, while $H^3_{\mathfrak{n}}(k[\Delta])_a \neq 0$ for $a \leq -2$. From the short exact sequence $0 \longrightarrow I \longrightarrow T \longrightarrow k[\Delta] \longrightarrow 0$ we get $H^i_{\mathfrak{n}}(I) = 0$ for $i \neq 2, 3, 4, 10$,

$$H^2_{\mathfrak{n}}(I)_a = \begin{cases} 0 & \text{for} \quad a \neq 0, \\ k & \text{for} \quad a = 0, \end{cases}$$

$$H^3_{\mathfrak{n}}(I)_a = \begin{cases} 0 & \text{for} \quad a > 0 \text{ and } a = -1, \\ k & \text{for} \quad a = 0 \text{ and } a \leq -2, \end{cases}$$

and $H^4_{\mathfrak{n}}(I)_a = 0$ for $a > -2$, $H^4_{\mathfrak{n}}(I)_a \neq 0$ for $a \leq -2$, $H^{10}_{\mathfrak{n}}(I)_a = 0$ for $a > -10$, $H^{10}_{\mathfrak{n}}(I)_a \neq 0$ for $a \leq -10$.

Let A be the idealization of the graded T-module $I(1)$ (see e.g. [N, p.2]). Since $I(1)$ is generated by elements of degree 1, A is a standard graded algebra over k. By the construction of the idealization we have a natural exact sequence of the form $0 \longrightarrow I(1) \longrightarrow A \longrightarrow T \longrightarrow 0$, where all homomorphisms are of degree 0. This exact sequence yields $H^i_{\mathfrak{m}}(A) = H^i_{\mathfrak{n}}(I(1))$ for $i \neq 10$ and the exact sequence $H^d_{\mathfrak{n}}(I(1)) \longrightarrow H^d_{\mathfrak{m}}(A) \longrightarrow H^d_{\mathfrak{n}}(T) \longrightarrow 0$. Using the above formula for $H^i_{\mathfrak{n}}(I)$ we get $H^i_{\mathfrak{m}}(A) = 0$ for $i \neq 2, 3, 4, 10$,

$$H^2_{\mathfrak{m}}(A)_a = \begin{cases} 0 & \text{for} \quad a \neq -1, \\ k & \text{for} \quad a = -1, \end{cases}$$

$$H^3_{\mathfrak{n}}(A)_a = \begin{cases} 0 & \text{for} \quad a > -1 \text{ and } a = -2, \\ k & \text{for} \quad a = -1 \text{ and } a \le -3, \end{cases}$$

and $H^4_{\mathfrak{m}}(A)_a = 0$ for $a > -3$, $H^4_{\mathfrak{m}}(A)_a \neq 0$ for $a \le -3$, $H^{10}_{\mathfrak{m}}(A)_a = 0$ for $a > -10$, $H^{10}_{\mathfrak{n}}(A)_{-10} \neq 0$. In particular, $a_i = -\infty$ for $i \neq 2, 3, 4, 10$, $a_2 = a_3 = -1$, $a_4 = -3$, $a_{10} = -10$. Applying Proposition 4.6 we obtain $a_i(R) = -\infty$ for $i \neq 4, 5, 11$ and $a_4(R) = a_5(R) = -3$, $a_{11}(R) = -10$. In particular, $a_4(R) = -3$ though $a_3 = -1$ and $a_4 = -3 < 0$.

5. Regularity of Rees algebras of maximal graded ideals

As in the last section, let $R = A[\mathfrak{m}t]$ be the Rees algebra of the maximal graded ideal $\mathfrak{m}$ of a standard graded algebra A over a field. The goal of this section is to estimate $a^*(R)$ and $\operatorname{reg}(R)$ in terms of $a^*(A)$ and $\operatorname{reg}(A)$.

Let n be the embedding dimension of A, that is, $n = \dim_k A_1$. We can consider A as a module over the polynomial ring $B = k[x_1, \ldots, x_n]$ and R as a module over the polynomial ring $S = k[x_1, \ldots, x_n, y_1, \ldots, y_n]$.

We have the following relationships between the invariants $a^*_j(A)$ and $a^*_j(R)$.

Proposition 5.1. *For any integer $j \ge 0$ we have*
(i) $a^*_{j-n}(A) - n \le a^*_j(R) \le a^*_j(A)$.
(ii) $a^*_j(R) = a^*_j(A)$ *if and only if* $a^*_j(A) \ge 0$ *or* $a^*_{j-1}(A) = a^*_j(A) \le -2$.

Proof. By Theorem 2.1(i) and Proposition 3.3 we have

$$a^*_j(R) = b^*_{2n-j}(R) - 2n \ge b^*_{2n-j}(A) - 2n = a^*_{j-n}(A) - n.$$

Since $G \cong A$, Theorem 3.5 implies $a_i(R) \le \max\{a_{i-1}(A), a_i(A)\}$ for all i. From this it follows that $a^*_j(R) \le a^*_j(A)$. So we obtain (i).

To prove (ii) we choose $i \le j$ such that $a^*_j(A) = a_i$.

If $a^*_j(A) \ge 0$, then $a_i(R) = a_i$ by Corollary 4.6. Hence $a^*_j(R) \ge a_i(R) = a^*_j(A)$. By (i) this implies $a^*_j(R) = a^*_j(A)$.

If $a^*_j(A) = -1$, $a_i(R) \le -2$ for all $i \le j$ by Proposition 4.6. Hence $a^*_j(R) \le -2 < a^*_j(A)$.

If $a^*_{j-1}(A) = a^*_j(A) \le -2$, we may assume that $i < j$. Then $a_{i+1} \le a_i \le -2$. Hence $a_{i+1}(R) = a_i$ by Proposition 4.6. Thus, $a^*_j(R) \ge a_{i+1}(R) = a^*_j(A)$. By (i) this implies $a^*_j(R) = a^*_j(A)$.

If $a^*_{j-1}(A) < a^*_j(A) \le -2$, we have $a_{j-1} < a_j = a^*_j(A) \le -2$. By Proposition 4.6, this implies $a_j(R) = a_{j-1} < a^*_j(A)$. By (i), $a^*_{j-1}(R) \le a^*_{j-1}(A)$. Therefore, $a^*_j(R) = \max\{a^*_{j-1}(R), a_j(R)\} < a^*_j(A)$. So we have proved (ii). ∎

Proposition 5.1 can be formulated in terms of the maximal shifts of the minimal free resolution of A as follows.

Corollary 5.2. *For any integer $j \ge 0$ we have*
(i) $b^*_j(A) \le b^*_j(R) \le b^*_{j-n}(A) + n$,
(ii) $b^*_j(R) = b^*_{j-n}(A) + n$ *if and only if* $b^*_{j-n}(A) \ge n$ *or* $b^*_{j-n+1}(A) = b^*_{j-n}(A) \le n - 2$.

Proof. By Theorem 2.1(i) we have

$$b_j^*(A) = a_{n-j}^*(A) + n,\ b_j^*(R) = a_{2n-j}^*(R) + 2n,\ b_{j-n}^*(A) = a_{2n-j}^*(A) + n.$$

Therefore, the conclusion follows from Proposition 5.1. $\square$

Theorem 5.3. (i) $a^*(A) - n \leq a^*(R) \leq a^*(A)$.
(ii) $a^*(R) = a^*(A)$ *if and only if* $a^*(A) \neq -1$.

Proof. From Proposition 5.1(i) we immediately obtain (i). To prove (ii) we assume first that $a^*(A) \neq -1$. Choose j such that $a^*(A) = a_{j-1}^*(A) = a_j^*(A)$. Then $a_j^*(R) = a_j^*(A)$ by Proposition 5.1(ii). It follows that

$$a^*(R) \geq a_j^*(A) = a^*(A).$$

By (i) this implies $a^*(R) = a^*(A)$. Now assume that $a^*(A) = -1$. Then $a_j^*(A) \leq -1$ for all j. If $a_j^*(A) \leq -2$, we have $a_j^*(R) \leq a_j^*(A) \leq -2$ by Theorem 5.1(i). If $a_j^*(A) = -1$, then Theorem 5.1(ii) implies $a_j^*(R) < a_j^*(A)$. Thus, $a^*(R) \leq -2$. So we have proved that $a^*(R) = a^*(A)$ if and only if $a^*(A) \neq -1$. $\square$

Example. Let $A = k[x_1, \ldots, x_n]$. From the formula for $a_i(A)$ and $a_i(R)$ in the last section we get $a^*(A) = -n$ while

$$a^*(R) = \begin{cases} -2 & \text{for} \quad n = 1, \\ -n & \text{for} \quad n > 1. \end{cases}$$

If $a^*(A) = -1$, we may expect that $a^*(R) = -2$. But that is not always the case. For instance, in the last example of Section 4 we have $a^*(A) = -1$ and $a^*(R) = -3$.

We may also formulate Theorem 5.3 in terms of the maximal shifts of the minimal free resolution of A as follows.

Corollary 5.4. (i) $b^*(A) \leq b^*(R) \leq b^*(A) + n$.
(ii) $b^*(R) = b^*(A) + n$ *if and only if* $b^*(A) \neq n - 1$.

Proof. By Corollary 2.2(i) we have $b^*(A) = a^*(A) + n$ and $b^*(R) = a^*(R) + 2n$. Hence the conclusion follows from Theorem 5.3. $\square$

Now we study the relationships between the partial regularities of a given graded algebra A and its Rees algebra R.

Proposition 5.5. *For any integer* $j \geq 0$ *we have*
(i) $\operatorname{reg}_{j-n}(A) \leq \operatorname{reg}_j(R) \leq \operatorname{reg}_j(A) + 1$.
(ii) $\operatorname{reg}_j(R) = \operatorname{reg}_j(A) + 1$ *if and only if there is an integer* $i < j$ *such that* $\operatorname{reg}_j(A) = a_i + i$ *and* $a_i \leq -2$.

Proof. By Theorem 2.1(ii) and Proposition 3.3 we have

$$\operatorname{reg}_j(R) = (2n - j)\text{-}\operatorname{reg}(R) \geq (2n - j)\text{-}\operatorname{reg}(A) = \operatorname{reg}_{j-n}(A).$$

Since $G \cong A$, Theorem 3.5 implies $a_i(R) \leq \max\{a_{i-1}(A), a_i(A)\}$ for all i. From this it follows that $\operatorname{reg}_j(R) \leq \operatorname{reg}_j(A) + 1$. So we obtain (i).

To prove (ii) we assume first that $\operatorname{reg}_j(R) = \operatorname{reg}_j(A) + 1$. Let $i \le j$ be an integer such that $\operatorname{reg}_j(R) = a_i(R) + i$. Then $a_i(R) = \operatorname{reg}_j(A) - i + 1$. On the other hand, $a_i(R) \le \max\{a_{i-1}, a_i\}$ by Proposition 3.2. Since $a_{i-1} \le \operatorname{reg}_j(A) - i + 1$ and $a_i \le \operatorname{reg}_j(A) - i$, we must have $a_i(R) = a_{i-1}$. This implies $\operatorname{reg}_j(A) = a_{i-1} + i - 1$ and, by Proposition 4.6, $a_{i-1} \le -2$.

Conversely, assume that there is an integer $i < j$ such that $\operatorname{reg}_j(A) = a_i + i$ and $a_i \le -2$. Then $a_{i+1} + i + 1 \le a_i + i$. Therefore, $a_{i+1} \le a_i - 1 < 0$. By Corollary 4.6, we get $a_{i+1}(R) = a_i$. Hence

$$\operatorname{reg}_j(R) \ge a_{i+1}(R) + i + 1 = a_i + i + 1 = \operatorname{reg}_j(A) + 1.$$

By (i) this implies $\operatorname{reg}_j(R) = \operatorname{reg}_j(A)$. $\square$

We may formulate Proposition 5.5(i) for the partial regularity $j\text{-}\operatorname{reg}(R)$ of Bayer, Charalambous, and Popescu (see Section 1). But, unlike the estimation for $b_j^*(R)$, we are not able to express the condition of Proposition 5.5(ii) in terms of the maximal shifts of the minimal free resolution of A.

Corollary 5.6. *For any integer $j \ge 0$ we have*
(i) $j\text{-}\operatorname{reg}(A) \le j\text{-}\operatorname{reg}(R) \le (j-n)\text{-}\operatorname{reg}(A) + 1$.
(ii) $j\text{-}\operatorname{reg}(R) = (j-n)\text{-}\operatorname{reg}(A) + 1$ *if* $(j-n)\text{-}\operatorname{reg}(A) = b_i(A) - i$ *and* $b_i(A) \le n-2$ *for some index $i > j - n$ at which A has an extremal Betti number.*

Proof. By Theorem 2.1(ii) we have

$$j\text{-}\operatorname{reg}(A) = \operatorname{reg}_{n-j}(A),\ j\text{-}\operatorname{reg}(R) = \operatorname{reg}_{2n-j}(R),\ (j-n)\text{-}\operatorname{reg}(A) = \operatorname{reg}_{2n-j}(A).$$

Therefore, (i) follows from Proposition 5.5(i). For (ii) we have $b_i(A) = a_{n-i} + n$ by Corollary 2.3, hence $\operatorname{reg}_{2n-j}(A) = a_{n-i} + n - i$ and $a_{n-i} \le -2$. By Theorem 5.5(ii) this implies $\operatorname{reg}_{2n-j}(R) = \operatorname{reg}_{2n-j}(A) + 1$. Thus, $j\text{-}\operatorname{reg}(R) = (j-n)\text{-}\operatorname{reg}(A) + 1$. $\square$

The following result which is an immediate consequence of Proposition 5.5 gives precise information on the value of the regularity of the Rees algebra of the maximal graded ideal.

Theorem 5.7. (i) $\operatorname{reg}(A) \le \operatorname{reg}(R) \le \operatorname{reg}(A) + 1$.
(ii) $\operatorname{reg}(R) = \operatorname{reg}(A) + 1$ *if and only if there is an integer i such that* $\operatorname{reg}(A) = a_i + i$ *and* $a_i \le -2$.

Example. Let $A = k[x_1, \ldots, x_n]$. We have $\operatorname{reg}(A) = 0$ with $a_i = -\infty$ for $i < n$ and $a_n = -n$. Therefore,

$$\operatorname{reg}(R) = \begin{cases} 0 & \text{for } n = 1, \\ 1 & \text{for } n > 1. \end{cases}$$

Let $T = A[z_1, \ldots, z_s]$ be a polynomial ring over A. Let R_s denote the Rees algebra of T with respect to the maximal graded ideal. It is well known that $\operatorname{reg}(T) = \operatorname{reg}(A)$. However, the regularities of the Rees algebras R_s and R need not to be the same. In fact, it may happen that $\operatorname{reg}(R) = \operatorname{reg}(A)$ but $\operatorname{reg}(R_s) = \operatorname{reg}(A) + 1$.

Corollary 5.8. *Let $i = \max\{j |\ \operatorname{reg}(A) = a_j + j\}$. Put $c = \max\{0, a_i - 2\}$. Then*

$$\operatorname{reg}(R_s) = \begin{cases} \operatorname{reg}(A) & \textit{for} \quad s < c, \\ \operatorname{reg}(A) + 1 & \textit{for} \quad s \geq c. \end{cases}$$

Proof. By Proposition 2.5 we have $a_{j+s}(T) = a_j - s$ for all $j \geq 0$. Since $\operatorname{reg}(T) = \operatorname{reg}(A)$, we have $\operatorname{reg}(T) = a_{j+s}(T) + (j+s)$ if and only if $\operatorname{reg}(A) = a_j + j$. If $s < c$, then $a_j - s > a_i - c \geq -2$ for all j with $\operatorname{reg}(A) = a_j + j$. ¿From this it follows that $a_{j+s}(T) > -2$ for all j with $\operatorname{reg}(T) = a_{j+s}(T) + (j+s)$. If $s \geq c$, then $\operatorname{reg}(T) = a_{i+s}(T) + (i+s)$ with $a_{i+s}(T) = a_i - s \leq -2$. Now we only need to apply Theorem 5.7(ii) to get the conclusion. □

There is the following sufficient condition for the equality $\operatorname{reg}(R) = \operatorname{reg}(A) + 1$ in terms of the maximal shifts of the minimal free resolution of A.

Corollary 5.9. $\operatorname{reg}(R) = \operatorname{reg}(A) + 1$ *if* $\operatorname{reg}(A) = b_i(A) - i$ *and* $b_i(A) \leq n - 2$ *for some index i at which A has an extremal Betti number.*

Proof. By Corollary 2.3 we have $b_i(A) = a_{n-i} + n$, hence $\operatorname{reg}(A) = a_{n-i}(A) + n - i$ and $a_{n-i} \leq -2$. By Proposition 5.5 this implies $\operatorname{reg}(R) = \operatorname{reg}(A) + 1$. □

Corollary 5.9 is not a necessary condition for the equality $\operatorname{reg}(R) = \operatorname{reg}(A) + 1$.

Example. Let $A = k[\Delta]$, where Δ is the simplicial complex on 7 vertices $\{1, 2, 3, 4, 5, 6, 7\}$ with the maximal faces $\{1, 2, 3\}$, $\{4, 5, 6\}$ and $\{5, 6, 7\}$. Using Hochster's formula for the local cohomology modules of $k[\Delta]$ (see e.g. [BH, Theorem 5.8]) we get $H^i_{\mathfrak{m}}(A) = 0$ for $i \neq 1, 3$, $H^1_{\mathfrak{m}}(A)_a = 0$ for $a \neq 0$, $H^3_{\mathfrak{m}}(A)_a \neq 0$ for $a \leq -2$ and $H^3_{\mathfrak{m}}(A)_a = 0$ for $a > -2$. From this it follows that $a_i(A) = -\infty$ for $i \neq 1, 3$, $a_1(A) = 0$ and $a_3(A) = -2$. Therefore, $\operatorname{reg}(A) = 1$ and, by Theorem 5.7, $\operatorname{reg}(R) = 2$. On the other hand, A has a 2-linear R-resolution since $\operatorname{reg}(A) = 1$. Hence the condition that $\operatorname{reg}(A) = b_i(A) - i$, and that A has an extremal Betti number at i is satisfied for $i = 6$. But we have $b_6(A) = 7 > n - 2 = 5$.

As applications, we will study the regularity of the Rees algebra $R_{\text{in}} := B[\mathfrak{m}_{\text{in}} t]$ of the algebra $A_{\text{in}} := B/(\text{in}(J))$, where $\text{in}(J)$ denotes the initial ideal of J with respect to an arbitrary term order and $\mathfrak{m}_{\text{in}}$ is the maximal graded ideal of A_{in}.

Proposition 5.10. (i) $a_i(R) \leq a_i(R_{\text{in}})$ *for all* $i \geq 0$,
(ii) $a^*(R) \leq a^*(R_{\text{in}})$,
(iii) $\operatorname{reg}(R) \leq \operatorname{reg}(R_{\text{in}})$.

Proof. We only need to prove (i). Note that $a_i \leq a_i(A_{\text{in}})$ by [Sb, Theorem 3.3]. If $a_i \geq 0$, then $a_i(A_{\text{in}}) \geq 0$. Applying Proposition 4.6 we obtain

$$a_i(R) = a_i \leq a_i(A_{\text{in}}) = a_i(R_{\text{in}}).$$

If $a_i < 0$ and $a_i(A_{\text{in}}) < 0$, then

$$\begin{aligned} a_i(R) &= \max\{a|\ a \leq -2 \text{ and } H^{i-1}_{\mathfrak{m}}(A)_a \neq 0\} \\ &\leq \max\{a|\ a \leq -2 \text{ and } H^{i-1}_{\mathfrak{m}}(A_{\text{in}})_a \neq 0\} \ = a_i(R_{\text{in}}). \end{aligned}$$

If $a_i < 0$ and $a_i(A_{\rm in}) \geq 0$, we apply Proposition 4.6 again to see that

$$a_i(R) \leq -2 < a_i(A_{\rm in}) = a_i(R_{\rm in}).$$

$\square$

For the generic initial ideal $\operatorname{Gin}(J)$ of I with respect to the reverse lexicographic term order, we set $A_{\rm Gin} := B/\operatorname{Gin}(J)$ and $R_{\rm Gin} := B[\mathfrak{m}_{\rm Gin}t]$, where $\mathfrak{m}_{\rm Gin}$ is the maximal graded ideal of $A_{\rm Gin}$. The following results show that in this case, $a^*(R_{\rm Gin})$ and $\operatorname{reg}(R_{\rm Gin})$ share the same lower and upper bounds of $a^*(R)$ and $\operatorname{reg}(R)$ as in Theorem 5.3 and Theorem 5.7. In particular, $\operatorname{reg}(R_{\rm Gin})$ differs from $\operatorname{reg}(R)$ at most by 1.

Proposition 5.11. (i) *$a^*(A) - n \leq a^*(R_{\rm Gin}) \leq a^*(A)$,*
(ii) *$a^*(R_{\rm Gin}) = a^*(A)$ if and only if $a^*(A) \neq -1$.*

Proof. By [T3, Corollary 1.4] we have $a^*(B/\operatorname{Gin}(J)) = a^*(A)$. Therefore, the conclusion follows from Theorem 5.3. $\square$

Proposition 5.12. (i) *$\operatorname{reg}(A) \leq \operatorname{reg}(R_{\rm Gin}) \leq \operatorname{reg}(A) + 1$,*
(ii) *$\operatorname{reg}(R_{\rm Gin}) = \operatorname{reg}(A) + 1$ if there is an integer i such that $\operatorname{reg}(A) = a_i + i$ and $a_i(A) \leq -2$.*

Proof. By [T3, Corollary 1.4] we have $\operatorname{reg}(A_{\rm Gin}) = \operatorname{reg}(A)$. Therefore, (i) follows from Theorem 4.3(i). By Theorem 5.7(ii), the condition of (ii) implies that $\operatorname{reg}(R) = \operatorname{reg}(A)+1$. By Proposition 5.10, we have $\operatorname{reg}(R) \leq \operatorname{reg}(R_{\rm Gin}) \leq \operatorname{reg}(A)+1$. Therefore, $\operatorname{reg}(R_{\rm Gin}) = \operatorname{reg}(A) + 1$. $\square$

Remark. In spite of the above results one may ask whether $a^*(R) = a^*(R_{\rm Gin})$ and $\operatorname{reg}(R) = \operatorname{reg}(R_{\rm Gin})$ always hold. Unfortunately, we were unable to settle this question.

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