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Involutive Directions and New Involutive Divisions

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Abstract—In this paper, we propose the concept of involutive direction as a vector representation for the concept of involutive division proposed by Gerdt and his co-workers. With this representation, most of the properties of involutive divisions such as Noetherity, Artinity, and constructivity, can be greatly simplified. A new algorithm to compute the involutive completion is also given. Based on the vector representation, two new types of involutive divisions are found and proved to be Noetherian, Artinian, and constructive. These new divisions may lead to new methods of finding integrability conditions of partial differential equations and computing Gröbner bases of polynomial ideals. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords—Involutive divisions, Involutive directions, Gröbner bases, Nonmultiplier prolongation, Involutive completion.

1. INTRODUCTION

There are three classical approaches to determine the integrability conditions of partial differential equations (PDEs): Janet's theory [1], Thomas' theory [2], and the formal theory of differential equations [3].

Based on Riquier's theorem [4], Janet proposed the involutivity conditions for orthonomic systems and designed an algorithm for their completion [1]. By his approach, independent variables are separated into two parts: multiplicative and nonmultiplicative, and in order to find new integrability conditions one need only to prolong given equations along with nonmultiplicative variables. The Riquier-Janet theory was developed by Ritt into the characteristic set method [5]. Schwarz clarified the Riquier-Janet theory and used it to develop programs for dealing with determining equations of symmetries of PDEs [6–8].

Thomas introduced another separation method for dividing independent variables into multiplicative and nonmultiplicative ones, and generalized the Riquier-Janet theory to nonorthonomic algebraic PDEs [2]. Combining Thomas' completion method with Ritt's characteristic set approach, Wu proposed the well-order principle and zero decomposition algorithms for nonlinear

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algebraic differential polynomial systems [9,10]. Wu's method was developed and used to solve polynomial systems [9,11,12], to prove theorems in geometries [13–15], and to simplify partial differential equations [16,17].

The third method in the formal theory of differential equations [3] allows one to formulate the involutivity intrinsically, in a coordinates independent way.

In [18], Wu showed that the theory of Thomas can be modified to give a new method for computing the Gröbner basis for polynomial ideals, which is different from that of Buchberger [19]. In [20], Zharkov and Blinkov extended Pommaret's theory to compute the Gröbner basis under certain conditions. In [21–23], Gerdt and Blinkov extracted the common properties of the three classical theories to introduce a concept of the involutive division. Furthermore, they gave, for a general involutive division, algorithms on completion of polynomials and linear differential systems to an involutive base [24]. This involutive base is the Gröbner basis under certain conditions.

In order to reveal further essential properties and to find new involutive divisions, we introduce a new concept *involutive direction*, which is a vector representation of the involutive division. Using this representation, most of the properties of the involutive division such as Noetherity, Artinity, and constructivity, can be greatly simplified. Some new properties are also found. A new algorithm to compute the involutive completion is also given. Based on the vector representation, two new types of involutive divisions are found and proved to be Noetherian, Artinian, and constructive. These new divisions may lead to new methods of finding integrability conditions of PDEs and computing Gröbner bases of polynomial ideals.

The rest of this paper is arranged as follows. In Section 2, we give a vector representation of the involutive division. The basic properties such as Noetherity, Artinity, and constructivity, are discussed in Section 3. A proof of the equivalence between the involutive direction and the involutive division can be found in Section 4. In Section 5, we generalize the Thomas and Janet directions to two types of directions. In Section 6, we give conclusion remarks.

2. BASIC NOTIONS

For any real vector α, we denote by α_i the i^{th} component of α, and for any two n-dimensional vectors α and β, we call the vector $(\alpha_1\beta_1, \dots, \alpha_n\beta_n)$ Hadamard product of α and β, denoted by $\alpha \circ \beta$. In this paper, we mainly consider exponent vectors, i.e., vectors whose components are all nonnegative integers. The set of all such vectors is denoted by $\mathbf{N}^n$. Let $\Delta_n = \{\alpha \in \mathbf{N}^n \mid \alpha_i = 0$ or $1,\ i = 1, \dots, n\}$. We introduce the following concepts.

DEFINITION 2.1. *δ is said to be a direction on $\mathbf{N}^n$ if, for every finite nonempty subset $\Gamma$of $\mathbf{N}^n$, a map $\delta^\Gamma : \Gamma \to \Delta_n$ can be given.*

For example, δ^*, defined as: for every finite subset Γ of $\mathbf{N}^n$, $\delta^{*\Gamma}(\alpha) = (1, 1, \dots, 1)$, $\forall\, \alpha \in \Gamma$, is a direction on $\mathbf{N}^n$. Clearly, the image $\delta^{*\Gamma}(\alpha)$ of α is independent of the set Γ which α belongs to, such directions are said to be global.

DEFINITION 2.2. *A direction δ on $\mathbf{N}^n$ is called an involutive direction if the following conditions are satisfied.*

(i) *For $\alpha, \beta \in \Gamma$, $\mu, \nu \in \mathbf{N}^n$, if $\alpha + \mu \circ \delta^\Gamma(\alpha) = \beta + \nu \circ \delta^\Gamma(\beta)$, then either $\beta - \alpha$ and $\delta^\Gamma(\alpha) - \delta^\Gamma(\beta)$ are both nonnegative, or $\alpha - \beta$ and $\delta^\Gamma(\beta) - \delta^\Gamma(\alpha)$ are both nonnegative.*

(ii) *If $\alpha \in \Sigma \subseteq \Gamma$, then $\delta^\Sigma(\alpha) - \delta^\Gamma(\alpha)$ is nonnegative.*

It is easy to examine that δ^* is not involutive for $n \geq 2$. We now consider the three classical methods of variable separations [1–3]. In our terminology they are called the Thomas direction, Janet direction, and Pommaret direction, respectively, which correspond in turn to Thomas division, Janet division, and Pommaret division, respectively, in [21]. For convenience, we define an operator $\mathbf{b}$ on $\mathcal{F}_n$, the set of all finite nonempty subset of $\mathbf{N}^n$, as follows:

$$\mathbf{b} : \Gamma \mapsto \gamma, \qquad \forall\, \Gamma \in \mathcal{F}_n,$$

where $\gamma_i = \max\{\beta_i \mid \beta \in \Gamma\}$, $i = 1, 2, \ldots, n$. We call $\mathbf{b}$ the *upper bound operator*, and the set

$$B(\Gamma) = \{\beta \in \mathbf{N}^n \mid \beta_i \leq \mathbf{b}_i(\Gamma),\ i = 1, \ldots, n\}$$

the bound of Γ.

EXAMPLE 2.3. The Thomas direction can be defined as follows:

$$\forall\, \alpha \in \Gamma, \delta_i^\Gamma(\alpha) = \begin{cases} 1, & \text{if } \alpha_i = \mathbf{b}_i(\Gamma), \\ 0, & \text{otherwise,} \end{cases}$$

$i = 1, 2, \ldots, n$, where $\delta_i^\Gamma(\alpha)$ and $\mathbf{b}_i(\Gamma)$ represent the i^{th} components of $\delta^\Gamma(\alpha)$ and $\mathbf{b}(\Gamma)$, respectively.

EXAMPLE 2.4. The Janet direction can be defined as follows:

$$\forall\, \alpha \in \Gamma, \qquad \begin{array}{l} \delta_1^\Gamma(\alpha) = \begin{cases} 1, & \text{if } \alpha_1 = \mathbf{b}_1(\Gamma), \\ 0, & \text{otherwise,} \end{cases} \\ \delta_i^\Gamma(\alpha) = \begin{cases} 1, & \text{if } \alpha_i = \mathbf{b}_i\left(\Gamma_{\alpha_1 \ldots \alpha_{i-1}}\right), \\ 0, & \text{otherwise,} \end{cases} \end{array}$$

where $\Gamma_{\alpha_1 \ldots \alpha_{i-1}} = \{\beta \in \Gamma \mid \beta_j = \alpha_j,\ j = 1, \ldots, i-1\}$, $i = 2, \ldots, n$.

EXAMPLE 2.5. The Pommaret direction is defined as

$$\forall\, \alpha \in \Gamma, \qquad \delta_i^\Gamma(\alpha) = \begin{cases} 1, & \text{if } i \geq L(\alpha), \\ 0, & \text{otherwise,} \end{cases} \qquad i = 1, \ldots, n,$$

where $L(\alpha)$ represents the number k such that the k^{th} component of α is the last nonzero component of α if $\alpha \neq 0$. $k = 0$ if $\alpha = 0$.

We will show that the three directions defined above are involutive. For the Thomas and Janet directions, the equality

$$\alpha + \mu \circ \delta^\Gamma(\alpha) = \beta + \nu \circ \delta^\Gamma(\beta) \tag{2.1}$$

implies $\alpha = \beta$, where $\alpha, \beta \in \Gamma$. Thus, (i) of Definition 2.2 is valid. Otherwise, we may suppose, without loss of generality, $\alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1}$, $\alpha_i > \beta_i$. According to the definition of the Thomas and Janet directions, we have $\delta_i^\Gamma(\beta) = 0$, whence $\alpha_i + \mu_i \delta_i^\Gamma(\alpha) = \beta_i \geq \alpha_i$ by (2.1), a contradiction. As for condition (ii) of Definition 2.2, note the fact: if $\alpha \in \Sigma \subseteq \Gamma$ then $\Sigma_{j_1 \ldots j_{i-1}} \subseteq \Gamma_{j_1 \ldots j_{i-1}}$. Therefore, $\mathbf{b}_i(\Sigma) \leq \mathbf{b}_i(\Gamma)$, $\mathbf{b}_i(\Sigma_{j_1 \ldots j_{i-1}}) \leq \mathbf{b}_i(\Gamma_{j_1 \ldots j_{i-1}})$, $j = 2, \ldots, n$. Hence, $\delta_i^\Gamma(\alpha) = 1$ implies $\delta_i^\Sigma(\alpha) = 1$, i.e., $\delta^\Sigma - \delta^\Gamma$ is nonnegative. So the Thomas and Janet directions are both involutive.

Pommaret direction is clearly global, so the condition (ii) of Definition 2.2 is satisfied. For any two vectors $\alpha, \beta \in \Gamma$, we suppose $k = L(\alpha) \leq L(\beta) = j$. Equation (2.1) implies $\alpha_1 = \beta_1, \ldots, \alpha_{k-1} = \beta_{k-1}$, $\alpha_k + \mu_k = \beta_k$, and $\alpha_{k+1} = 0, \ldots, \alpha_n = 0$ by the definition of Pommaret direction. Hence, $\beta - \alpha$ and $\delta^\Gamma(\alpha) - \delta^\Gamma(\beta)$ are both negative. Pommaret direction is also involutive.

DEFINITION 2.6. *A set Γ is called auto-reduced with respect to a direction δ, or δ-auto-reduced if (2.1) implies $\alpha = \beta$ for every pair of $\alpha, \beta \in \Gamma$.*

Any finite set is auto-reduced with respect to the Thomas and Janet directions.

3. NOETHERIAN, ARTINIAN, AND CONSTRUCTIVE DIRECTIONS

From now on, δ represents an involutive direction, $<_{\text{lex}}$ represents the lexicographical order.

3.1. Noetherian Directions

For $\alpha \in \Gamma$ and $\mu \in N^n$, we call $\alpha + \mu \circ \delta^\Gamma(\alpha)$ a prolongation of α. The set of all prolongations of α is denoted by $P_\delta^\Gamma(\alpha)$. Let $P_\delta(\alpha) = \bigcup_{\alpha \in \Gamma} P_\delta^\Gamma(\alpha)$.

DEFINITION 3.1. *A finite subset Γ of $\mathbf{N}^n$ is called complete with respect to δ, if $P_\delta(\Gamma) = P^*(\Gamma)$, where $P^*(\Gamma) = \{\alpha + \mu \mid \alpha \in \Gamma,\ \mu \in \mathbf{N}^n\}$.*

Clearly, $\Gamma' = P_\delta(\Gamma) \cap B(\Gamma)$ is complete with respect to the Thomas and Janet directions, and $\Gamma' \supseteq \Gamma$. For a finite set Γ, if there exists a finite set Γ' such that

(a) Γ' is complete with respect to δ, and $\Gamma' \supseteq \Gamma$;
(b) $P_\delta(\Gamma') = P^*(\Gamma)$.

then Γ is said to be *finitely generated* with respect to δ, Γ' is called a completion of Γ. If every finite set is finitely generated with respect to δ, then δ is said to be *Noetherian*.

The Thomas and Janet directions are Noetherian, but Pommaret direction is not.

3.2. Artinian Directions

For convenience, we often omit Γ and δ in $\delta^\Gamma(\alpha)$ and $P_\delta(\Gamma)$, respectively. Denote by $\delta_i(\alpha)$ the i^{th} component of $\delta(\alpha)$, and by $\delta^{(i)}$ the vector whose i^{th} component is 1 and the other components are 0. An index i is called a multiplier (or nonmultiplier) if $\delta_i(\alpha) = 1$ (or $\delta_i(\alpha) = 0$). If i is a nonmultiplier of α, we call $\alpha + \delta^{(i)}$ a nonmultiplicative prologation of α.

DEFINITION 3.2. *$\alpha, \beta \in \Gamma$, β is said to be a pseudo-divisor of α if there exists an index i and a vector $\mu \in \mathbf{N}^n$, such that*

$$\alpha + \delta^{(i)} = \beta + \mu \circ \delta(\beta), \qquad \delta_i(\alpha) = 0. \tag{3.1}$$

LEMMA 3.3. *If (3.1) holds, then*

(T) *$\alpha + \delta^{(i)} = \beta$, i.e., $\alpha_i + 1 = \beta_i$, $\alpha_j = \beta_j$, $j \neq i$, are valid for the Thomas direction.*
(J) *$\alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1}$, $\alpha_i + 1 = \beta_i$, are valid for Janet direction.*
(P) *$L(\alpha) > L(\beta)$, or $L(\alpha) = L(\beta)$ and $\alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1}$, $\alpha_i + 1 = \beta_i$, are valid for Pommaret direction.*

PROOF. For the Thomas direction, if $\mu_j \delta_j(\beta) \neq 0$, then $\beta_j < \alpha_j$ for $j \neq i$ by (3.1). So $\beta_j < \mathbf{b}_j(\Gamma)$, $\delta_j(\beta) = 0$, a contradiction. If $\mu_i \delta_i(\beta) \neq 0$, then $\beta_i \leq \alpha_i < \mathbf{b}_i(\Gamma)$ since $\delta_i(\alpha) = 0$. So $\delta_i(\beta) = 0$, a contradiction. We have $\alpha_i + 1 = \beta_i$, $\alpha_j = \beta_j$, $j \neq i$.

For the Janet direction, $\beta_j \leq \alpha_j$ for $j \neq i$ by (3.1). If there exists $k < i$, such that $\beta_1 = \alpha_1, \ldots, \beta_{k-1} = \alpha_{k-1}$, $\beta_k < \alpha_k$, then $\beta_k < \mathbf{b}_k(\Gamma_{\beta_1 \ldots \beta_{k-1}})$, whence $\delta_k(\beta) = 0$. We have $\alpha_k = \beta_k$ by (3.1), a contradiction. So $\alpha_j = \beta_j$ for $j < i$. Since $\delta_i(\alpha) = 0$, $\mu_i \delta_i(\beta) \neq 0$ implies that $\beta_i \leq \alpha_i < \mathbf{b}_i(\Gamma_{\beta_1 \ldots \beta_{i-1}})$. Hence, $\delta_i(\beta) = 0$, a contradiction.

For the Pommaret direction, if $L(\alpha) \leq L(\beta)$, then $\delta_i(\alpha) = 0$ implies $i < L(\alpha)$, whence $\alpha_j \geq \beta_j$ for $j \geq L(\alpha)$ by (3.1). But $\alpha_j = 0$, whence $\beta_j = 0$ for $j > L(\alpha)$, so $L(\alpha) = L(\beta)$. Since $i < L(\beta)$, $\delta_j(\beta) = 0$ for $j \leq i$, we have $\alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1}$, $\alpha_i = \beta_i$ by (3.1). ∎

Consider a sequence

$${}^{(1)}\beta,\ {}^{(2)}\beta,\ \ldots,\ {}^{(k)}\beta, \ldots, \tag{3.2}$$

where the ${}^{(i)}\beta \in \Gamma$. If ${}^{(i+1)}\beta$ is a pseudo-divisor of ${}^{(i)}\beta$ for $i = 1, 2, \ldots$, then (3.2) is said to be a *pseudo-divisor sequence* of Γ. If for every finite set Γ, every pseudo-divisor sequence of Γ is finite, then δ is said to be *Artinian*. Since Γ is a finite set, the property that 'every pseudo-divisor sequence is finite' is equivalent to the property that 'every pseudo-divisor sequence consists of distinct elements'.

For the Thomas and Janet directions, β is a pseudo-divisor of α implies $\alpha <_{\text{lex}} \beta$. For Pommaret direction, β is a pseudo-divisor of α implies $L(\alpha) > L(\beta)$, or $L(\alpha) = L(\beta)$ and $\alpha <_{\text{lex}} \beta$. So the three directions are Artinian directions since relations $<_{\text{lex}}$ and $>$ are both transitive.

THEOREM 3.4. *Let δ be an Artinian direction. A finite set Γ is complete with respect to δ if and only if*

$$\delta_i(\gamma) = 0 \Rightarrow \gamma + \delta^{(i)} \in P(\Gamma), \tag{3.3}$$

for all $\gamma \in \Gamma$.

PROOF. The necessity is clear. For the sufficiency, we suppose $P^*(\Gamma) \setminus P(\Gamma) \neq \emptyset$, and consider any element γ of it. Set $\gamma = {}^{(1)}\beta + {}^{(1)}\nu$, ${}^{(1)}\beta \in \Gamma$. Since $\gamma \notin P(\Gamma)$, there is an index i_1, such that ${}^{(1)}\nu_{i_1} \neq 0$, and $\delta_{i_1}({}^{(1)}\beta) = 0$. So ${}^{(1)}\beta + {}^{(1)}\nu = {}^{(1)}\beta + \delta^{(i_1)} + ({}^{(1)}\nu - \delta^{(i_1)})$. By (3.3), ${}^{(1)}\beta + \delta^{(i_1)} \in P(\Gamma)$, i.e., there is a ${}^{(2)}\beta \in \Gamma$, such that ${}^{(1)}\beta + \delta^{(i_1)} = {}^{(2)}\beta + {}^{(2)}\mu \circ \delta({}^{(2)}\beta)$, whence ${}^{(2)}\beta$ is a pseudo-divisor of ${}^{(1)}\beta$. We have ${}^{(1)}\beta + {}^{(1)}\nu = {}^{(2)}\beta + {}^{(2)}\nu \notin P(\Gamma)$. By this way we may get an infinite pseudo-divisor sequence of Γ. This is in contradiction with Artinian property of δ. ∎

For a given Γ, if (3.3) holds for all $\gamma \in \Gamma$, we say Γ is locally complete. In [17], the authors gave a proof of Theorem 3.4 for Janet direction only. A special kind of complete set, called closed set, was introduced. An algorithm to compute the minimal closed set Γ' such that $\Gamma' \supseteq \Gamma$ and $P(\Gamma') = P^*(\Gamma)$, which is called closure of Γ, was given for a finite set Γ.

3.3. Constructive Directions

DEFINITION 3.5. *A nonmultiplicative prolongation $\alpha + \delta^{(i)}$ of $\Gamma \in \mathcal{F}_n$ is said to be critical if the following conditions are satisfied:*

(c) $\alpha + \delta^{(i)} \notin P(\Gamma)$;

(d) *if $\alpha + \delta^{(i)} = \beta + \delta^{(j)} + \gamma$, where $\beta + \delta^{(j)}$ is also a nonmultiplicative prolongation of Γ and $\gamma \neq 0$, then $\beta + \delta^{(j)} \in P(\Gamma)$.*

A direction δ is said to be *constructive* if, for every finite set Γ, no critical prolongation $\alpha + \delta^{(i)}$ of Γ can be expressed as

$$\alpha + \delta^{(i)} = \beta + \mu \circ \delta^{\Gamma}(\beta) + \nu \circ \delta^{\Gamma'}(\beta'), \tag{3.4}$$

where $\beta \in \Gamma$, $\beta' = \beta + \mu \circ \delta^{\Gamma}(\beta)$, $\Gamma' = \Gamma \cup \{\beta'\}$.

The Thomas, Janet, and Pommaret directions are all constructive. For the Thomas direction, we suppose that $\alpha + \delta^{(i)}$ is a critical prolongation and (3.4) holds. If $\delta_j^{\Gamma}(\beta) = 0$, then $\beta'_j = \beta_j < \mathbf{b}_j(\Gamma) \leq \mathbf{b}_j(\Gamma')$. Hence, $\delta_j^{\Gamma'}(\beta') = 0$. Setting $\mu' = \nu \circ \delta^{\Gamma'}(\beta')$, we have $\alpha + \delta^{(i)} = \beta + (\mu + \mu') \circ \delta^{\Gamma}(\beta) \in P(\Gamma)$, which is in contradiction with (c) of Definition 3.5. For Pommaret direction, $\delta_j^{\Gamma}(\beta) = 0$ implies $j < L(\beta) \leq L(\beta')$. So $\delta_j^{\Gamma'}(\beta') = 0$, and the remaining part is similar to the proof for the Thomas direction.

As for Janet direction, choose α as high as possible with respect to $<_{\text{lex}}$, such that $\alpha + \delta^{(i)}$ is a critical prolongation, and (3.4) holds. We claim that $\alpha <_{\text{lex}} \beta$. Otherwise one may suppose $\alpha_1 = \beta_1, \ldots, \alpha_{k-1} = \beta_{k-1}$, $\alpha_k > \beta_k$. If $k \geq i$, then $\alpha_1 = \beta_1 = \beta', \ldots, \alpha_{i-1} = \beta_{i-1} = \beta'_{i-1}$, $\alpha_i \geq \beta_i$ by (3.4). Since $\delta_i^{\Gamma}(\alpha) = 0$, $\alpha_i < \mathbf{b}_i(\Gamma_{\alpha_1 \ldots \alpha_{i-1}}) = \mathbf{b}_i(\Gamma_{\beta_1 \ldots \beta_{i-1}})$, whence $\delta_i^{\Gamma}(\beta) = 0$. Hence, $\beta'_i = \beta_i < \mathbf{b}_i(\Gamma_{\beta_1 \ldots \beta_{i-1}}) \leq \mathbf{b}_i(\Gamma'_{\beta'_1 \ldots \beta'_{i-1}})$, and $\delta_i^{\Gamma'}(\beta') = 0$. By (3.4), $\alpha_i + 1 = \beta_i$, a contradiction. If $k < i$, then by (3.4), $\alpha_1 = \beta_1 = \beta', \ldots, \alpha_{k-1} = \beta_{k-1} = \beta'_{k-1}$, $\alpha_k > \beta_k$. This implies $\delta_k^{\Gamma}(\beta) = 0$, $\beta'_k = \beta_k$, $\delta_k^{\Gamma'}(\beta') = 0$, and $\alpha_k = \beta_k$, which is a contradiction.

Since $\alpha + \delta^{(i)} \notin P(\Gamma)$, there is a j such that $\nu_j \delta_j^{\Gamma'}(\beta') \neq 0$ and $\delta_j^{\Gamma}(\beta) = 0$. Hence, $\beta' \neq \beta$. (3.4) can be rewritten as

$$\alpha + \delta^{(i)} = \beta + \delta^{(j)} + \gamma, \tag{3.5}$$

where $\gamma = \mu \circ \delta^{\Gamma}(\beta) + (\nu \circ \delta^{\Gamma'}(\beta') - \delta^{(j)}) \neq 0$. By (d) of Definition 3.5, $\beta + \delta^{(j)} \in P(\Gamma)$. Setting $\beta + \delta^{(j)} = {}^{(1)}\alpha + \sigma \circ \delta^{\Gamma}({}^{(1)}\alpha)$, ${}^{(1)}\alpha \in \Gamma$, by Lemma 3.3 we have $\beta <_{\text{lex}} {}^{(1)}\alpha$, and

$$\alpha + \delta^{(i)} = {}^{(1)}\alpha + {}^{(1)}\gamma, \qquad \alpha <_{\text{lex}} {}^{(1)}\alpha, \tag{3.6}$$

where ${}^{(1)}\gamma = \sigma \circ \delta^{\Gamma}({}^{(1)}\alpha) + \gamma \neq 0$. Since $\alpha + \delta^{(i)} \notin P(\Gamma)$, we have ${}^{(1)}\gamma_{i_1} \neq 0$, such that $\delta_{i_1}^{\Gamma}({}^{(1)}\alpha) = 0$. Rewrite (3.6) as $\alpha + \delta^{(i)} = {}^{(1)}\alpha + \delta^{(i_1)} + ({}^{(1)}\gamma - \delta^{(i_1)})$. By th choice of α, ${}^{(1)}\gamma - \delta^{(i_1)} \neq 0$, whence ${}^{(1)}\alpha + \delta^{(i_1)} \in P(\Gamma)$. Setting ${}^{(1)}\alpha + \delta^{(i_1)} = {}^{(2)}\alpha + \tau \circ \delta^{\Gamma}({}^{(2)}\alpha)$, ${}^{(2)}\alpha \in \Gamma$. By Lemma 3.3 we have ${}^{(1)}\alpha <_{\text{lex}} {}^{(2)}\alpha$, and

$$\alpha + \delta^{(i)} = {}^{(2)}\alpha + {}^{(2)}\gamma, \qquad \alpha <_{\text{lex}} {}^{(2)}\alpha, \tag{3.7}$$

where ${}^{(2)}\gamma = \tau \circ \delta^{\Gamma}({}^{(2)}\alpha) + ({}^{(1)}\gamma - \delta^{(i_1)}) \neq 0$. In this way, we would get an infinite sequence

$$\alpha <_{\text{lex}} \beta <_{\text{lex}} {}^{(1)}\alpha <_{\text{lex}} {}^{(2)}\alpha <_{\text{lex}} \cdots$$

of Γ, which is in contradiction with the finiteness of Γ.

THEOREM 3.6. *Let δ be Artinian and constructive. Any completion Γ' of a finite set Γ contains all critical prolongations of Γ.*

PROOF. Suppose that there exists a critical prolongation $\alpha+\delta^{(i)}$ of Γ, which does not belong to Γ'. Since $\alpha+\delta^{(i)} \in P^*(\Gamma)=P(\Gamma')$, there is a $\beta' \in \Gamma'$, such that $\alpha+\delta^{(i)}=\beta'+\nu\circ\delta^{\Gamma'}(\beta')$. Set $\nu'=\nu\circ\delta^{\Gamma'}(\beta')$, $\Gamma''=\Gamma\cup\{\beta'\}$. By (ii) of Definition 2.2, $\delta^{\Gamma''}(\beta')-\delta^{\Gamma'}(\beta')$ is nonnegative. Hence, $\nu'=\nu'\circ\delta^{\Gamma''}(\beta')$, and

$$\alpha+\delta^{(i)}=\beta'+\nu', \qquad \nu'\neq 0. \tag{3.8}$$

We claim that $\beta'\in P(\Gamma)$. Since $\beta'\in\Gamma'\subseteq P^*(\Gamma')=P^*(\Gamma)$, it can be rewritten as

$$\beta'={}^{(1)}\beta+{}^{(1)}\gamma, \qquad {}^{(1)}\beta\in\Gamma. \tag{3.9}$$

If there is a ${}^{(1)}\gamma_j\neq 0$, such that $\delta_j^{\Gamma}({}^{(1)}\beta)=0$, substituting $\beta'={}^{(1)}\beta+\delta^{(j)}+({}^{(1)}\gamma-\delta^{(j)})$ into (3.8) we have $\alpha+\delta^{(i)}={}^{(1)}\beta+\delta^{(j)}+({}^{(1)}\gamma-\delta^{(j)})+\nu'$. Then ${}^{(1)}\beta+\delta^{(j)}\in P(\Gamma)$ since $\alpha+\delta^{(i)}$ is critical and $\nu'\neq 0$. Let ${}^{(1)}\beta+\delta^{(j)}={}^{(2)}\beta+\sigma\circ\delta^{\Gamma}({}^{(2)}\beta)$, ${}^{(2)}\beta\in\Gamma$. Then ${}^{(2)}\beta$ is a pseudo-divisor of ${}^{(1)}\beta$, and

$$\beta'={}^{(2)}\beta+{}^{(2)}\gamma, \tag{3.10}$$

where ${}^{(2)}\gamma=\sigma\circ\delta^{\Gamma}({}^{(2)}\beta)+({}^{(1)}\gamma-\delta^{(j)})$. For $\beta'={}^{(2)}\beta+{}^{(2)}\gamma$, we discuss about ${}^{(2)}\gamma$ as about ${}^{(1)}\gamma$ for $\beta'={}^{(1)}\beta+{}^{(1)}\gamma$, and so on. Because δ is Artinian, we can achieve that, by a finite number of steps, there exits a $\beta\in\Gamma$ such that $\beta'=\beta+\mu$, and $\mu_k\neq 0$ implies $\delta_k^{\Gamma}(\beta)=1$, $k=1,\dots,n$. So $\beta'=\beta+\mu\circ\delta^{\Gamma}(\beta)\in P(\Gamma)$. Substituting $\beta'=\beta+\mu\circ\delta(\beta)$ into (3.8), we have $\alpha+\delta^{(i)}=\beta+\mu\circ\delta(\beta)+\nu'\circ\delta^{\Gamma''}(\beta')$, which is in contradiction with the constructivity of δ. ∎

4. EQUIVALENCE BETWEEN INVOLUTIVE DIRECTION AND INVOLUTIVE DIVISION

4.1. An Equivalent Theorem

In [21], an axiomatic definition on involutive division was given. Let $\mathbf{M}$ be the set of all n-variables monomials.

DEFINITION 4.1. *An involutive division L on $\mathbf{M}$ is given, if for any finite monomial set $U\subset\mathbf{M}$ and any $u\in U$ there is a submonoid $L(u,U)$ of $\mathbf{M}$ satisfying the conditions*

(a) *if $w\in L(u,U)$ and $v\mid w$, then $v\in L(u,U)$;*
(b) *if $u,v\in U$ and $uL(u,U)\cap vL(u,U)\neq 0$, then $u\in vL(v,U)$ or $v\in uL(u,U)$;*
(c) *if $v\in U$ and $v\in uL(u,U)$, then $L(v,U)\subseteq L(u,U)$;*
(d) *if $V\subseteq U$, then $L(u,U)\subseteq L(u,V)$ for all $u\in V$.*

For convenience, we denote by x^α the monomial $x_1^{\alpha_1}x_2^{\alpha_2}\dots x_n^{\alpha_n}$, where $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)$. It is clear that the map $\varphi:\alpha\mapsto x^\alpha$ is an isomorphism between monoids $(\mathbf{N}^n,+)$ and $(\mathbf{M},\cdot)$. We will prove the next theorem.

THEOREM 4.2. *The involutive direction in Definition 2.2 and the involutive division in Definition 4.1 are equivalent under the map φ.*

PROOF. Let δ be an involutive direction on $\mathbf{N}^n$. For any element u of a finite subset U of $\mathbf{M}$, let $\Gamma=\varphi^{-1}(U)$, and $u=x^\alpha$. We need to prove that the set

$$L(u,U)=\left\{x^\beta \mid \beta=\mu\circ\delta^{\Gamma}(\alpha),\ \mu\in\mathbf{N}^n\right\} \tag{4.1}$$

is a submonoid of $(\mathbf{M},\cdot)$, and (a), (b), (c), (d) hold. Hence, L is an involutive division on $\mathbf{M}$.

Taking $\mu=0$, then $\beta=0$, $x^\beta=1\in L(u,U)$. If $v,w\in L(u,U)$, let $v=x^\beta$, $w=x^\gamma$, where $\beta=\mu\circ\delta^{\Gamma}(\alpha)$, $\gamma=\nu\circ\delta^{\Gamma}(\alpha)$. Then $\beta+\gamma=(\mu+\nu)\circ\delta^{\Gamma}(\alpha)$, whence $vw=x^{\beta+\gamma}\in L(u,U)$. Hence, $L(u,U)$ is a submonoid of $(\mathbf{M},\cdot)$.

If $w \in L(u,U)$, $v \mid w$, set $w = x^\gamma$, $\gamma = \nu \circ \delta^\Gamma(\alpha)$. Then $\delta_k^\Gamma(\alpha) = 0$ implies $\beta_k = 0$. Hence, $\beta = \beta \circ \delta^\Gamma(\alpha)$, $v = x^\beta \in L(u,U)$, and (a) holds.

For $u, v \in U$, set $u = x^\alpha$, $v = x^\beta$, $\alpha, \beta \in \Gamma$. If $uL(u,U) \cap vL(v,U) \neq \emptyset$, supposing $w = x^\gamma \in uL(u,U) \cap vL(v,U)$, then $\gamma = \alpha + \mu \circ \delta^\Gamma(\alpha) = \beta + \nu \circ \delta^\Gamma(\beta)$, $\mu, \nu \in \mathbf{N}^n$. By (ii) of Definition 2.2, we may suppose, without loss of generality, that $\delta^\Gamma(\alpha) - \delta^\Gamma(\beta)$ and $\beta - \alpha$ are both nonnegative. Let $\nu' = \nu \circ \delta^\Gamma(\beta)$. Then $\nu \circ \delta^\Gamma(\beta) = \nu' \circ \delta^\Gamma(\alpha)$. Hence, $\alpha + (\mu - \nu') \circ \delta^\Gamma(\alpha) = \beta$. Let $\mu' = (\mu - \nu') \circ \delta^\Gamma(\alpha)$. Then $\mu' \in \mathbf{N}^n$, and $\beta = \alpha + \mu' \circ \delta^\Gamma(\alpha)$. We have $v \in uL(u,U)$, which implies (b).

For $u, v \in U$ let $u = x^\alpha$, $v = x^\beta$, $\alpha, \beta \in \Gamma$. If $v \in uL(u,U)$, then $\beta = \alpha + \mu \circ \delta^\Gamma(\alpha)$, i.e., $\beta + 0 \circ \delta^\Gamma(\beta) = \alpha + \mu \circ \delta^\Gamma(\alpha)$. By (ii) of Definition 2.2, $\delta^\Gamma(\alpha) - \delta^\Gamma(\beta)$ is nonnegative since $\beta - \alpha$ is nonnegative. For $w \in L(v,U)$, let $w = x^\gamma$. Then $\gamma = \nu \circ \delta^\Gamma(\beta) = \nu' \circ \delta^\Gamma(\alpha)$ (ν', taken as above). Hence, $w \in L(u,U)$. We have $L(v,U) \subseteq L(u,U)$, which implies (c).

For $u \in V \subseteq U$, let $u = x^\alpha$, $\Sigma = \varphi^{-1}(V)$. We have $\alpha \in \Sigma \subseteq \Gamma$, and $\delta^\Sigma(\alpha) - \delta^\Gamma(\alpha)$ is nonnegative by (ii) of Definition 2.2. Similar to the above proof, we derive $L(u,U) \subseteq L(u,V)$, which implies (d).

Conversely, given an involutive division L on $\mathbf{M}$, we will construct an involutive direction δ on $\mathbf{N}^n$. For any element α of any finite subset Γ of $\mathbf{N}^n$, setting $U = \{x^\beta \mid \beta \in \Gamma\}$, $u = x^\alpha$, we define $\delta^\Gamma(\alpha)$ as follows:

$$\delta_k^\Gamma(\alpha) = \begin{cases} 1, & \text{if } x_k \text{ effectively appears in some elements of } L(u,U), \\ 0, & \text{otherwise,} \end{cases} \tag{4.2}$$

$k = 1, 2, \ldots, n$. Clearly δ^Γ is a map from Γ to Δ_n. Now we begin to prove that conditions (i), (ii) of Definition 2.2 are satisfied.

Let $\alpha, \beta \in \Gamma$, and $\alpha + \mu \circ \delta^\Gamma(\alpha) = \beta + \nu \circ \delta^\Gamma(\beta)$. Let $u = x^\alpha$, $v = x^\beta$. Then $u, v \in U$. We claim that $\bar{u} = x^{\mu \circ \delta^\Gamma(\alpha)} \in L(u,U)$. In fact, if $\delta_k^\Gamma(\alpha) = 1$, x_k effectively appears in some element, say u', of $L(v,U)$, whence $x_k \in L(u,U)$ by (a) of Definition 4.1. Since $L(u,U)$ is a monoid, we have $\bar{u} = x_1^{\mu_1 \delta_1^\Gamma(\alpha)} \ldots x_k^{\mu_k \delta_k^\Gamma(\alpha)} \ldots x_n^{\mu_n \delta_n^\Gamma(\alpha)} \in L(u,U)$. Similarly $\bar{v} = x^{\nu \circ \delta^\Gamma(\beta)} \in L(v,U)$. So $u\bar{u} = v\bar{v} \in uL(u,U) \cap vL(v,U)$. By (b) of Definition 4.1, either $u \in vL(v,U)$, or $v \in uL(u,U)$. In the case $v \in uL(u,U)$, $\beta - \alpha$ is nonnegative and $L(v,U) \subseteq L(u,U)$. Hence, $\delta^\Gamma(\alpha) - \delta^\Gamma(\beta)$ is nonnegative. And likewise for the case $u \in vL(v,U)$. Then (i) is satisfied.

For $\alpha \in \Sigma \subseteq \Gamma$. Let $V = \{x^\beta \mid \beta \in \Sigma\}$. Then $u = x^\alpha \in V \subseteq U$. By (d) of Definition 4.1, $L(u,U) \subseteq L(u,V)$. Hence, $\delta^\Sigma(\alpha) - \delta^\Gamma(\alpha)$ is nonnegative. Then (ii) is satisfied. ∎

Table 1. Corresponding relations between involutive directions and involutive divisions.

$(\mathbf{N}^n, +) \xrightarrow{\varphi} (\mathbf{M}, \cdot) : \alpha \mapsto u = x^\alpha$
$\Gamma \longrightarrow U$, $\delta^\Gamma \longrightarrow L_U$
$\delta_i(\alpha) = 1$, or $0 \longrightarrow ux_i = u \times x_i$ or not
$\alpha + \mu \circ \delta(\alpha) \longrightarrow u \times v_1$, $v_1 = x^{\mu \circ \delta(\alpha)}$
$\{\mu \circ \delta(\alpha) \mid \mu \in \mathbf{N}^n\} \longrightarrow L(u,U)$, $P_\delta(\Gamma) \longrightarrow C_L(U)$, $P^*(\Gamma) \longrightarrow C(U)$
δ is Artinian $\longrightarrow L$ is continuous, δ is Artinian and constructive $\longrightarrow L$ is constructive

4.2. An Improved Completion Algorithm

In [21,22], the authors gave an algorithm to determine a minimal involutive completion for a given finite set of monomial. In their algorithm, a given finite set of monomials was enlarged to its completion by adding one monomial each step. By Theorem 3.6, we can enlarge a given noncomplete finite set, by adding all critical prolongations in each step, to a minimal completion of the given set. It is easy to see that deciding whether ux_j is critical is as simple as deciding whether ux_j is the lowest element with respect to a given ordering. Based on these considerations, we give an improved completion algorithm. A variable x_i is called multiplicative is i is a mutiplier. Denote the set of nonmultiplicative variables of u in V by $NM_L(u, V)$.

IMPROVED COMPLETION ALGORITHM

Input: U, a finite monomial set
Output: $\tilde{U}$, a minimal completion of U
begin
$\tilde{U} := U$
while exist $u \in \tilde{U}$ and $x \in NM_L(u, \tilde{U})$,
such that $u \cdot x$ has no involutive divisors in $\tilde{U}$ **do**
choose all critical prolongations of $\tilde{U}$, say, $\bar{U}$
$\tilde{U} := \tilde{U} \cup \bar{U}$
end
end

For example, consider the Thomas division and a set $U = \{xy, y^2, z\}$. By their algorithm, taking the lexicographical ordering with $z < y < x$,

$$\begin{aligned} U = \{xy, y^2, z\} &\to \{xy, y^2, z, yz\} \to \{xy, y^2, z, yz, y^2z\} \to \{xy, y^2, z, yz, y^2z, xz\} \\ &\to \{xy, y^2, z, yz, y^2z, xz, xyz\} \to \{xy, y^2, z, yz, y^2z, xz, xyz, xy^2\} \\ &\to \{xy, y^2, z, yz, y^2z, xz, xyz, xy^2, xy^2z\} = \tilde{U}_T. \end{aligned}$$

By the improved algorithm,

$$\begin{aligned} U = \{xy, y^2, z\} &\to \{xy, y^2, z, xy^2, xz, yz\} \to \{xy, y^2, z, xy^2, xz, yz, xyz, y^2z\} \\ &\to \{xy, y^2, z, yz, y^2z, xz, xyz, xy^2, xy^2z\} = \tilde{U}_T. \end{aligned}$$

By adding a set of monomials in each step, our algorithm may finish the completion process in less number of steps.

5. GENERALIZATION OF THE THOMAS AND JANET DIRECTIONS

In [22], Gerdt *et al.* gave two new involutive divisions, Division I and Division II, different from that of Thomas, Janet, and Pommaret. We will show how to represent them using involutive directions.

EXAMPLE 5.1. DIVISION I. Let U be a finite monomial set. The variable x_i is nonmultiplicative for $u \in U$ if there is a $v \in U$, such that

$$x_{i_1}^{d_1} \dots x_{i_m}^{d_m} u = \operatorname{lcm}(u, v), \qquad 1 \le m \le [n/2], \qquad d_j > 0 \quad (1 \le j \le m),$$

and $x_i \in \{x_{i_1}, \dots, x_{i_m}\}$. In other words, for $\alpha \in \Gamma$, $\delta_i^\Gamma(\alpha) = 0$ iff there exists a $\beta \in \Gamma$, such that $\beta_i > \alpha_i$ and the number of positive components of $\beta - \alpha$ ranges from 1 to $[n/2]$. In the next section, we will generalize this concept.

EXAMPLE 5.2. DIVISION II. For monomial $u = x_1^{d_1} \dots x_n^{d_n}$, the variable x_i is multiplicative if $d_i = d_{\max}(u)$, where $d_{\max} = \max\{d_1, \dots, d_n\}$. In other words, for $\alpha \in \Gamma$, $\delta_i^\Gamma(\alpha) = 1$ iff α_i is the maximal component of α.

5.1. The Thomas Type Directions

We denote by $N_p(\alpha)$ the number of positive components of α. Let l be an integer not less than $[n/2]$. For any finite subset Γ of $\mathbf{N}^n$, set

$$B_l^\Gamma(\alpha) = \{\beta \in \Gamma \mid 1 \le N_p(\beta - \alpha) \le l\}. \tag{5.1}$$

DEFINITION 5.3. *A Thomas type direction, δ on $\mathbf{N}^n$ is defined as*

$$\delta_i^\Gamma(\alpha) = 1 \text{ iff } \beta_i \leq \alpha_i, \qquad \text{for all } \beta \in B_l^\Gamma(\alpha). \tag{5.2}$$

In other words, $\delta_i^\Gamma(\alpha) = 0$ iff there is a $\beta \in B_l^\Gamma(\alpha)$, s.t. $\beta_i > \alpha_i$. This direction is denoted by T_l.

Clearly, when $l = [n/2]$, δ is just the direction corresponding to Division I of Gerdt *et al.* The Thomas direction corresponds to the case $l = n$. T_l is Noetherian since $P^*(\Gamma) \cap B(\Gamma)$ is a completion of Γ. We will discuss the involutivity, Artinity and constructivity of T_l.

T_l is involutive. Set $\delta = T_l$. For $\alpha, \beta \in \Gamma$, let

$$\alpha + \mu \circ \delta(\alpha) = \beta + \nu \circ \delta(\beta), \tag{5.3}$$

where $\mu, \nu \in \mathbf{N}^n$. If there are indices i and j, such that $\alpha_i > \beta_i$, and $\alpha_j < \beta_j$, then either $\alpha \in B_l^\Gamma(\beta)$ or $\beta \in B_l^\Gamma(\alpha)$ since $l \geq [n/2]$. But $\alpha \in B_l^\Gamma(\beta)$, with $\alpha_i > \beta_i$, implies $\delta_i(\beta) = 0$, whence $\beta_i = \alpha_i + \mu_i\delta_i(\alpha) \geq \alpha_i$ by (5.3), a contradiction. And likewise for $\beta \in B_l^\Gamma(\alpha)$. We proved either $\alpha - \beta$ or $\beta - \alpha$ is nonnegative. Let us suppose that $\alpha - \beta$ is nonnegative. If $\delta_k(\beta) = 0$, by the definition, there exists a vector $\gamma \in B_l^\Gamma(\beta)$, such that $\gamma_k > \beta_k$. But by (5.3) $\alpha_k \leq \beta_k$, $\gamma_k > \alpha_k$, $\gamma \in B_l^\Gamma(\alpha)$ since $\alpha - \beta$ is nonnegative, whence $\delta_k(\alpha) = 0$, a contradiction. Hence, $\delta(\beta) - \delta(\alpha)$ is nonnegative, the condition (i) of Definition 2.2 is satisfied. The condition (ii) is clearly satisfied since $B_l^\Sigma(\alpha) \subseteq B_l^\Gamma(\alpha)$ for $\alpha \in \Sigma \subseteq \Gamma$. Therefore, δ is involutive.

T_l is constructive. For any critical prolongation $\alpha + \delta^{(i)}$ of Γ, let us suppose that

$$\alpha + \delta^{(i)} = \beta + \mu \circ \delta^\Gamma(\beta) + \nu \circ \delta^{\Gamma'}(\beta') \tag{5.4}$$

holds for some $\beta \in \Gamma$, where $\beta' = \beta + \mu \circ \delta^\Gamma(\beta)$, $\Gamma' = \Gamma \cup \{\beta'\}$. If $\delta_k^\Gamma(\beta) = 0$, then $\beta'_k = \beta_k$, and there exists a vector $\gamma \in B_l^\Gamma(\beta)$, such that $\gamma_k > \beta_k = \beta'_k$. This implies $\gamma \in B_l^{\Gamma'}(\beta')$, and $\delta_k^{\Gamma'}(\beta') = 0$. Setting $\nu' = \nu \circ \delta^{\Gamma'}(\beta')$, we have $\alpha + \delta^{(i)} = \beta + (\mu + \nu') \circ \delta^\Gamma(\beta) \in P_\delta(\Gamma)$, a contradiction.

T_l is Artinian. To prove that we first give a lemma. For $\beta \in \Gamma$, $\beta \oplus \gamma$ means that there exists $\mu \in \mathbf{N}^n$, such that $\gamma = \mu \circ \delta^\Gamma(\beta)$.

LEMMA 5.4. *Let δ be a direction defined in Definition 5.3. If*

$$\alpha + \sigma = \beta \oplus \gamma, \qquad \sigma \circ \gamma = 0, \qquad \gamma \neq 0, \tag{5.5}$$

where $\alpha, \beta \in \Gamma$, $\sigma, \gamma \in \mathbf{N}^n$, then $N_p(\gamma) > l$.

It is easy to prove Lemma 5.4, since $N_p(\gamma) \leq l$ implies $\alpha \in B_l^\Gamma(\beta)$, which would be in contradiction with $\beta \oplus \gamma$.

PROOF OF ARTINIAN PROPERTY OF T_l. Given any pseudo-divisor sequence S of Γ

$$^{(1)}\alpha, {}^{(2)}\alpha, \ldots, {}^{(k)}\alpha, \ldots, \tag{5.6}$$

where

$$^{(k)}\alpha + \delta^{(i_k)} = {}^{(k+1)}\alpha \oplus {}^{(k+1)}\gamma, \qquad \delta_{i_k}\left({}^{(k)}\alpha\right) = 0, \tag{5.7}$$

$k = 1, 2, \ldots$. If all $^{(k+1)}\gamma$ are zero, then $^{(k)}\alpha <_{\text{lex}} {}^{(k+1)}\alpha$ for all k. Hence, $^{(i)}\alpha \neq {}^{(j)}\alpha$ for $i \neq j$. Now we suppose that there is at least one $^{(k+1)}\gamma \neq 0$. Extract from S such a subsequence S'

$$^{(1)}\beta, {}^{(2)}\beta \ldots {}^{(j)}\beta \ldots, \tag{5.8}$$

where $^{(j)}\beta = {}^{(k_j+1)}\alpha \in S'$ satisfying $^{(k_j+1)}\gamma \neq 0$. We will prove $^{(i)}\beta \neq {}^{(j)}\beta$ for $i \neq j$.

Consider ${}^{(1)}\beta$ and ${}^{(2)}\beta$. Let

$$\begin{aligned} {}^{(k_1)}\alpha + \delta^{(i_{k_1})} = {}^{(k_1+1)}\alpha \oplus {}^{(k_1+1)}\gamma, {}^{(k_1+1)}\alpha + \delta^{(i_{k_1+1})} = {}^{(k_1+2)}\alpha, \dots \\ {}^{(k_2-1)}\alpha + \delta^{(i_{k_2-1})} = {}^{(k_2)}\alpha, {}^{(k_2)}\alpha + \delta^{(i_{k_2})} = {}^{(k_2+1)}\alpha \oplus {}^{(k_2+1)}\gamma. \end{aligned} \tag{5.9}$$

Then

$$ {}^{(k_1+1)}\alpha + \delta^{(i_{k_1+1})} \dots + \delta^{(i_{k_2})} = {}^{(k_2+1)}\alpha \oplus {}^{(k_2+1)}\gamma. \tag{5.10}$$

Consider $\delta^{(i_{k_2})}$ and ${}^{(k_2+1)}\gamma$. If $\delta^{(i_{k_2})} \circ {}^{(k_2+1)}\gamma \neq 0$, we may eliminate $\delta^{(i_{k_2})}$ from both sides of (5.10),

$$ {}^{(k_1+1)}\alpha + \delta^{(i_{k_1+1})} \dots + \delta^{(i_{k_2-1})} = {}^{(k_2+1)}\alpha \oplus \gamma'.$$

If $\delta^{(i_{k_2})} \circ {}^{(k_2+1)}\gamma = 0$, we set $\gamma' = {}^{(k_2+1)}\gamma$. Then consider $\delta^{(i_{k_2-1})}$ and γ', and so on. Let $\delta^{(i_h)}$ be the last one eliminated. Then $k_1 + 1 \leq h \leq k_2$, (5.10) is reduced to

$$ {}^{(h)}\alpha + \sigma' = {}^{(k_2+1)}\alpha \oplus \gamma, \qquad \delta^{(i_h)} \circ {}^{(k_2+1)}\gamma \neq 0, \tag{5.11}$$

or simply

$$ {}^{(1)}\beta + \sigma = {}^{(2)}\beta \oplus \gamma, \qquad \sigma \circ \gamma = 0. \tag{5.12}$$

We claim that $\gamma \neq 0$, and $\sigma \circ \delta({}^{(1)}\beta) = 0$.

If $\gamma = 0$, then $h < k_2$, ${}^{(h)}\alpha + \sigma' = {}^{(k_2+1)}\alpha$, ${}^{(h+1)}\alpha + \sigma' = {}^{(k_2+1)}\alpha + \delta^{(i_h)}$, and $\delta^{(i_h)} \circ \sigma' = 0$, whence ${}^{(h+1)}\alpha_{i_h} > {}^{(k_2+1)}\alpha_{i_h}$, ${}^{(h+1)}\alpha \in B_l({}^{(k_2+1)}\alpha)$, $\delta_{i_h}({}^{(k_2+1)}\alpha) = 0$. This is in contraction with $\delta^{(i_h)} \circ {}^{(k_2+1)}\gamma \neq 0$, and ${}^{(k_2+1)}\alpha \oplus {}^{(k_2+1)}\gamma$. By Lemma 5.4, $N_p(\gamma) > l$. If $\sigma \neq 0$, then $N_p(\sigma) \leq l$, ${}^{(2)}\beta \in B_l({}^{(1)}\beta)$. So $\sigma \circ \delta({}^{(1)}\beta) = 0$.

We may treat any pair of ${}^{(j+1)}\beta$ and ${}^{(j+2)}\beta$, $j = 1, \dots$ similar to ${}^{(1)}\beta$ and ${}^{(2)}\beta$. We have

$$ {}^{(1)}\beta + {}^{(1)}\sigma = {}^{(2)}\beta \oplus {}^{(2)}\tau, \dots, {}^{(j+1)}\beta + {}^{(j+1)}\sigma = {}^{(j+2)}\beta \oplus {}^{(j+2)}\tau, \tag{5.13}$$

where ${}^{(k)}\sigma \circ {}^{(k)}\tau = 0$, ${}^{(k-1)}\sigma \circ {}^{(k)}\tau = 0$, $N_p({}^{(k)}\tau) > l$, $k = 2, 3, \dots$. Next we prove that for any $j > 1$, there are vectors ${}^{(j)}\zeta$, ${}^{(j)}\xi \in \mathbf{N}^n$, such that

$$ {}^{(1)}\beta + {}^{(j)}\zeta = {}^{(j)}\beta + {}^{(j)}\xi, \qquad N_p({}^{(j)}\xi) > l, \qquad {}^{(j)}\zeta \circ {}^{(j)}\xi = 0. \tag{5.14}$$

It is obvious for $j = 2$. Suppose, by induction, that (5.14) holds for j. Consider the case $j + 1$. By (5.13),

$$ {}^{(1)}\beta + {}^{(j)}\sigma + {}^{(j)}\zeta = ({}^{(j+1)}\beta \oplus {}^{(j+1)}\tau) + {}^{(j)}\xi, \qquad {}^{(j)}\sigma \circ {}^{(j+1)}\tau = 0, \qquad {}^{(j)}\zeta \circ {}^{(j)}\xi = 0. \tag{5.15}$$

Let ${}^{(j)}\zeta = \lambda + {}^{(j)}\zeta'$, ${}^{(j+1)}\tau = \lambda + {}^{(j+1)}\tau'$, such that ${}^{(j)}\zeta' \circ {}^{(j+1)}\tau' = 0$. Since $N_p({}^{(j)}\zeta) \leq l < N_p({}^{(j+1)}\tau)$, ${}^{(j+1)}\tau' \neq 0$. Similarly, we may reduce ${}^{(j)}\sigma$ and ${}^{(j)}\xi$ to ${}^{(j)}\sigma'$ and ${}^{(j)}\xi'$, such that ${}^{(j)}\sigma' \circ {}^{(j)}\xi' = 0$, ${}^{(j)}\xi' \neq 0$, whence (5.15) can be rewritten as

$$ {}^{(1)}\beta + {}^{(j+1)}\zeta = {}^{(j+1)}\beta + {}^{(j+1)}\xi, \tag{5.16}$$

where ${}^{(j+1)}\zeta = {}^{(j)}\sigma' + {}^{(j)}\zeta'$, ${}^{(j+1)}\xi = {}^{(j+1)}\tau' + {}^{(j)}\xi' \neq 0$, ${}^{(j+1)}\zeta \circ {}^{(j+1)}\xi = 0$. If $N_p({}^{(j+1)}\xi) \leq l$, then ${}^{(1)}\beta \in B_l({}^{(j+1)}\beta)$. For ${}^{(j+1)}\tau'_k \neq 0$, $\delta_k({}^{(j+1)}\beta) = 1$, since ${}^{(j+1)}\beta \oplus {}^{(j+1)}\tau'$ and ${}^{(1)}\beta_k > {}^{(j+1)}\beta_k$, we have $\delta_k({}^{(j+1)}\beta) = 0$, a contradiction. The assertion has been proven. And likewise for any pair of ${}^{(i)}\beta$ and ${}^{(j)}\beta$, $(j > i)$. So S' consists of distinct terms.

If there were two identical terms in sequence S, one would construct easily a sequence T, such that the subsequence T'(extract from T as S' from S) contains two identical terms, which is in contradiction with what we proved above. So the sequence S consists of distinct terms, and T_l is Artinian. ∎

Table 2. Examples for Thomas type directions.

Exponent Vectors	Directions		
	T_4 = Thomas	T_3	T_2 =Division II
(2,2,2,1)	(1,1,1,1)	(1,1,1,1)	(1,1,1,1)
(1,1,0,0)	(0,0,0,0)	(1,1,0,0)	(1,1,0,0)
(0,1,0,0)	(0,0,0,0)	(0,1,0,0)	(0,1,1,1)
(1,1,1,1)	(0,0,0,1)	(0,0,0,1)	(1,1,1,1)

5.2. The Janet Type Directions

To generalize the concept of Janet direction on $\mathbf{N}^n$, we introduce *ordered dissection of a positive integer*. An ordered dissection with length l of a positive integer n is a vector $(n_1, n_2, \dots, n_l)$, where the n_i are positive integers, such that

$$n = n_1 + n_2 + \cdots + n_l. \tag{5.17}$$

Set $s_0 = 0, s_k = n_1 + \cdots + n_k, k = 1, \dots, l$.

DEFINITION 5.5. *For any ordered dissection $(n_1, n_2, \dots, n_l)$ of n, we define a direction δ on $\mathbf{N}^n$: for every finite subset Γ of $\mathbf{N}^n$, $\forall \beta \in \Gamma$,*

$$\begin{aligned} &\delta_i^\Gamma(\beta) = 1 \text{ iff } \beta_i = \mathbf{b}_i(\Gamma), && \text{if } i \le s_1, \\ &\delta_i^\Gamma(\beta) = 1 \text{ iff } \beta_i = \mathbf{b}_i\left(\Gamma_{\beta_1 \dots \beta_{s_k}}\right), && \text{if } s_k < i \le s_{k+1}, \end{aligned} \tag{5.18}$$

$k = 1, \dots, l-1$, where $\Gamma_{\beta_1 \dots \beta_m}$ is the same as in Example 2.4.

We will prove that δ is Noetherian, Artinian, and constructive. The direction such defined is called *Janet type direction*, denoted by $J_{(n_1, \dots, n_l)}$.

At first, δ is involutive. $\forall \alpha, \beta \in \Gamma$, if $\alpha + \mu \circ \delta^\Gamma(\alpha) = \beta + \nu \circ \delta^\Gamma(\beta)$, then $\alpha = \beta$. Otherwise, we suppose, without loss of generality, $\alpha_1 = \beta_1, \dots, \alpha_{i-1} = \beta_{i-1}$, $\alpha_i > \beta_i$. If $i \le s_1$, then $\delta_i^\Gamma(\beta) = 0$, whence $\beta_i = \alpha_i + \mu_i \delta_i^\Gamma(\alpha) \ge \alpha_i$, a contradiction. If $s_k < i \le s_{k+1}$, then $\Gamma_{\beta_1 \dots \beta_{s_k}} = \Gamma_{\alpha_1 \dots \alpha_{s_k}}$, whence $\delta_i^\Gamma(\beta) = 0$, a contradiction. So the condition (i) of Definition 2.2 is satisfied. The condition (ii) is clearly satisfied. So δ is involutive. δ is Noetherian, since $B(\Gamma) \cap P^*(\Gamma)$ is a completion of Γ for any finite set Γ.

Secondly, we prove that δ is Artinian. The following lemma is important.

LEMMA 5.6. *For all $\alpha, \beta \in \Gamma$, if $s_{k-1} < i \le s_k$, and*

$$\delta_i^\Gamma(\alpha) = 0, \qquad \alpha + \delta^{(i)} = \beta + \nu \circ \delta^\Gamma(\beta) \tag{5.19}$$

then $\beta_i = \alpha_i + 1$, $\beta_j = \alpha_j$, for $j \ne i$ and $j \le s_k$.

PROOF. For $j \le i - 1$, we claim $\alpha_j = \beta_j$, which implies $\alpha \in \Gamma_{\beta_1 \dots \beta_{s_{k-1}}}$ since $s_{k-1} < i \le s_k$. Otherwise, suppose that $\alpha_1 = \beta_1, \dots, \alpha_{j-1} = \beta_{j-1}$, $\alpha_j > \beta_j$. Then $\delta_j^\Gamma(\beta) = 0$ (similar to the above argument). By (5.19), $\alpha_j = \beta_j$, a contradiction. Similarly, $\alpha_j = \beta_j$ for $i < j \le s_k$ since $\alpha \in \Gamma_{\beta_1 \dots \beta_{s_{k-1}}}$. As for i, if $\nu_i \delta_i^\Gamma(\beta) \ne 0$, then $\delta_i(\beta) = 1$, $\beta_i \le \alpha_i < \mathbf{b}_i(\Gamma_{\alpha_1 \dots \alpha_{s_{k-1}}}) = \mathbf{b}_i(\Gamma_{\beta_1 \dots \beta_{s_{k-1}}})$. We derived $\delta_i^\Gamma(\beta) = 0$, a contradiction. So $\alpha_i + 1 = \beta_i$, and the proof is completed. ∎

By Lemma 5.6, β is a pseudo-divisor of α implies $\alpha <_{\text{lex}} \beta$. Hence, δ is Artinian. The proof of constructivity of δ is similar to the proof for Janet direction; we omit it here.

Table 3. Examples for Janet type directions.

Exponent Vectors	Directions			
	$J_{(3)}$ = Thomas	$J_{(2,1)}$	$J_{(1,2)}$	$J_{(1,1,1)}$ = Janet
(1,1,1)	(0,0,0)	(0,0,1)	(0,0,1)	(0,0,1)
(1,2,1)	(0,1,0)	(0,1,1)	(0,1,1)	(0,1,1)
(2,1,0)	(0,0,0)	(0,0,0)	(0,1,0)	(0,1,0)
(2,1,1)	(0,0,0)	(0,0,1)	(0,1,1)	(0,1,1)
(3,0,2)	(1,0,1)	(1,0,1)	(1,0,1)	(1,0,1)
(3,1,1)	(1,0,0)	(1,0,1)	(1,1,0)	(1,1,1)

6. CONCLUSION

The vector representation of involutive division is useful to study the structure of involutive divisions and to find new divisions. So far we know the Thomas type, Janet type, Pommaret, induced divisions [23] and Division II. All divisions listed above are 'good' in the sense that they are Artinian and constructive. It is interesting to see whether there exist Pommaret type and type (II) divisions.

REFERENCES

1. J. Janet, Sur les systèmes d'equations aux dérivées partielles, *J. Math. Pure et Appl.* **3**, 65–151, (1920).
2. J. Thomas, *Differential Systems*, Amer. Math. Soc., New York, (1937).
3. J.F. Pommaret, *Systems of Partial Differential Equations and Lie Pseudo-Groups*, Gordon & Breach, New York, (1978).
4. C. Riquier, *Les Systems d'Équations Aux Dérivées Partielles*, Gauthies-Villars, Paris, (1910).
5. J.F. Ritt, *Differential Algebra*, Amer. Math. Soc., New York, (1950).
6. F. Schwarz, The Riquier-Janet theory and its application to nonlinear evolution equations, *Physica D* **11**, 243–251, (1984).
7. F. Schwarz, An algorithm for determining the size of symmetry groups, *Computing* **49**, 95–115, (1992).
8. F. Schwarz, Janet bases for symmetry groups, In *Proc. of the 3rd ASCM '98*, pp. 131–133.
9. W.T. Wu, Basic principles of mechanical theorem proving in geometries, *J. of Sys. Sci. & Math. Sci.* **4** (3), 207–235, (1984).
10. W.T. Wu, On the foundation of algebraic differential geometry, *J. Sys. & Math. Sci.* **2** (4), 290–312, (1989).
11. X.S. Gao and S.C. Chou, Solving parametric algebraic systems, In *Proc. ISSAC'92*, pp. 335–341, ACM Press, (1992).
12. X.S. Gao and S.C. Chou, A zero structure theorem for differential parametric systems, *J. Symbolic Computation* **16**, 585–595, (1993).
13. S.C. Chou, *Mechanical Geometry Theorem Proving*, D. Reidel, Dordrecht, Netherlands, (1988).
14. S.C. Chou and X.S. Gao, Automated reasoning in differential geometry and mechanics using the characteristic set method, Part I: An improved version of Ritt-Wu's decomposition algorithm; Part II: Mechanical theorem proving, *J. Automated Reasoning* **10**, 161–189, (1993).
15. D.M. Wang, Reasoning about geometric problems using an elimination method, In *Automated Practical Reasoning: Algebraic Approaches*, (Edited by J. Pfalzgraf and D. Wang), pp. 147–185, Springer, Wien, (1995).
16. Z.M. Li and D.M. Wang, Coherent, regular and simple system in zero decompositions of partial differential system, *J. Sys. & Math. Sci.* **12** (5), 43–60, (1999).
17. H.Q. Zhang and Y.F. Chen, A method of reducing over-determined systems of differential equations, *J. of Dalian University of Technology* **39** (2), 35–42, (1999).
18. W.T. Wu, On the construction of Gröebner basis of a polynomial ideal based on Riquier-Janet theory, In *MM-Preprints, No. 5*, pp. 5–22, MMRC, Academia Sinica, (1990).
19. B. Buchberger, Gröbner bases: An algorithmic method in polynomial ideal theory, In *Recent Trends in Multidimensional Systems Theory*, (Edited by N.K. Bose), D. Reidel, Dordrecht, Netherlands, (1985).
20. A.Yu. Zharkov and Yu.A. Blinkov, Involutive approach to investigating polynomial systems, *Math. Comp. in Simulation* **42**, 323–332, (1996).
21. V.P. Gerdt and Yu.A. Blinkov, Involutive bases of polynomial ideals, *Math. Comp. in Simulation* **45**, 519–541, (1998).
22. V.P. Gerdt and Yu.A. Blinkov, Minimal involutive bases, *Math. Comp. in Simulation* **45**, 543–560, (1998).
23. V.P. Gerdt, Involutive division technique: Some generalizations and optimizations, *CASS '98* (to appear).
24. V.P. Gerdt, Completion of linear differential systems to involution, In *Proc. of CASC '99*, pp. 115–138.
25. V.L. Topunov, Reducing systems of linear differential equations to a passive form, *Acta Appl. Math.* **16**, 191–206, (1989).