AAECC **9**, 117–123 (1998)

AAECC
Applicable Algebra in
Engineering, Communication
and Computing


# On the Relation Between Gröbner and Pommaret Bases

**Daniel Mall**

Department of Mathematics, ETH Zürich, CH-8092 Zürich, Switzerland

Received: October 29, 1997; revised version: April 15, 1998

**Abstract.** The reduced Pommaret basis and the reduced Gröbner basis of an ideal $I$ with respect to a fixed admissible term order $\prec$ differ in general. A necessary and sufficient criterion for the coincidence of these bases is given.

**Keywords:** Pommaret bases, Gröbner bases, Stable ideals, Borel ideals.

## 1 Introduction

A Pommaret basis is a special kind of a, in general, non-reduced Gröbner basis. The recent interest in this subject comes from the fact that there is an algorithm - the Zharkov-Blinkov (ZB) algorithm - which computes Pommaret bases, i.e., Gröbner bases, and which sometimes performs better than Buchberger's algorithm. From a theoretical viewpoint Pommaret bases establish links to classical subjects, e.g., differential equations (cf. [9]). Recently, Apel (cf. [1]) found interesting connections to the theory of *non-commutative* Gröbner bases, and Gerdt and Blinkov (cf. [7]) and Apel (cf. [2]) introduced and treated generalisations of Pommaret bases.

The purpose of this note is to give an intrinsic description of the ideals and admissible term orders for which reduced Pommaret bases and reduced Gröbner bases coincide.

More precisely, we will prove that, given an ideal $I$ and an admissible term order $\prec$, the reduced Pommaret basis $P_\prec(I)$ and the reduced Gröbner basis $R_\prec(I)$ coincide if and only if the initial ideal $in_\prec(I)$ is *stable*.

It turns out that the initial ideal with respect to an arbitrary admissible term order is stable for any *homogeneous* ideal in *generic* position. However, the reader should observe that the ideals met in practice are, in general, generated by sparse polynomials, and are not in generic position.

We gather the notation and some definitions (cf. [12] and [9]).

*Notation:* We denote the integers $\{0, 1, \ldots\}$ with $\mathbb{N}$.

Let $k$ be a field and $S := k[x_0, \ldots, x_\ell]$. The vector space of the homogeneous polynomials of degree $d$ in $S$ is denoted by $S_d$. If $I \subset S$ is a homogeneous ideal we denote $I \cap S_d$ by $I_d$ and for a set $F$ of homogeneous polynomials $F_d$ denotes the set $F \cap S_d$.

The set of terms $\{x_0^{n_0} \cdot \ldots \cdot x_\ell^{n_\ell} \mid n_i \in \mathbb{N}, i = 0, \ldots, \ell\}$ is denoted by $T^{\ell+1}$ (or simply by $T$) and the set of admissible term orders by $TO_{\ell+1}$ or $TO$ (see for example Robbiano (cf. [10]) and Weispfenning (cf. [11])).

If $f \in S$ then $supp(f)$ denotes the set of *terms* occurring with non-zero constant coefficient in the distributive normal form of $f$; for $\prec \in TO_{\ell+1}$, $in_\prec f$ denotes the greatest *term* in $supp(f)$ with respect to $\prec$ and $cf(f, t)$ denotes the coefficient of the term $t$ in $f$. For a set of polynomials $G$, $in_\prec G$ is the set $\{in_\prec f \mid f \in G\}$, and the ideal generated by $G$ is denoted $\langle G \rangle$.

$R_\prec(I)$ denotes the reduced Gröbner basis of $I$ with respect to the admissible term order $\prec$.

We denote by $NF(f, G)$ the set of the usual normal forms of $f$ modulo $G$ (cf. [3]).

We assume a fixed ordering on the set of variables, e.g., $x_0 > \cdots > x_\ell$. A variable $x_i$ is called *multiplicative*, $x_i \in Mult_>(u)$, for the term $u \in T$ with respect to the ordering $>$ if $x_i$ is smaller than or equal to the smallest variable occurring in $u$. Otherwise, $x_i$ is *non-multiplicative* for $u$ with respect to $>$, $x_i \in Nonmult_>(u)$. An admissible term order $\prec \in TO$ is always assumed to induce the ordering $>$ on the set of variables.

Let $t, u, v \in T^{\ell+1}$. We write $t \sim u \times v$ if $t = uv$ and if all variables in $v$ are multiplicative for $u$ or if $v = 1$. If $g \in S$, we write also $gv \sim g \times v$ if $in_\prec(g)v \sim in_\prec(g) \times v$.

A term $u$ is called a *Pommaret divisor* for the term $t$ if there exists a term $v$ such that $t \sim u \times v$.

We say that a polynomial $f$ is *reducible* to $h$ modulo $G$ *in the sense of Pommaret* if there exists an element $g \in G$ and a term $v$ such that $in_\prec(g) \cdot v \sim in_\prec(g) \times v$ and $h = f - a/b \cdot gv$ with $a := cf(f, in_\prec(g)v) \neq 0$ and $b := cf(g, in_\prec(g))$. The step from $f$ to $h$ is called a Pommaret reduction (P-reduction).

A polynomial $f$ is in *Pommaret normal form* modulo $G$ if for each term in $f$ there are no Pommaret divisors in $\{in_\prec(g) \mid g \in G\}$. A polynomial $h$ is a *Pommaret normal form* of $f$ modulo $G$ if there exists a chain of Pommaret reductions from $f$ to $h$ and $h$ is in Pommaret normal form. We denote the set of all Pommaret normal forms of $f$ modulo $G$ by $NF_P(f, G)$.

A set of polynomials $G$ is *reduced* (in the sense of Pommaret) if for all $g \in G$ we have $\{g\} = NF_P(g, G \backslash \{g\})$.

The *prolongation* of a polynomial $g \in k[x_0, \ldots, x_\ell]$ by the variable $x_i$ is the product $x_i g$. If $x_i \in Nonmult(in_\prec(g))$ then the prolongation is called non-multiplicative, otherwise multiplicative.

## 2 Pommaret bases

In the following we assume that every admissible term order $\prec \in TO$ induces the ordering $x_\ell \prec \cdots \prec x_0$ on the set of variables.

**Definition 2.1.** *A set of polynomials $G$ is called a (reduced)* Pommaret basis *(of $\langle G\rangle$) with respect to $\prec \in TO$ (cf.* [12] *and* [9]*) if it is reduced in the sense of Pommaret and*

$$\forall g \in G\ \forall x \in Nonmult(in_\prec(g))\ NF_P(g \cdot x, G) = \{0\}. \tag{1}$$

There are examples of ideals and admissible term orders with no finite reduced Pommaret basis.

**Definition 2.2.** *Let $\prec \in TO$ and let $I \subset k[x_0, \ldots, x_\ell]$ be a monomial ideal. A generating set $M := \{m_1, m_2, \ldots\} \subset T$ of $I$ is called P-generating for $I$ with respect to $\prec$ if for all terms $t \in I$ there is a $m \in M$ and a term $v \in T$ such that $t \sim m \times v$ with respect to $\prec$.*

The reader should observe that only the ordering of the variables induced by $\prec$ is relevant.

**Example 2.3.** *Let $I \subset k[x, y]$ be the monomial ideal generated by the set $M := \{x^4, x^2y\}$ and let $\prec \in TO$ be an admissible term order with $y \prec x$. Then $M$ is not a P-generating set for $I$ with respect to $\prec$, since there are no $m \in M$ and $v \in T$ such that $x^3y \sim m \times v$ with respect to $\prec$. On the other hand, the set $\tilde{M} := M \cup \{x^3y\}$ is P-generating for $I$ with respect to $\prec$.*

**Lemma 2.4.** *Let $I \in k[x_0, \ldots, x_\ell]$ be a monomial ideal and $\prec \in TO$ an admissible term order. Then there is a unique minimal P-generating set for $I$ with respect to $\prec$.*

*Proof.* Let $L$ be the minimal generating set consisting of monomials of $I$ and $d_1$ the smallest degree of an element in $L$. For any $d \geq d_1$, we construct a set $M_d$ of terms of degree $d$ such that $M := \cup_{d \geq d_1} M_d$ is the wanted minimal P-generating set for $I$ with respect to $\prec$. We put $M_{d_1} := L_{d_1}$. Assume we have already constructed $M_d$ for $d < d_0$ with $d_1 < d_0$. Let $M_{d_0}$ be the set consisting of the elements $t$ of $I_{d_0} \cap T$ for which there are no $m \in \cup_{d_1 \leq d < d_0} M_d$ and no $v \in T$ such that $t \sim m \times v$ with respect of $\prec$. The set $M$ is P-generating and it is minimal with respect to this property. QED

**Remark 2.5.** *There are monomial ideals and admissible term orders such that their minimal P-generating sets are infinite. An example of a monomial ideal with an infinite minimal P-generating set is $\langle x^2y\rangle \subset k[x, y]$ if $\prec \in TO$ is chosen such that $y \prec x$.*

**Lemma 2.6.** *Let there be given an arbitrary ideal $I \in k[x_0, \dots, x_\ell]$ and an admissible term order $\prec \in TO$. Then the initial terms of the elements of a reduced Pommaret basis of $I$ with respect to $\prec$ form the minimal P-generating set with respect to $\prec$ for the initial ideal $in_\prec(I)$.*

*Proof.* Let $G$ be a reduced Pommaret basis of $I$ with respect to $\prec$. The lemma is proven, if we can show that for any $in_\prec(f)$ with $f \in I$ there is an element $g \in G$ such that $in_\prec(f) \sim in_\prec(g) \times v$ with $v \in T$. Since $G$ is a Gröbner basis, there is a $t \in T$ and an element $g \in G$ such that $in_\prec(f) = t\ in_\prec(g)$. There are $t_{01}, t_{02} \in T$ such that $t = t_{01}t_{02}$ with $t_{01}$ a product of non-multiplicative variables for $in_\prec(g)$ with respect to $\prec$ and $t_{02}$ a product of multiplicative variables for $in_\prec(g)$ with respect to $\prec$. Hence, we have $in_\prec(f) \sim (t_{01}\ in_\prec(g)) \times t_{02}$. Let $x_i$ be the smallest variable in $t_{01}$. Since $G$ is a Pommaret basis, there is a $g_1 \in G$ such that $x_i\ in_\prec(g) \sim in_\prec(g_1) \times t_{12}$. The variables occurring in $t_{02}$ are multiplicative for $in_\prec(g_1)$. This implies that we obtain $in_\prec(f) \sim (t_{01}/x_i\ in_\prec(g_1)) \times (t_{12}t_{02})$. Hence, by induction over the number of variables in $t_{01}$, we eventually find a $g_r \in G$ with $in_\prec(f) \sim in_\prec(g_r)) \times (t_{r2} \cdots t_{12}t_{02})$. QED

**Proposition 2.7.** *Let $I \in k[x_0, \dots, x_\ell]$ be an arbitrary ideal and $\prec \in TO$ an admissible term order. Then there is at most one reduced Pommaret basis of $I$ with respect to $\prec$ denoted by $P_\prec(I)$.*

*Proof.* Let $P_1(I)$ and $P_2(I)$ be two different reduced Pommaret bases of $I$ with respect to $\prec$. By Lemma 2.6 and Lemma 2.4 the sets of the initial terms of the elements of the reduced Pommaret bases $P_1(I)$ and $P_2(I)$ coincide. Since, by assumption, $P_1(I)$ and $P_2(I)$ are different, there are $g_1 \in P_1(I)$ and $g_2 \in P_2(I)$ with $in_\prec(g_1) = in_\prec(g_2)$ and $g_1 \neq g_2$. This implies that $g_1 - g_2 \in I$ and cannot be reduced (in the sense of Pommaret) neither by $P_1(I)$ nor by $P_2(I)$. This is a contradiction, since any Pommaret basis can reduce any element of $I$ to zero in the sense of Pommaret (cf. Theorem 6 in [12]). QED

**Definition 2.8.** *Let $\prec \in TO$. A monomial ideal is called* P-generated *with respect to $\prec$ if its minimal generating set is P-generating for $I$ with respect to $\prec$.*

**Definition 2.9.** *A set of terms $M \subset T$ is called* stable *with respect to $\prec$ if for all terms $t \in M$ the following holds: if $x_j$ is the smallest variable occuring in $t$ then we have $x_i t/x_j \in M$ for all $x_i \in Nonmult(t)$.*

*A monomial ideal $I$ is called* stable *with respect to $\prec$ if $I \cap T$ is stable with respect to $\prec$.*

**Remark 2.10.** *Let $I$ be a monomial ideal and $M$ be its minimal generating system. The ideal $I$ is stable with respect to $\prec$ if and only if for all $t \in M$*

*it follows that* $x_i t/x_j \in I$ *for* $x_j$ *the smallest variable occuring in* $t$ *and all* $x_i \in Nonmult(t)$.

**Example 2.11.** *The monomial ideal* $\langle x^3, x^2y, xy^2, y^3, y^2z\rangle \subset k[x, y, z]$ *is stable with respect to every admissible term order* $\prec$ *with* $z \prec y \prec x$.

**Remark 2.12.** *The above definition is a slight generalisation of the stable ideals introduced by Eliahou and Kervaire (cf.* [5]*).* Borel ideals *(cf. Definition 2.17) build a strictly smaller class of monomial ideals than the stable ideals, if the characteristic of the field* $k$ *is zero.*

**Lemma 2.13.** *Let* $I \in k[x_0, \ldots, x_\ell]$ *be a monomial ideal and* $\prec \in TO$ *an admissible term order. Then* $I$ *is stable with respect to* $\prec$ *if and only if* $I$ *is P-generated with respect to* $\prec$.

*Proof.* Let $M$ be a minimal generating set of $I$.
a) Assume that $I$ is stable. We show that $M$ is a P-generating set. Let $t \in I$ a term. This implies that there is a $m \in M$ and a term $v \in T$ such that $t = mv$. Let $x_j$ be the smallest variable with respect to $\prec$ which occurs in $m$. If there is a $x_i \in Nonmult(m)$ which occurs in $v$ then we write

$$t = (x_i m/x_j)(x_j v/x_i). \tag{2}$$

Since $I$ is stable $\tilde{t} := x_i m/x_j \in I$. Hence there is a $\tilde{m} \in M$ and a $\tilde{v} \in T$ such that $\tilde{t} = \tilde{m}\tilde{v}$ and $t = \tilde{m}\tilde{v}(x_j v/x_i)$. We can repeat this process and after finitely many steps we obtain $\hat{m} \in M$ and $\hat{v} \in T$ such that $t \sim \hat{m} \times \hat{v}$.

b) Assume that $I$ is P-generated. Let $t = x_0^{a_0} \cdots x_j^{a_j} \in I$ with $x_0 \succ \cdots \succ x_j$ and $a_j > 0$. There is a $m \in M$ and a term $v \in T$ such that $t \sim m \times v$. This implies that $x_j | v$ unless $v = 1$. We have to show that if $i < j$ then $x_i t/x_j \in I$. This follows immediately if $v \neq 1$. Assume $t = m$. Then $x_i \in Nonmult(m)$ and by hypothesis there is a $\tilde{m} \in M$ and a $\tilde{v} \in T$ such that $x_i t \sim \tilde{m} \times \tilde{v}$. Since $M$ is a minimal generating set it follows that $\tilde{v} \neq 1$ and the claim follows. QED

Let $F := \{f_1, \ldots, f_s\}$ be a set of arbitrary polynomials in $k[x_0, \ldots, x_\ell]$ and $\prec$ an admissible term order. If there is a Pommaret reduction of a polynomial $g \in k[x_0, \ldots, x_\ell]$ with respect to $F$ and $\prec$, then there is a Buchberger reduction of $g$ with respect to $F$ and $\prec$. The converse is not true in general.

**Lemma 2.14.** *Let* $F := \{f_1, \ldots, f_s\}$ *be a set of arbitrary polynomials in* $k[x_0, \ldots, x_\ell]$ *and* $\prec$ *an admissible term order such that* $\langle in_\prec(F)\rangle$ *is stable. Then for any Buchberger reduction at a term of a polynomial* $g \in k[x_0, \ldots, x_\ell]$ *with respect to* $F$ *there is a Pommaret reduction with respect to* $F$ *at the same term of* $g$.

*Proof.* Let $g \longrightarrow g - \alpha t f_i$ be a Buchberger reduction with $\alpha \in k, t \in T$. We have to show that there are $v \in T$, $f_j \in F$ such that $t\, in_\prec(f_i) \sim in_\prec(f_j) \times v$. By Lemma 2.13, $\langle in_\prec(F)\rangle$ is P-generated and the claim follows. QED

**Theorem 2.15.** *Let $I \in k[x_0, \dots, x_\ell]$ be an arbitrary ideal and $\prec \in TO$ an admissible term order. Then the reduced Gröbner basis $R_\prec(I)$ is a Pommaret basis if and only if $in_\prec(I)$ is stable with respect to $\prec$.*

*Proof.* The *only if* part follows from Lemma 2.13 and Lemma 2.6. Without loss of generality, we may assume that $x_\ell \prec \cdots \prec x_0$. We have to show that for all $g \in R_\prec(I)$ and all $x_i \in Nonmult(in_\prec(g))$ it follows that $NF_P(x_i g, R_\prec(I)) = \{0\}$. We put $h := x_i g$. Since $R_\prec(I)$ is a Gröbner basis we have $NF(h, R_\prec(I)) = \{0\}$. We show that $NF_P(h, R_\prec(I)) = \{0\}$. Assume that there is $f \in NF_P(h, R_\prec(I))$ with $f \neq 0$. Since $NF(f, R_\prec(I)) = \{0\}$ there are $g \in R_\prec(I), t \in T^{\ell+1}$ and $\alpha \in k$ such that $\tilde{f} := f - \alpha t g$ is a reduction in the usual sense. By Lemma 2.14, there must be a $\tilde{g} \in R_\prec(I)$ and a term $\tilde{t}$ such that $t in_\prec(g) \sim in_\prec(\tilde{g}) \times \tilde{t}$. Hence there is a Pommaret reduction of $f$ and $f$ is not a Pommaret normal form modulo $R_\prec(I)$, a contradiction. QED

**Remark 2.16.** *Given an ideal $I \in k[x_0, \dots, x_\ell]$ and an admissible term order $\prec \in TO$. As V. Gerdt pointed out to the author, Proposition 3.11 in* [7] *implies that $R_\prec(I)$ is a Janet basis if $in_\prec(I)$ is stable with respect to $\prec$ (cf. Definition 5 of Section 3 in* [7]*). Moreover, in this situation any two reduced (in the sense of Pommaret) Janet bases of $I$ with respect to $\prec$ coincide.*

*Notation.* We denote the set of all reduced Gröbner bases of an ideal $I$ by $\mathscr{R}(I)$ and the set of all finite reduced Pommaret bases by $\mathscr{P}(I)$.

We let the general linear group $GL(\ell+1, k)$ operate on $S$ in the following canonical way. If $g = (g_{lk}) \in GL(\ell+1, k)$ then

$$x_i \longmapsto g x_i := \sum_{j=0}^{\ell} g_{ji} x_j. \tag{3}$$

If $I \subset k[x_0, \dots, x_\ell]$ is an ideal and $g \in GL(\ell+1)$, we denote the ideal $\langle \{gf \mid f \in I\}\rangle$ by $gI$.

The Borel subgroup of the upper triangular matrices is denoted by $B(\ell+1)$.

**Definition 2.17.** *An ideal $J \subset k[x_0, \dots, x_\ell]$ is called a Borel ideal, if it is invariant under the action of the Borel subgroup $B(\ell+1) \subset GL(\ell+1, k)$, i.e., for all $g \in B(\ell+1)$ we have $gI \subseteq I$.*

**Remark 2.18.** *The importance of the notion of Borel ideal comes from the following well-known result by Galligo and Grauert (cf. e.g.,* [6],[8]*): for any homogeneous ideal $I$ there is a Zariski-open set in $GL(k, \ell+1)$, denoted by*

*$GT(I)$, such that for any linear transformation $g \in GT(I)$ the finitely many initial ideals of $I$ are Borel ideals which are independent from the chosen linear transformation $g \in GT(I)$. Roughly speaking, the finitely many initial ideals of a homogeneous ideal in "generic coordinates" are well defined Borel ideals (cf. e.g.,* [4] *p.348).*

From Remark 2.18, Remark 2.12 and Theorem 2.15, the following statement follows.

**Corollary 2.19.** *Let $k$ be a field of characteristic zero. If $I \in k[x_0, \ldots, x_\ell]$ is a homogeneous ideal in generic position, then $\mathscr{R}(I) = \mathscr{P}(I)$. In particular, for all $\prec \in TO$ there exists a finite Pommaret basis of $I$ and the set of all finite reduced Pommaret bases is finite.*

**Remark 2.20.** *If $k$ is a field of characteristic $p > 0$, then there are Borel ideals which are not stable, e.g., $I = \langle x^p, y^p \rangle \subset k[x, y, z]$.*

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