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Combinatorial Aspects of Total Weight Orders over Monomials of Fixed Degree

ANDREA VIETRI
Department of Mathematics, Università La Sapienza, Roma, Piazzale A. Moro 2, 00185 Roma, Italy. e-mail: vietri@mat.uniroma1.it

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Abstract. Among all the restrictions of weight orders to the subsets of monomials with a fixed degree, we consider those that yield a *total* order. Furthermore, we assume that each weight vector consists of an increasing tuple of weights. Every restriction, which is shown to be achieved by some monomial order, is interpreted as a suitable linearization of the poset arising by the intersection of all the weight orders. In the case of three variables, an enumeration is provided. For a higher number of variables, we show a necessary condition for obtaining such restrictions, using deducibility rules applied to homogeneous inequalities. The logarithmic version of this approach is deeply related to classical results of Farkas type, on systems of linear inequalities. Finally, we analyze the linearizations determined by sequences of prime numbers and provide some connections with topics in arithmetic.

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1. Introduction

We denote by $\mathbf{N}$ the set of nonnegative integers. If $k \in \mathbf{N}$, the elements of $\mathbf{N}^{k+1}$ can be naturally put in one-to-one correspondence with the monomials in $k+1$ variables $x_0, \dots, x_k$ by associating $\underline{u} \in \mathbf{N}^{k+1}$ to $x^{\underline{u}} := x_0^{u_0}, x_1^{u_1}, \dots, x_k^{u_k}$. In this paper, we analyze certain linear orders defined over the monomials $x^{\underline{u}}$ of a fixed degree, in a fixed number of variables. In particular, we deal with *monomial orders* and *weight orders*.

DEFINITION 1.1. If F is a field, a *monomial order* on $F[x_0, \dots, x_k]$ is any relation $<$ on $\mathbf{N}^{k+1}$ or, equivalently, any relation on the set of monomials $\{x^{\underline{u}}, \underline{u} \in \mathbf{N}^{k+1}\}$, satisfying:

(i) $<$ is a total order.
(ii) If $\underline{u} < \underline{v}$ and $\underline{z} \in \mathbf{N}^{k+1}$, then $\underline{u} + \underline{z} < \underline{v} + \underline{z}$.
(iii) $<$ is a well-ordering.

Monomial orders are basic ingredients in the theory of Gröbner bases of polynomial ideals (see, for example, [1]).

DEFINITION 1.2. Let $\underline{r} := (r_0, \ldots, r_k)$ be a vector in $\mathbf{R}^{k+1}$, whose coordinates are positive. Then, for any $\underline{u}, \underline{v} \in \mathbf{N}^{k+1}$, define

$$\underline{u} <_{\underline{r}} \underline{v} \quad \text{iff } \underline{r} \cdot \underline{u} < \underline{r} \cdot \underline{v}.$$

The relation $<_{\underline{r}}$ is called the *weight order* determined by $\underline{r}$.

If $r_0, \ldots, r_k$ are linearly independent over $\mathbf{Q}$, then $<_{\underline{r}}$ is a total order, and conversely. In this case, as a consequence of Dickson's Lemma, we have that $<_{\underline{r}}$ is in fact a monomial order (see, for example, [1], pp. 69–72). Robbiano ([5]) has shown that every monomial order on $\mathbf{N}^s$ is the lexicographic product of s weight orders. Therefore, weight orders contain much information related to monomial orders. In this paper, we will be concerned with the weight orders determined by some *increasing* sequence of weights; in other words, if $\underline{r}$ is the tuple determining the weight order, then we assume that $r_i < r_j$ for any $i < j$. Our object of study is the intersection over all possible choices of such weight orders restricted to each subset of monomials of a fixed degree h in $k+1$ variables. Using the above rephrasing, in place of monomials we consider the corresponding subset of $(k+1)$-tuples, namely $D_h^k \subset \mathbf{N}^{k+1}$. Theorem 2.3 characterizes each intersection as a partial order $(D_h^k, \ll)$ which has a simple combinatorial definition. Among all the linearizations of a fixed D_h^k we focus on the restrictions of weight orders. These particular linearizations, which we term *β-linearizations*, turn out to be achieved by some monomial order whose restriction over the monomials of lower degree is uniquely determined by the linearization itself. Subsequently, the β-linearizations are interpreted as system of inequalities of the form

$$x_0^{u_0}, x_1^{u_1}, \ldots, x_k^{u_k} < x_0^{v_0}, x_1^{v_1}, \ldots, x_k^{v_k},$$

with $\sum_i u_i = \sum_i v_i$.

In Section 3 we enumerate all the β-linearizations of a fixed D_h^2. The case $k \geq 3$ is investigated in Sections 4 and 5 with the help of a further class of posets. Although our approach does not provide any enumeration in this case, it leads to a necessary condition for the existence of such linearizations. This condition can be easily checked in the cases D_2^3, D_3^3 and D_4^3. More precisely, we exhibit a class of inferential rules which generate all the inequalities deducible from a single inequality. The idea of describing inferences by means of certain rules has been extensively developed, so far, in the case of linear systems (see, for example, [2, 8, 10]). Since the logarithmic version of the above inequalities consists of linear inequalities, our necessary condition is in fact derivable from a classical result, namely the Kuhn–Fourier Theorem (see [8]). The cited theorem provides a characterization for solvable systems of linear inequalities (for this reason, the theorem is classified as a result of 'Farkas type'). In the present work, we provide an independent proof of the above necessary condition. Our argument yields, as an immediate consequence, a better understanding of deducibility rules in our specific context. In Section 6, we show that prime numbers in place of real numbers are enough to define all the weight orders $<_{\underline{r}}$. In Section 7, we describe some curious symmetries of certain β-linearizations, which have been a further motivation to develop the present analysis.

In the related case of binary strings, monomial orderings have been studied by Maclagan (see [3]), obtaining a complete enumeration for smaller values of the length. A concept similar to β-linearization, namely the *coherent Boolean order*, has been investigated in the cited paper. Monomial orderings on binary strings are a basic ingredient for the construction of Gröbner bases over exterior algebras.

2. The Posets D_h^k and their β-Linearizations

We define a class of finite partially ordered sets $(D_h^k, \ll)$.

DEFINITION 2.1. If $h \in \mathbf{N}^+$ and $k \in \mathbf{N}$, D_h^k stands for the set of $(k+1)$-tuples $\underline{u} = (u_0, \dots, u_k)$ such that $u_i \in \mathbf{N}$ for all i, and $\sum_i u_i = h$. Given $\underline{u}$, $\underline{u}'$ in D_h^k, we write $\underline{u} \ll^* \underline{u}'$ (equivalently, $\underline{u}' \gg^* \underline{u}$) if $u_i = u_i' + 1$, $u_{i+1} = u_{i+1}' - 1$ for some index $i \leq k-1$, and $u_j = u_j'$ otherwise. The relation $\ll^*$ is extended by reflexivity and transitivity so as to obtain a relation $\ll$. If $\underline{u} \ll \underline{u}'$ (resp. $\underline{u} \gg \underline{u}'$), we say that $\underline{u}$ is *under* (resp. *over*) $\underline{u}'$.

By defining $r(\underline{u}) := \sum_{0 \leq i \leq k} i \cdot u_i$, we have $\underline{u} \ll^* \underline{u}' \Rightarrow r(\underline{u}) = r(\underline{u}') - 1$, whence $\ll$ is antisymmetric. Furthermore, $r(h, 0, \dots, 0) = 0$, which means that $r(\underline{u})$ counts the elementary steps connecting $\underline{u}$ to the minimum; in particular, their number does not depend on the path chosen. Thus, each $(D_h^k, \ll)$ is a ranked poset. It can be easily shown that each D_h^k is a lattice. The proof of this fact is postponed to Section 5. Throughout this paper we will often denote vectors by sequences of entries with no parentheses, nor commas (e.g. $u_0 u_1 \dots u_k$). In Figures 1 and 2, the Hasse diagrams of D_2^2, D_2^3 and D_4^3 are shown (from left to right, instead of the usual top-down representation).

Notice that $(u_0, \dots, u_k) \ll (v_0, \dots, v_k)$ if and only if $(u_k, u_{k-1}, \dots, u_0) \gg (v_k, v_{k-1}, \dots, v_0)$. The next result is a basic tool for proving the subsequent theorem.

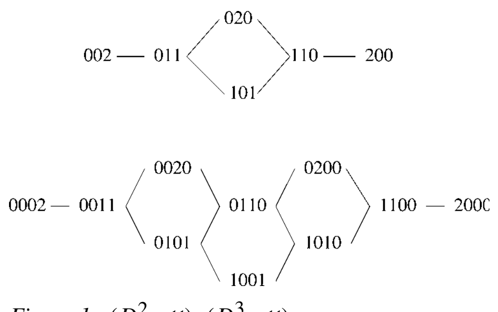

Figure 1. $(D_2^2, \ll)$, $(D_2^3, \ll)$.

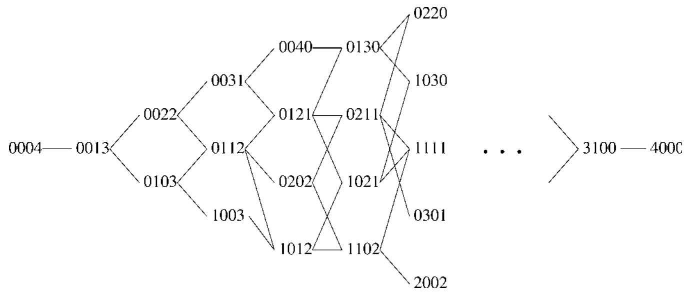

Figure 2. $(D^3_4, \ll)$.

LEMMA 2.2. *$\underline{u} \ll \underline{v}$ if and only if $\sum_{j\geq i} u_j \leq \sum_{j\geq i} v_j$ for all $i \leq k-1$.*

Proof. ($\Rightarrow$) Suppose that $\underline{u} \ll \underline{v}$. Then, there exist $\underline{z}_0, \ldots, \underline{z}_n$ such that

$$\underline{u} = \underline{z}_0 \ll^* \underline{z}_1 \ll^* \cdots \ll^* \underline{z}_n = \underline{v}.$$

If $n = 0$ there is nothing to prove. Otherwise, let $z_{\gamma,j}$ denote the jth entry of $\underline{z}_\gamma$. Since the definition of $\ll^*$ implies that $\sum_{j\geq i} z_{n-1,j} \leq \sum_{j\geq i} z_{n,j}$ for any $i \leq k-1$, by induction on n we obtain $\sum_{j\geq i} z_{0,j} \leq \sum_{j\geq i} z_{n,j}$ for all $i \leq k-1$.

($\Leftarrow$) If $\underline{u} \neq \underline{v}$, then consider the rightmost coordinate where $\underline{u}$ and $\underline{v}$ differ. Let i be the index for this coordinate. The inequality $\sum_{j\geq i} u_j \leq \sum_{j\geq i} v_j$ implies that $u_i = v_i - t$ for some $t > 0$. Thus, we may replace $\underline{v}$ by $\underline{v}'$ such that $v'_i = v_i - t$, $v'_{i-1} = v_{i-1} + t$ and $v'_j = v_j$ for $j \neq i, i-1$. Now $\underline{v}'$ still satisfies the hypothesis, but this vector agrees with $\underline{u}$ in more coordinates than $\underline{v}$, and $\underline{v}' \ll \underline{v}$. If $i = 2$, then $\underline{v}' = \underline{u}$, so we are done. For larger i, we use induction on the position i defined above, since we may replace $\underline{v}$ by $\underline{v}'$ to reduce from i to $i-1$, and the transitivity of $\ll$ yields $\underline{u} \ll \underline{v}$. □

THEOREM 2.3. *Given $\underline{u} \neq \underline{v}$ in D^k_h, $\underline{u} \ll \underline{v}$ if and only if*

$$\prod_{i=0}^{k} q_i^{u_i} < \prod_{i=0}^{k} q_i^{v_i} \tag{1}$$

for every increasing sequence of $k+1$ real numbers $0 < q_0 < q_1 < \cdots < q_k$.

Proof. Suppose that $\underline{u} \ll \underline{v}$. Let $n > 0$ be such that $\underline{u} = \underline{z}_0 \ll^* \underline{z}_1 \ll^* \cdots \ll^* \underline{z}_n = \underline{v}$. We reason by induction on n. Let us set $\underline{w} := \underline{z}_{n-1}$ and define a as the unique index such that $v_a = w_a + 1$, $v_{a-1} = w_{a-1} - 1$; by also using the inductive hypothesis we get

$$\prod_i q_i^{u_i} \leq \prod_i q_i^{w_i} < \prod_i q_i^{w_i} \cdot \frac{q_a}{q_{a-1}} = \prod_i q_i^{v_i}.$$

Conversely, it is enough to prove that if $\underline{u}$ and $\underline{v}$ are incomparable, then there exists a sequence $0 < q_0 < \cdots < q_k$ such that

$$\prod_i q_i^{u_i} > \prod_i q_i^{v_i}. \tag{2}$$

Indeed, if $\underline{u} \gg \underline{v}$, then we use the previous argumentation to get (2). Let b be the greatest index such that $u_b \neq v_b$ and assume that $u_b > v_b$. A sequence satisfying (2) is built as follows: fix an increasing sequence $\{q_i, 0 \le i < b\}$; then, choose $q_b > q_{b-1}$ in such a way that $q_b > \prod_{i=0}^{b-1} q_i^{v_i - u_i}$ and, if necessary, complete the sequence with any increasing sequence of numbers greater than q_b. By doing so, we obtain a suitable sequence, because

$$\prod_{i=0}^{b-1} q_i^{u_i} \cdot q_b^{u_b} > \prod_{i=0}^{b-1} q_i^{v_i} \cdot q_b^{u_b - 1} \geq \prod_{i=0}^{b-1} q_i^{v_i} \cdot q_b^{v_b}$$

and for each index $j > b$ (if any) $u_j = v_j$. Now, assume that $u_b < v_b$. Let i be some index contradicting the assertion of Lemma 2.2. Write U, V, respectively, in place of $\sum_{j \ge i} u_j$, $\sum_{j \ge i} v_j$ and define $0 < q_0 < \cdots < q_{i-1}$ in any way (notice that $i \geq 1$). Let $\delta > 1$ be such that

$$\delta \prod_{j<i} q_j^{u_j} > \prod_{j<i} q_j^{v_j}. \tag{3}$$

Using the hypothesis $U > V$, choose q_i in such a way that $q_i^{\frac{U}{V}-1} > \sqrt[V]{\delta}$, or equivalently $q_i^{\frac{U}{V}} / \sqrt[V]{\delta} > q_i$. Now, if necessary, complete the sequence under the condition

$$q_i < q_{i+1} < \cdots < q_k \leq \frac{q_i^{\frac{U}{V}}}{\sqrt[V]{\delta}}.$$

We get

$$\prod_{j \ge i} q_j^{u_j} \geq q_i^U \geq \delta q_k^V \geq \delta \prod_{j \ge i} q_j^{v_j},$$

and also using (3), we can conclude. □

As a consequence of the above theorem we obtain the following characterization of each $(D_h^k, \ll)$ in terms of weight orders.

COROLLARY 2.4. *For every fixed D_h^k, the relation $\ll$ is the intersection of all the weight orders $<_{\underline{r}}$, restricted to D_h^k, such that $r_0 < \cdots < r_k$.*

Proof. If $\underline{u} \ll \underline{v} \neq \underline{u}$ and $<_{\underline{r}}$ is such a weight order, then

$$\prod_i (e^{r_i})^{u_i} < \prod_i (e^{r_i})^{v_i} \Rightarrow \sum_i r_i u_i < \sum_i r_i v_i \Rightarrow \underline{u} <_{\underline{r}} \underline{v},$$

where the first inequality is due to Theorem 2.3. On the other hand, if $\underline{u} \not\ll \underline{v}$, then the same theorem enables us to find an increasing sequence of real numbers $\{s_i\}$ such that $\prod_i s_i^{u_i} \geq \prod_i s_i^{v_i}$. Furthermore, we may assume that $s_0 > 1$, because any sequence $\{\alpha s_i\}$, with $\alpha > 0$, clearly induces the same inequality of $\{s_i\}$. Since $\underline{u} \not<_{(\log s_0,\ldots,\log s_k)} \underline{v}$, the vector $\underline{u}$ does not precede $\underline{v}$ in the intersection of all the weight orders. $\square$

If $h = 1$ or $k \leq 1$, then D_h^k is easily seen to be totally ordered by $\ll$. Let us suppose that $h, k \geq 2$ and consider the set

$$W_h^k := \{\{\underline{u}, \underline{v}\} \in D_h^k \times D_h^k \ : \ \underline{u} \not\ll \underline{v} \ \wedge \ \underline{u} \not\gg \underline{v}\}.$$

If $L = \{\underline{u}, \underline{v}\} \in W_h^k$ and $\alpha \in \mathbf{N}^+$, then we define $\alpha L := \{\alpha\underline{u}, \alpha\underline{v}\}$. Notice that $\alpha L \in W_{\alpha h}^k$, by Lemma 2.2. Furthermore, we set $\overline{L} := \{\underline{v}, \underline{u}\}$ and say that $\overline{L}$ is equal to L with *reversed orientation*.

DEFINITION 2.5. A *β-linearization* of $(D_h^k, \ll)$ is an extension of $\ll$ to a total order, obtained by defining for each $\{\underline{u}, \underline{v}\} \in W_h^k$

$$\underline{u} \ll \underline{v} \Leftrightarrow \prod_{i=0}^{k} q_i^{u_i} < \prod_{i=0}^{k} q_i^{v_i},$$

where $\underline{q}$ is a fixed increasing sequence of $k+1$ positive real numbers, which necessarily yields only *strict* inequalities. We say that $\underline{q}$ *induces* the β-linearization $\lambda_{\underline{q}}$.

Theorem 2.3 implies that the notion of β-linearization is well-defined. Since the multiplication of $\underline{q}$ by any positive number does not change the corresponding β-linearization, we may assume that $q_0 > 1$. Thus, we see that $\lambda_{\underline{q}}$ corresponds to the weight order $<_{\log q_0,\ldots,\log q_k}$, restricted to D_h^k. Observe that the linear independence over $\mathbf{Q}$ of $\{\log q_i\}$ translates to

$$(q_0^{z_0} \cdot q_1^{z_1} \cdots q_k^{z_k} = 1, \ z_i \in \mathbf{Z}) \ \Rightarrow \ z_i = 0 \ \forall i.$$

It may happen that $\underline{q}$ induces a β-linearization of some D_h^k, whereas it yields only a *partial* ordering over a certain $D_{h'}^k$, because some element $L \in W_{h'}^k$ gets no orientation. For example, the above situation occurs when

$$h = 2, \qquad h' = 3, \qquad k = 2, \qquad \underline{q} = (2, 4, 32), \qquad L = \{(1, 0, 2), (0, 3, 0)\},$$

or when

$$h = 3, \qquad h' = 2, \qquad k = 2, \qquad \underline{q} = (2, 4, 8), \qquad L = \{(1, 0, 1), (0, 2, 0)\}.$$

Nevertheless, it turns out that each β-linearization $\lambda_{\underline{q}}$ is always the restriction of some total weight order $<_{\underline{r}}$. To see this, let us assume that the $(k+1)$-tuple $\underline{q}$ (with $q_0 > 1$) is such that $\sum_i c_i \log q_i = 0$ for some rational numbers $\{c_i\}$. Notice that $\log q_k$ can vary in a suitable neighborhood $X \ni \log q_k$ without altering the related β-linearization (here we use the finiteness of D_h^k). Since

$$|X| > \aleph_0 = |\{\rho : \{\log q_0, \ldots, \log q_{k-1}, \rho\} \text{ are lin. dependent over } \mathbf{Q}\}|,$$

it follows that we can find some $x \in X$ such that $<_{(\log q_0,\ldots,\log q_{k-1},x)}$ is a total weight order over the whole $\mathbf{N}^{k+1}$. Furthermore, by the definition of monomial order (see Definition 1.1(ii)) we may easily deduce that every total weight order, which extends some fixed β-linearization, is *uniquely* determined over $\bigcup_{\theta \le h} D^k_\theta$. To summarize, we have thus established the

PROPERTY 2.6. *Every β-linearization of D^k_h can be extended to some total weight order on $\mathbf{N}^{k+1}$. Any two such extensions coincide over $\bigcup_{\theta \le h} D^k_\theta$.*

Given the real numbers $0 < q_0 < \cdots < q_k$, let $A^k_h(q_0, \ldots, q_k)$ stand for the subset of $\mathbf{R}^+$ made of all the products of h elements chosen in $\{q_i\}$ with repetitions allowed. If $|A^k_h(q_0, \ldots, q_k)| = |D^k_h|$ (that is, if $\underline{q}$ induces a β-linearization over D^k_h) the two structures are given a natural bijection, while the relation $\ll$ is a first indicator of how much the position of the totally ordered elements in $(A^k_h(q_0, \ldots, q_k), <)$ is conditioned by the underlying combinatorial structure.

The obstruction to the choice of a β-linearization among all the available linearizations can be interpreted by means of systems of inequalities. For example, let us consider D^2_2 or D^3_2, depicted in Figure 1. In the former case only one pair is incomparable, namely $\{(0, 2, 0), (1, 0, 1)\}$. That is, the order $\ll$ captures almost completely the behavior of $(A^2_2(x, y, z), <)$ for any $0 < x < y < z$, because we can find both $0 < x' < y' < z'$ such that $x'z' < y'^2$ and $0 < x'' < y'' < z''$ such that $x''z'' > y''^2$; hence, the totally ordered set has one of the following forms:

$$z^2 > yz > y^2 > xz > xy > x^2; \quad z^2 > yz > xz > y^2 > xy > x^2.$$

Instead, in the latter case, five pairs are incomparable, whence there are at most 2^5 ways of obtaining the final linearization. Nonetheless, some obstructions (more or less evident) actually reduce the choices. Thus, we need to know which systems of inequalities are satisfiable among the 32 ones, of the form

$$\begin{array}{lll} z^2 *_1 yw, & yz *_2 xw, & y^2 *_3 xz, \\ z^2 *_4 xw, & y^2 *_5 xw, & \end{array}$$

where $*_i \in \{<, >\}$ and $0 < x < y < z < w$. Theorem 2.3 does not face this question, for it only deals with single inequalities. Nevertheless, it ensures us that all the combinatorial obstructions are in fact arithmetical constraints. We formalize the above discussion by associating to each $L \in W^k_h$ the inequality

$$\beta_L \equiv \prod_{i=0}^{k} x_i^{u_i} < \prod_{i=0}^{k} x_i^{v_i}.$$

A β-linearization of $(D^k_h, \ll)$ can be interpreted as a suitable choice of one inequality between β_L and $\beta_{\overline{L}}$, for each element L.

DEFINITION 2.7. Let $\Omega \subseteq \mathbf{R}^+$. The inequality $\beta_{L'}$ is Ω-*derivable from* β_L if $\beta_{L'}$ is satisfied by every increasing sequence $\{q_0 < \cdots < q_k,\ q_i \in \Omega\}$ which satisfies β_L. We write $\beta_L \Rightarrow_\Omega \beta_{L'}$. If $\Omega = \mathbf{R}^+$ we simply write $\beta_L \Rightarrow \beta_{L'}$ and say that $\beta_{L'}$ is *derivable from* β_L. An analogous terming is used for systems of inequalities in place of single inequalities.

Every β-linearization is to some extent related by the above derivability conditions. In the next three sections we will therefore investigate such conditions, in order to get some more knowledge about the allowed linearizations.

3. The β-Linearizations of D_h^2

In this section we enumerate the β-linearizations of $(D_h^2, \ll)$ for any fixed $h \geq 2$. Let us denote the variables by p, q, r. If $L \in W_h^2$, then β_L is equivalent to either $p^a r^b < q^{a+b}$ or $p^{a'} r^{b'} > q^{a'+b'}$, with $a + b \leq h, a > 0, b > 0$, and conversely. Thus, a satisfiable system $\{\beta_L : L \in W_h^2\}$ can be set up in at most $2^{h(h-1)/2}$ ways.

LEMMA 3.1. *The system*

$$S\colon \begin{cases} p^a r^b < q^{a+b}, \\ p^{a'} r^{b'} > q^{a'+b'}, \end{cases}$$

having positive real numbers as exponents, can be satisfied by some real numbers $0 < p_0 < q_0 < r_0$ *in place of* p, q, r *respectively, if and only if* $b/a < b'/a'$. *Under this condition, each inequality of the form* $p^{a''} r^{b''} < q^{a''+b''}$ (*resp.* $p^{a''} r^{b''} > q^{a''+b''}$) *with* $a'', b'' > 0$ *is derivable from* S *if and only if* $b''/a'' \leq b/a$ (*resp.* $b''/a'' \geq b/a$).

Proof. An equivalent condition for a sequence $0 < p < q < r$ satisfying S is

$$\left(\frac{q}{p}\right)^{\gamma'} < \frac{r}{q} < \left(\frac{q}{p}\right)^{\gamma}, \tag{4}$$

with $\gamma := a/b$, $\gamma' := a'/b'$. Clearly, (4) does not hold if $\gamma \leq \gamma'$. On the other hand, when $\gamma > \gamma'$, a suitable sequence can be produced by choosing any $0 < p < q$ and subsequently finding a number r such that (4) holds. We will prove only the case $<$ of the second assertion (the other one is similar). Suppose that $b''/a'' \leq b/a$. By the first inequality of S we get

$$p_0^{a''} r_0^{a''b/a} < q_0^{a''+a''b/a}. \tag{5}$$

Since $a''b/a - b'' \geq 0$, the provable inequality

$$r_0^{a''b/a-b''} \geq q_0^{a''b/a-b''}$$

yields, together with (5), the desired inequality. On the contrary, if $b''/a'' > b/a$ we show the existence of $0 < p_1 < q_1 < r_1$ which do not verify the requested inequality, though they satisfy S. To this end, set $\gamma'' := a''/b''$ and observe that γ'' and γ' are both smaller than γ. Set $\overline{\gamma} := \max(\gamma', \gamma'')$. Then, the condition

$$\left(\frac{q_1}{p_1}\right)^{\overline{\gamma}} < \frac{r_1}{q_1} < \left(\frac{q_1}{p_1}\right)^{\gamma}$$

is satisfiable, by the previous argumentation; moreover, the solutions are suitable for S, though $p_1^{a''} r_1^{b''} > q_1^{a''+b''}$. □

COROLLARY 3.2. *The system*

$$S\colon \begin{cases} p^{a_i} r^{b_i} < q^{a_i+b_i}, & 1 \le i \le I, \\ p^{a'_j} r^{b'_j} > q^{a'_j+b'_j}, & 1 \le j \le J, \end{cases}$$

where all the exponents are positive real numbers, can be satisfied by $0 < p_0 < q_0 < r_0$ *in place of* p, q, r *respectively, if and only if*

$$\max\left\{\frac{b_i}{a_i}, 1 \le i \le I\right\} =: M < M' := \min\left\{\frac{b'_j}{a'_j}, 1 \le j \le J\right\},$$

having defined $M := 0$ *if* $I = 0$, $M' = +\infty$ *if* $J = 0$ *and using the rules of extended arithmetic. Under this condition, the inequalities of the form* $p^c r^d < q^{c+d} : c, d > 0$ *and those of the form* $p^{c'} r^{d'} > q^{c'+d'} : c', d' > 0$ *can be derived from* S *if and only if* $d/c \le M$ *and* $d'/c' \ge M'$*, respectively.*

The easy proof of the corollary is omitted. As a consequence, we can enumerate the β-linearizations of $(D_h^2, \ll)$: they depend uniquely by the choice of M, which assumes either any rational value b/a with $b \ge 1, a \ge 1, a + b \le h - 1$, or the value zero; in this case, we have $M' = 1/(h-1)$. In general, M' is the number following M in $(\mathbf{Q}^+ \cup \{+\infty\}, <)$ among the admitted values. If $M = h - 1$, then $M' = \infty$. By the previous analysis we get the

COROLLARY 3.3. *The* β*-linearizations of* $(D_h^2, \ll), h \ge 2$*, are indexed by the rational numbers of the form* b/a*, with* $b \ge 0, a > 0, a + b \le h - 1$*. Thus, their number is equal to* $2\sum_{1 \le i \le h-1} \phi(i)$*, where* ϕ *is the Euler function.*

Proof. We prove only the second assert, by induction on h. If $h = 2$, then the admitted rational numbers are 0 and 1; hence the basis holds, because $2 = 2\phi(1)$. Let us assume that $h = H > 2$ and that the assert holds if $h = H - 1$. The admitted rational numbers whose numerator is equal to $H - 1$ are as many as the numbers smaller than $H - 1$ and coprime with it. Thus, there exist $\phi(H - 1)$ such numbers. We do a similar calculation with the denominator in place of the numerator. Now, using the inductive hypothesis, we can conclude. □

In the previous section we have shown that every β-linearization of D_h^k is induced by the restriction to $\bigcup_{\theta \le h} D_\theta^k$ of some total weight order, and that the restriction does not depend on the order. On the other hand, the above corollary has the following two consequences:

PROPERTY 3.4. *If* $h > h'$*, then there exist several restrictions of total weight orders to* $\bigcup_{\theta \le h'} D_\theta^2$*, which yield the same* β*-linearization over* D_h^2*.*

Proof. Since the restrictions to $\bigcup_{\theta \leq h} D_\theta^2$ of all the total weight orders $<_{\underline{r}}$, with $r_0 < r_1 < r_2$, correspond to all the β-linearizations of D_h^2, the number of such restrictions is a strictly increasing function of h. Therefore, at least two restrictions defined over $\bigcup_{\theta \leq h'} D_\theta^2$ must coincide over $\bigcup_{\theta \leq h} D_\theta^2$. $\square$

If $k > 2$, a weaker result can be established.

PROPERTY 3.5. *If $k \geq 3$, then for every $h \geq 2$ there exists some $h' > h$ such that two distinct restrictions of total weight orders to $\bigcup_{\theta \leq h'} D_\theta^k$ yield the same β-linearization over D_h^k.*

Proof. A fixed β-linearization $\lambda_{\underline{s}}$ of $D_{h'}^k$ induces the β-linearization $\lambda_{(s_0,s_1,s_2)}$ on $D_{h'}^2 \simeq \{\underline{u} \in D_{h'}^k \ : \ u_i = 0 \ \forall i > 2\}$. Thus, the restrictions of total weight orders to $\bigcup_{\theta \leq h'} D_\theta^k$ are at least as many as the β-linearizations of $D_{h'}^2$. Since there are finitely many restrictions of total weight orders to $\bigcup_{\theta \leq h} D_\theta^k$, whereas the number of β-linearizations of $D_{h'}^2$ increases with h', we get the assert. $\square$

As a further application of the above corollary, let us consider the poset D_3^2 and in particular the three pairs of elements having rank 2, 3, 4 respectively (since D_3^2 and D_2^3 are isomorphic posets, the reader may refer to Figure 1 using different labels). The linearizations of this poset are at least 2^3, because the above pairs lie on distinct levels. The obstruction is therefore evident, being $8 > 4 = |\{0, 1/2, 1, 2\}|$.

We may consider sequences of the form $1 < \xi < \eta$ in order to represent all the β-linearizations in the right upper quadrant of $\mathbf{R}^2$, over the bisector. In the case $h = 3$, Figure 3 provides such a representation. The sectors I, ..., IV are bounded (clockwise) by the curves $\xi = 1, \eta = \xi^3, \eta = \xi^2, \eta = \xi^{3/2}$ and the bisector (we have used different scalings for the two axes). The points lying in the interior of a sector represent all the triples which induce the same β-linearization. As a further suggestion, we have drawn the dotted line L which represents all the sequences $\{x_i\}$ such that $x_1 - x_0 = x_2 - x_1$. We see that L lies in the third and fourth sector; this means that the arithmetical progressions made of $k + 1 = 3$ numbers induce only two linearizations among the four available.

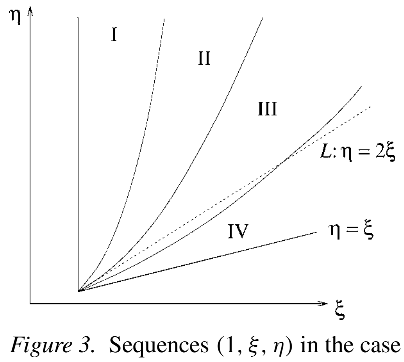

Figure 3. Sequences $(1, \xi, \eta)$ in the case $h = 3$.

More generally, for any $h \geq 2$, it can be easily shown that each sector is defined by $\xi^{1+a'/b'} < \eta < \xi^{1+a/b}$, where $b'/a' = M'$.

We conclude this section with a worth mentioning fact.

PROPERTY 3.6. *The β-linearizations of D_h^2, with $h \geq 3$, form a proper subclass of all the linearizations.*

Proof. The distinct pairs $\{(1, 0, h-1), (0, 2, h-2)\}$, $\{(h-1, 0, 1), (h-2, 2, 0)\}$ give rise to the same inequality, namely $pr < q^2$. This is of course an obstruction to the free choice of an orientation. On the other hand, all the four choices give rise to some linearization. □

4. Results for the Case $k \geq 3$

The previous discussion seems more difficult to set up when $k \geq 3$, because β_L cannot be reduced to a unique form. Hence, in this section we introduce a further class of posets, in order to describe at least a necessary condition for obtaining a β-linearization in every D_h^k. The main result (Theorem 4.3) provides a characterization of all the inequalities β_M which are derivable by some inequality β_L, where L, M belong to the same set W_h^k. Actually, this result might be established with few difficulties using the Kuhn–Fourier Theorem (see also the Introduction), which asserts that a system of linear relations is solvable if every *legal linear dependence* leads to a relation always true, as for example $0 < 3$ or $0 = 0$ (by definition, a system possesses a legal linear dependence if, using some correct linear combinations, one can obtain a new relation with all zero coefficient of the variables). In our proof, the concept of linear combination is used as well. Furthermore, the techniques developed in our argumentation bear some consequences (namely, Lemma 4.6 and Corollary 4.8) which do not seem to be easily derivable from the classical result. These consequences are worth mentioning in the present context; indeed, in Section 5 we avail of the cited lemma, whereas the corollary is a sharpening of the Kuhn–Fourier Theorem in the particular case of the system

$$\begin{aligned}&\beta_L,\\&0 < x_0 < x_1 < \cdots < x_k.\end{aligned}$$

Let $\mathcal{W}_h^k$ stand for the subset of $\bigcup_{2\leq\theta\leq h} W_\theta^k$ with the property that

$$\forall\theta\ \forall L \in W_\theta^k \exists! M \in \mathcal{W}_h^k, \exists \underline{c} : L = M + \{\underline{c}, \underline{c}\}, c_i \geq 0\ \forall i.$$

Every inequality β_L, $L \in W_\theta^k$, is therefore equivalent to a unique β_M, $M \in \mathcal{W}_h^k$, whose degree is the smallest admitted. The subset $\mathcal{W}_h^k$ can be also defined by the simpler condition $\{\underline{u}, \underline{v}\} \in W_\theta^k : u_i v_i = 0\ \forall i$, e.g.

$$\{(1011), (0300)\} \in \mathcal{W}_3^3, \qquad \{(102), (021)\} \notin \mathcal{W}_3^2, \qquad \{(101), (020)\} \in \mathcal{W}_3^2.$$

Clearly $\mathcal{W}_h^k \subset \mathcal{W}_{h'}^k$ if $h < h'$. Let $\{\underline{e}_i\}_{0\le i\le k}$ stand for the canonical basis of $\mathbf{R}^{k+1}$ and E_h^k be the set of $(k+1)$-tuples $\underline{u} = (u_0, \ldots, u_k)$ such that each u_i is an integer and $\sum_i u_i = h$. We define four classes of maps from $\mathcal{W}_h^k$ to $\bigcup_{1\le\theta\le h+1}(E_\theta^k \times E_\theta^k)$ as follows. If $L \in \mathcal{W}_h^k$, then

$$\begin{aligned}
&A_i(L) := L + \{\underline{e}_i - \underline{e}_{i+1}, \underline{0}\}, \qquad 0 \le i < k;\\
&B_i(L) := L + \{\underline{0}, -\underline{e}_i + \underline{e}_{i+1}\}, \qquad 0 \le i < k;\\
&C_i(L) := L + \{\underline{e}_i, \underline{e}_{i+1}\}, \qquad 0 \le i < k;\\
&\overline{C}_i(L) := L - \{\underline{e}_i, \underline{e}_{i-1}\}, \qquad 0 < i \le k.
\end{aligned}$$

In the sequel, we will extend the symbol $\{\underline{u}, \underline{v}\}$ to the pairs of elements of E_h^k. Let Z denote the class of all the maps defined above. We endow $\mathcal{W}_h^k$ with the following partial order $<$. Firstly, we write $L <^* L'$ if $f(L') = L$ for some $f \in Z$. Then, the relation $<$ is defined as the reflexive and transitive closure of $<^*$. An argument similar to the one adopted for $(D_h^k, \ll)$ can be used for proving that $(\mathcal{W}_h^k, <)$ is ranked and that $<$ is in particular antisymmetric. However, these new posets have no maximum nor minimum in general (they may be even disconnected, as $\mathcal{W}_2^3$ in Figure 5). The rank function is defined as $r(\{\underline{u}, \underline{v}\}) := \delta + \sum_i i(u_i - v_i)$, for some suitable integer δ.

The following relation does not extend $<$ as a partial order, for we will quickly check that it is not antisymmetric.

DEFINITION 4.1. Let $L, M \in \mathcal{W}_h^k$. L is *weakly preceding* M ($L <^w M$) if for some $H \ge h$ there exist $2n$ positive integers $\{a_i\}, \{b_i\}$, $i = 0, \ldots, n-1$, and $n+1$ elements $\{L = L_0, L_1, \ldots, L_n = M\}$ such that $a_i L_i < b_i L_{i+1}$ for all $i \le n-1$, in $(\mathcal{W}_H^k, <)$. The sequence $\{L_i\}$ is termed an *inferential sequence* related to L, M.

Remark 4.2. $<^w$ is not antisymmetric because for example $L <^w 2L <^w L$. It is still transitive, as it could be easily seen. Moreover, $L < M$ clearly implies $L <^w M$.

In the following theorem we show that the maps belonging to the class Z give rise to all the possible inequalities which can be derived from an initial inequality β_M. Thus, we exhibit a class of algorithms for generating all the logical consequences of β_M related to some fixed D_h^k.

THEOREM 4.3. *$\beta_M \Rightarrow \beta_L$ if and only if $L <^w M$.*

The proof of the 'only if' part of the theorem will be split into two lemmas. Let l_i, $0 \le i \le k-1$, denote the $(k+1)$-tuple $\underline{e}_{i+1} - \underline{e}_i$. Observe that E_0^k is a k-dimensional $\mathbf{Z}$-module having $\{l_i\}$ as a basis. Given $L := \{\underline{u}, \underline{v}\}$, $M := \{\underline{u}', \underline{v}'\}$ in $\mathcal{W}_h^k$, let us define $\underline{z} := \underline{v} - \underline{u}$, $\underline{z}' := \underline{v}' - \underline{u}'$. Then, for some integers $\{a_i\}, \{b_i\}$, $\underline{z} = \sum_i a_i l_i$, $\underline{z}' = \sum_i b_i l_i$. We will write $\langle 0 : a_0, 1 : a_1, \ldots, k-1 : a_{k-1}\rangle$ in place of L, with the indices not necessarily increasing (this notation will be useful in the proof of

Proposition 5.1). Also notice that the hypothesis of incomparability yields easily $a_i < 0 < a_j$ for some i, j, as well as $b_s < 0 < b_t$ for some s, t. The following claim will be useful in the next section:

PROPERTY 4.4.

$$a_i = \sum_{r=0}^{i} (u_r - v_r), \quad \textit{for all } i \leq k-1.$$

The assert can be easily proved by induction, using the relations $-a_0 = v_0 - u_0, a_0 - a_1 = v_1 - u_1, a_1 - a_2 = v_2 - u_2, \dots, a_{k-2} - a_{k-1} = v_{k-1} - u_{k-1}$.

LEMMA 4.5. *If some positive rational number q satisfies $qa_i \geq b_i$ (resp. $qa_i \leq b_i$) for all i, then $L <^w M$ (resp. $L >^w M$).*

Proof. We deal with the first case. If $q = c/d$ we have

$$\begin{aligned}
L <^w cL &= c\{\underline{u}, \underline{v}\} = d\{\underline{u}', \underline{v}'\} + \{c\underline{u} - d\underline{u}', c\underline{u} - d\underline{u}'\} + \\
&\quad + \left\{\underline{0}, c\sum_i a_i l_i - d\sum_i b_i l_i\right\} \\
&= dM + \{c\underline{u} - d\underline{u}', c\underline{u} - d\underline{u}'\} + \sum_i (ca_i - db_i)\{\underline{e}_i, \underline{e}_{i+1}\} - \\
&\quad - \sum_i (ca_i - db_i)\{\underline{e}_i, \underline{e}_i\} \\
&= dM + \{\underline{t}, \underline{t}\} + \sum_i \alpha_i \{\underline{e}_i, \underline{e}_{i+1}\}
\end{aligned}$$

for some $\underline{t}$, where $\alpha_i := ca_i - db_i \geq 0\ \forall i$. We use induction on $\sum_i \alpha_i$ for proving that $cL < dM$. If that sum is zero, then $\alpha_i = 0\ \forall i$ and we get $\underline{t} = \underline{0}$ by the following argument: if $t_i \geq 0\ \forall i$ then $t_i = 0\ \forall i$ by the definition of $\mathcal{W}_h^k$; else, if $t_i < 0$ for some i, then cL has some negative entry because $u_i' v_i' = 0\ \forall i$; this is absurd. In conclusion, we get $cL = dM$. Now assume that $\alpha_s > 0$ for some s. Define $N := \{\underline{x}, \underline{y}\} := cL - \{\underline{e}_s, \underline{e}_{s+1}\}$. First suppose that N has only non-negative entries. We claim that $N \in \mathcal{W}_{ch-1}^k$. Indeed, if by contradiction N was a pair of comparable vectors, then by Lemma 2.2 $\sum_{j\geq i} x_i - y_i \geq 0\ \forall i$ (the other case does not arise in this context, having subtracted $\{\underline{e}_i, \underline{e}_{i+1}\}$ from an incomparable pair). On the other hand, the same lemma implies that $\sum_{j\geq I}(du_j' - dv_j') < 0$ for some index I. Therefore, because

$$N = dM + \{\underline{t}, \underline{t}\} + \sum_{j\neq s} \alpha_j \{\underline{e}_j, \underline{e}_{j+1}\} + (\alpha_s - 1)\{\underline{e}_s, \underline{e}_{s+1}\},$$

the same index I is such that $\sum_{j\geq I}(x_j - y_j) < 0$, which is absurd. Moreover, $x_{s+1} = u_{s+1} = 0, y_s = v_s = 0$. Thus, $N \in \mathcal{W}_{ch-1}^k$. Since $\sum_{j\neq s} \alpha_j + \alpha_s - 1 < \sum_j \alpha_j$, it follows that $N < dM$ by induction. Finally, since $cL = C_s(N) < N$, we have that $cL < dM$, by transitivity. Now assume that N has some negative entry. Then, necessarily, one of the following three cases holds: $x_s = -1, y_{s+1} \geq 0$; $x_s \geq 0, y_{s+1} = -1$; $x_s = -1, y_{s+1} = -1$. In the first case, define $N' := cL + \{\underline{0}, \underline{e}_s - \underline{e}_{s+1}\}$. This element has no negative entries, because $y_{s+1} = 0$ implies that $cv_{s+1} > 0$. Furthermore,

$$N' = dM + \{\underline{t} + \underline{e}_s, \underline{t} + \underline{e}_s\} + \sum_{j \neq s} \alpha_j \{\underline{e}_j, \underline{e}_{j+1}\} + (\alpha_s - 1)\{\underline{e}_s, \underline{e}_{s+1}\}.$$

Therefore, reasoning as above, we see that $N' \in \mathcal{W}_{ch}^k$, and applying induction we get $N' < dM$. Finally, using the map B_s, we obtain $cL = B_s(N') < N'$ and consequently $cL < dM$. In the second case we consider the element $cL + \{-\underline{e}_s + \underline{e}_{s+1}, \underline{0}\}$ in place of N', and reason as above. In the third case we use $N'' := cL + \{\underline{e}_{s+1}, \underline{e}_s\}$, observing that

$$\begin{aligned} N'' = dM &+ \{\underline{t} + \underline{e}_s + \underline{e}_{s+1}, \underline{t} + \underline{e}_s + \underline{e}_{s+1}\} + \\ &+ \sum_{j \neq s} \alpha_j \{\underline{e}_j, \underline{e}_{j+1}\} + (\alpha_s - 1)\{\underline{e}_s, \underline{e}_{s+1}\}. \end{aligned}$$

The case $qa_i \leq b_i$ in the assertion of the lemma is treated similarly. $\square$

Before introducing the second lemma, we give some definitions. The system $\{\beta_L, \beta_{\overline{M}}\}$ can be written as

$$1 < \prod_{i=0}^{k-1} \left(\frac{x_{i+1}}{x_i}\right)^{a_i}, \qquad 1 > \prod_{i=0}^{k-1} \left(\frac{x_{i+1}}{x_i}\right)^{b_i}. \tag{6}$$

Indeed,

$$\beta_L \equiv 1 < \prod_i x_i^{v_i - u_i} \equiv 1 < x_0^{-a_0} \cdot x_1^{a_0 - a_1} \cdots x_{k-1}^{a_{k-2} - a_{k-1}} \cdot x_k^{a_{k-1}},$$

and we reason similarly with $\beta_{\overline{M}}$. It is useful to consider the variables $y_i := x_{i+1}/x_i$, where each y_i is supposed to be a real number greater than 1. Notice that every given vector $\underline{y}$ can always be associated to some suitable sequence $\{x_i\}$. Thus, we write the system as

$$1 < \prod_i y_i^{a_i}, \qquad 1 > \prod_i y_i^{b_i} \equiv \begin{cases} \phi(\underline{y}) > 1, \\ \psi(\underline{y}) < 1, \end{cases}$$

having introduced the functions ϕ, ψ. We do similarly with $\{\beta_{\overline{L}}, \beta_M\}$.

LEMMA 4.6. *If there exists no $q \in \mathbf{Q}^+$ such that $qa_i \leq b_i$ (resp. $qa_i \geq b_i$) for all i, then $\beta_L \not\Rightarrow \beta_M$ (resp. $\beta_M \not\Rightarrow \beta_L$).*

Proof. In order to prove the first assert, let us assume the following hypothesis:

$$(H) \equiv b_i \leq 0 < a_i \text{ or } b_i < 0 \leq a_i \text{ for some index } i.$$

Observe that the hypothesis of the lemma is implied by the given condition. Since $\phi = \hat{\phi} y_i^{a_i}$, $\psi = \hat{\psi} y_i^{b_i}$ with $\hat{\phi}, \hat{\psi}$ not depending on y_i, a sequence $\underline{Y}$ such that $\phi(\underline{Y}) > 1$, $\psi(\underline{Y}) < 1$ is easily obtained when $b_i < 0 < a_i$, by choosing Y_i large enough. Else, if $b_i = 0$ then ψ does not depend on y_i and Theorem 2.3 enables us to find a suitable sequence $\underline{Y}'$ such that $\psi(\underline{Y}') < 1$; now, if necessary, increase Y_i' so as to get $\phi(\underline{Y}') > 1$. If $a_i = 0$ a similar argument is used. Now suppose that (H) does not hold and that $(a_i = 0 \Rightarrow b_i \geq 0)$, $(b_i = 0 \Rightarrow a_i \leq 0)$. We claim that for some indices i, j the following five conditions hold: $a_i > 0, b_i > 0$, $a_j := -A_j < 0, b_j := -B_j < 0$ and $a_i B_j > A_j b_i$. The first four conditions are satisfiable because L, M belong to $\mathcal{W}_h^k$ and therefore $a_i > 0, b_j < 0$ for some i, j; if we suppose that the last condition does not hold whenever the others are satisfied, then a number q which contradicts the hypothesis can be exhibited by defining

$$q := \min\left\{\frac{b_i}{a_i} : a_i > 0, b_i > 0\right\}.$$

Indeed, firstly this definition implies that $qa_j \leq b_j$, also in the case that a_j and b_j are both negative. To prove this, consider $a_i, b_i > 0$ such that $q = b_i/a_i$; then, $qa_j = -qA_j \leq -qB_j a_i/b_i = -B_j = b_j$, so we are done. Furthermore, the remaining case $a_i \leq 0 \leq b_i$ is trivially satisfied by q. For showing that $\beta_L \not\Rightarrow \beta_M$ it is enough to exhibit an element $\underline{Z}$ such that $\phi(\underline{Z}) > 1 = \psi(\underline{Z})$, because we can subsequently obtain the desired sequence by slightly modifying $\underline{Z}$ (using a continuity argument over $(1, \infty)^k$). Let us write ϕ, ψ respectively as $\tilde{\phi} y_i^{a_i} y_j^{a_j}$, $\tilde{\psi} y_i^{b_i} y_j^{b_j}$. Thus, $\tilde{\phi}, \tilde{\psi}$ do not depend on y_i and y_j. According to Theorem 2.3, we can find two sequences which satisfy β_M and $\beta_{\overline{M}}$ respectively. As a consequence, again due to continuity, there exists a sequence $\underline{Y}$ which makes $\psi(\underline{Y})$ equal to 1. Let us set $K := \tilde{\psi}(\underline{Y}) = Y_j^{B_j}/Y_i^{b_i}$; if we choose any number $Z_j \geq Y_j$ and define

$$Z_i := \left(\frac{Z_j^{B_j}}{K}\right)^{1/b_i}, \qquad Z_s := Y_s \ \forall i \neq s \neq j,$$

then we obtain a correct sequence $\underline{Z}$ still with the property that $\psi(\underline{Z}) = 1$. Moreover, for some positive number C,

$$\phi(\underline{Z}) = \left(\frac{Z_j^{B_j}}{K}\right)^{a_i/b_i} \cdot Z_j^{-A_j} \tilde{\phi}(\underline{Y}) = C Z_j^{a_i B_j/b_i - A_j}$$

and if we choose Z_j large enough, the condition $a_i B_j > A_j b_i$ implies the existence of a sequence $\underline{Z}'$ which makes $\phi(\underline{Z})$ greater than 1. The second assert can be similarly established. $\square$

Proof of Theorem 4.3. If $L \not<^w M$ then $\beta_M \not\Rightarrow \beta_L$ as a consequence of the two lemmas above. Moreover, the 'if' part is easily derivable by the definition of weak comparability. That is, each map of the class Z is easily seen to represent some correct inference between the involved inequalities; thus, β_L can be deduced from β_M through a finite sequence of inferences. □

COROLLARY 4.7. *If $L, M, \{a_i\}, \{b_i\}$ are as above, then the following facts are equivalent:*

(a) $L <^w M <^w L$.
(b) $\beta_L \Leftrightarrow \beta_M$.
(c) *There exist $q, q' \in \mathbf{Q}^+$ such that $qa_i \leq b_i, q'a_i \geq b_i$ for all i.*
(d) $M = qL$ *for some* $q \in \mathbf{Q}^+$.

Proof. The equivalence between **a** and **b** is a straightforward consequence of Theorem 4.3, as well as the two lemmas above, together with the 'if' part of the theorem, make **a** and **c** equivalent. **d** implies of course **b**. It suffices to prove that **c** implies **d**. By the assumption, it follows that $a_i b_i \geq 0$ for all i. Furthermore, since $a_s > 0, a_t < 0$ for some indices s, t, we have

$$q' \;\geq\; \max\left\{\frac{b_i}{a_i}, a_i > 0\right\} \geq \min\left\{\frac{b_i}{a_i}, a_i > 0\right\} \geq q,$$

$$q' \;\leq\; \min\left\{\frac{b_i}{a_i}, a_i < 0\right\} \leq \max\left\{\frac{b_i}{a_i}, a_i < 0\right\} \leq q,$$

whence $q = q'$. Therefore, we obtain $b_i = qa_i \; \forall i$, that is $v_i' - u_i' = q(v_i - u_i) \; \forall i$. The last equality, together with $u_i v_i = u_i' v_i' = 0 \; \forall i$, leads to the conclusion. □

The argumentation used in the proof of Lemma 4.5 yields the

COROLLARY 4.8. *The relation $<^w$ can be defined more simply through*

$$L <^w M \equiv \exists c, d, H \in \mathbf{N}^+ : cL, dM \in \mathcal{W}_H^k, cL < dM.$$

In other words, if we consider the logarithmic version of our inequalities, the above claim characterizes the derivable inequalities by means of a *proper subclass* of all the allowed linear combinations. In particular, the multiplication by positive numbers (namely, d and $1/c$) need to be performed only in the beginning and in the end respectively, whereas the intermediate steps (related to the order $<$) represent summations with the inequalities $x_i < x_j, i < j$, together with possible subtractions of the equalities $x_i = x_i$.

The proof of the following fact does not require any result of the above ones. It might have been obtained directly by the definition of $\ll$. Anyway, the formalism developed in this section will add some more clearness to the argumentation.

LEMMA 4.9. *Each $(D_h^k, \ll)$ is a lattice.*

Proof. Let $\underline{u}, \underline{v}$ be incomparable elements of a fixed D_h^k. Then, we have that $\underline{v} - \underline{u} \in E_0^k$. Hence

$$\underline{u} + \sum_{i \in I} a_i l_i = \underline{v} + \sum_{j \in J} b_j l_j =: \underline{z},$$
$$\exists I \neq \emptyset \neq J,\ I \cap J = \emptyset,\ a_i > 0\ \forall i,\ b_j > 0\ \forall j.$$

Notice that $\underline{u}, \underline{v}$ are both under $\underline{z}$. We will show that $\underline{z} = \underline{u} \vee \underline{v}$, by proving that any $\underline{w} \neq \underline{z}$ which is over these two elements, is also over $\underline{z}$. By the hypothesis, there exist some sets S, T such that

$$\underline{w} - \underline{u} = \sum_{s \in S} A_s l_s, \qquad \underline{w} - \underline{v} = \sum_{t \in T} B_t l_t, \qquad A_s > 0\ \forall s,\ B_t > 0\ \forall t,$$

and $S \cap T$ is possibly non-empty. Therefore,

$$\sum_{t \in T} B_t l_t - \sum_{s \in S} A_s l_s = \sum_{j \in J} b_j l_j - \sum_{i \in I} a_i l_i,$$

whence, for example, $I \subseteq S$ and $A_i \geq a_i$ for all $i \in I$. It follows that

$$\begin{aligned}
\underline{w} &= \underline{u} + \sum_{s \in S} A_s l_s \\
&= \underline{u} + \sum_{s \in I} A_s l_s + \sum_{s \in S \setminus I} A_s l_s \\
&= \underline{u} + \sum_{s \in I} a_s l_s + \sum_{s \in I} (A_s - a_s) l_s + \sum_{s \in S \setminus I} A_s l_s \\
&= \underline{z} + \sum_{s \in I} (A_s - a_s) l_s + \sum_{s \in S \setminus I} A_s l_s \gg \underline{z}.
\end{aligned}$$

Similarly, the greatest lower bound can be proved to exist for every pair. □

5. A Sharper Result for D_h^3, $h \leq 4$

Here we show that the weak comparability in $\mathcal{W}_4^3$ can be defined by means of inferential sequences which remain inside $\mathcal{W}_4^3$ itself. Moreover, the extension from $<$ to $<^w$ carries only a slight change to the structure, whereas in the cases $\mathcal{W}_h^3$, $h = 2, 3$, the relation $<^w$ collapses to $<$.

PROPOSITION 5.1. *Let $L = \{\underline{u}, \underline{v}\}$, $M = \{\underline{u}', \underline{v}'\}$ be elements of W_3^4. If $L <^w M$ then, an inferential sequence $\{L_0, \ldots, L_n\}$ related to L, M can be chosen in such a way that $L_i \in \mathcal{W}_3^4$ for all i.*

Proof. Since $\beta_M \Rightarrow \beta_L$, Lemma 4.6 guarantees the existence of a positive rational number q such that $qa_i \geq a_i'$ for all i, where as usual $\underline{v} - \underline{u} = \sum_{0 \leq i \leq 2} a_i l_i$,

$\underline{v}' - \underline{u}' = \sum_{0 \le i \le 2} a'_i l_i$. If $q = 1$, then we get the assertion because $L < M$ after a brief calculation. Now assume that $q := \min\{\overline{q} : \overline{q} a_i \ge a'_i \ \forall i\} > 1$; let a_i be negative, for some index i. Then, $a'_i < a_i$ (otherwise $qa_i < a'_i$) and $q \le a'_i/a_i$. Furthermore, there exists some positive element a_j. We claim that $a'_s - a'_t \le 4$ for all s, t. Indeed, $a'_1 - a'_0 = u'_1 - v'_1 \le 4$, and a similar equality holds whenever s, t are consecutive. Furthermore, by Property 4.4, $a_2 - a_0 = u'_1 - v'_1 + u'_2 - v'_2 \le 4$, and we are done. As a consequence we have that $a_i \ge -3$, for otherwise every $a_t \le 0$ is actually equal to zero, whence $L \notin \mathcal{W}_3^4$; the same argument yields $a_j \le 3$ (of course nothing changes if we replace a_s with a'_s).

Thus, we may distinguish three cases. (A) $a_i = -2, a'_i = -3$. Then, any a'_j positive is equal to 1, which implies that $q = 1$ is large enough to provide the weak comparability (notice that $a'_j > 0$ implies $a_j > 0$, because $qa_j \ge a'_j$ by hypothesis). It follows that $L < M$. (B) $a_i = -1, a'_i = -3$. We use the same argumentation. (C) $a_i = -1, a'_i = -2$. Then, the hypothesis $q > 1$ implies that at least one couple $\{a_j, a'_j\}$ is equal to $\{1, 2\}$ (here we also use $a'_j \le 2$). Let k denote the remaining index. Firstly, let us suppose that $a'_k \le 0$; then, the hypothesis $qa_k \ge a'_k$ implies that $a_k \ge -1$, because in any case $a'_k \ge -2$ (otherwise $a'_j - a'_k \ge 5$). If $a_k = -1$, then $a'_k = -2$ and we get the assert; otherwise, if $a_k \ge 0$, then we decrease it to 0 (if $a_k > 0$), multiply by 2 the element obtained (namely $\langle i : -1, j : 1, k : 0\rangle$) and, if $a'_k < 0$, decrease 0 to a'_k. Hence we obtain the element $\langle i : -2, j : 2, k : a'_k\rangle$. Now suppose that $a'_k > 0$. Then, $a_k > 0$, and one can easily check that k, j must be consecutive; moreover, $a'_k \le 2$ (otherwise $a'_k - a'_i \ge 5$). It follows that $a'_k = 2$ is an admitted value (this is not trivial only when $a'_k = 1$); we can therefore decrease a_k to 1, in case, then multiply by 2 the new element and finally decrease the k-th entry (equal to 2) to a'_k, if necessary. Since the decreasing procedure gives rise to a greater element (with regard to the order $<$), than the whole procedure yields a suitable sequence. The case $0 < q < 1$ is interpreted as $M >^w L$ together with $r := 1/q > 1$ such that $ra_i \le a'_i$ for all i. Now an argument similar to the one above leads to the conclusion. □

In Figure 4 we provide a representation of $(\mathcal{W}_4^3, <^w)$ based on a horizontal Hasse diagram of $(\mathcal{W}_4^3, <)$. The reader should identify the top with the bottom, so as to get a cylinder. The pairs within the grey connected regions are the ones which contradict antisymmetry. Also the thick segments come from the extension of $<$ by the weak comparability, though they are compatible with the rank function of the initial poset. Notice that the extension involves pairs which contain always one element of $\mathcal{W}_4^3 \setminus \mathcal{W}_3^3$. Therefore, the weak comparability is precisely the relation $<$ in the cases $\mathcal{W}_h^3,\ h = 2, 3$.

It is worth observing that Theorem 4.3 sheds very little light on the classification of β-linearizations, essentially because it takes account of the only combinatorial structure of $(D_h^k, \ll)$, leaving aside almost all the restrictions due to the inferences among inequalities. The following brief analysis aims to emphasize this aspect.

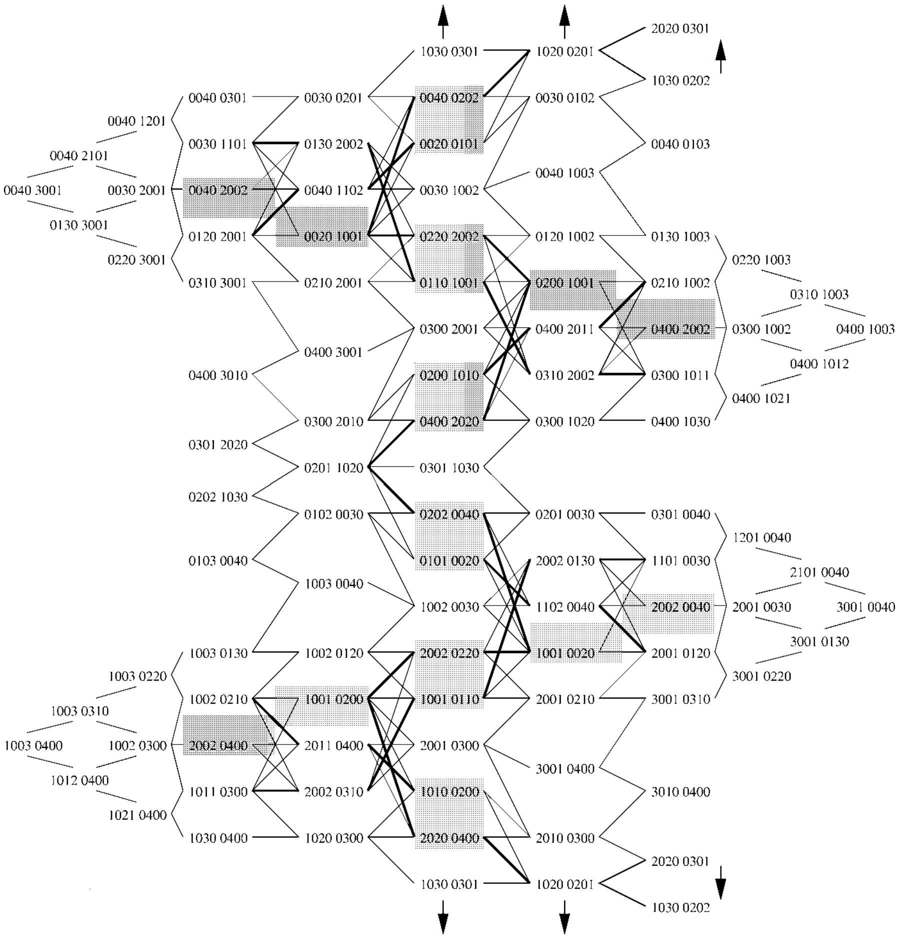

Figure 4. $(\mathcal{W}_4^3, <^w)$.

DEFINITION 5.2. A set $S \subseteq (\mathcal{W}_h^k, <^w)$ is an *ideal* if

$$(L \in S, M <^w L) \quad \text{implies} \quad M \in S,$$

for every pair L, M.

In particular, if $<^w$ is a partial order, the notions of ideal and *order ideal* coincide. Theorem 4.3 implies that a fixed β-linearization of $(D_h^k, \ll)$, say $\lambda_{\underline{s}}$, may be associated to an ideal of $(\mathcal{W}_h^k, <^w)$ which contains either L or $\overline{L}$ and not both, for every L. This ideal is defined as $\{L : \beta_L(\underline{s}) \text{ holds }\}$. If $k = 3$ and $h = 2, 3$, then every ideal is an order ideal of $(\mathcal{W}_h^3, <)$. Nevertheless, many ideals are not associated to any β-linearization. Let us consider for example $(\mathcal{W}_2^3, <)$, represented in Figure 5. If we denote by L, M, N the elements with the same rank in the left component, from left to right, then an elementary calculation yields $\beta_L \wedge \beta_N \Rightarrow \beta_M$ and $\beta_{\overline{L}} \wedge \beta_{\overline{N}} \Rightarrow \beta_{\overline{M}}$, which is an obstruction to the choice of order ideals representing

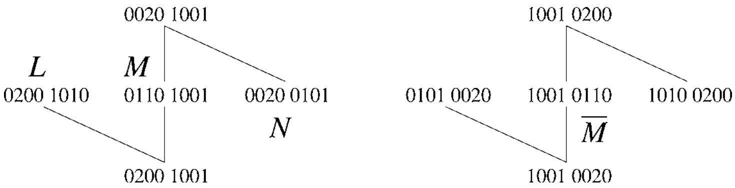

Figure 5. $(\mathcal{W}_2^3, <) = (\mathcal{W}_2^3, <^w)$.

β-linearizations, e.g. $\{L, \overline{M}, N, \{0200, 1001\}, \{1001, 0020\}\}$ is not allowed. We hope that some basic information is nested in some fixed level, as the middle one in the above case. Probably, Theorem 4.3 together with an adequate investigation on levels may lead to some more interesting conclusion. Notice that the above counterexample is valid for every $h \geq 2, k \geq 3$. In the general case, the structure depicted in Figure 5 is properly contained in $\mathcal{W}_h^k$, and the *ideal generated* by $L, N, \overline{M}$ is not allowed.

Following the above example, we conclude this section with the natural generalization of Property 3.6.

PROPERTY 5.3. *If $k \geq 3$ and $h \geq 2$, the β-linearizations of $(D_h^k, \ll)$ are properly contained in the class of all the linearizations of the poset.*

Proof. Assume that $h = 2H$. Then, the three pairs of elements

$$\{2H\underline{e}_1, H(\underline{e}_0 + \underline{e}_2)\}, \qquad \{H(\underline{e}_1 + \underline{e}_2), H(\underline{e}_0 + \underline{e}_3)\}, \qquad \{2H\underline{e}_2, H(\underline{e}_1 + \underline{e}_3)\}$$

have rank equal to $2H, 3H, 4H$ respectively. One can therefore construct 2^3 linearizations by firstly orienting these pairs. On the other hand, the obstruction carried by the related inequalities (similar to the above one) reduces the choice of the three orientations for obtaining a β-linearization. The same proof can be adapted to the odd case, by adding 1 to some fixed coordinate throughout. $\square$

6. The Interpretation inside N

If $p_0 < p_1 < \cdots < p_k$ are prime numbers, then the weight order determined by $(\log p_0, \ldots, \log p_k)$ is a total order. This is a straightforward consequence of the Unique Factorization Theorem. Indeed,

$$\sum_{i=0}^{k} c_i \log p_i = 0 \Leftrightarrow \prod_{i=0}^{k} p_i^{c_i} = 1 \Leftrightarrow c_i = 0 \; \forall i,$$

for otherwise, if the second equivalence did not hold, by separating the positive exponents from the negative ones we could prove the existence of some natural number having two distinct factorizations. As a consequence, $<_{(\log p_0, \ldots, \log p_k)}$ induces a β-linearization in each D_h^k. In this section we prove the following ‘prime number’ version of Theorem 2.3.

THEOREM 6.1. *Given* $\underline{u} \neq \underline{v}$ *in* D_h^k, $\underline{u} \ll \underline{v}$ *if and only if* (1) *holds for any increasing sequence of* $k+1$ *prime numbers* $q_0 < q_1 < \cdots < q_k$. *Thus, for any fixed* D_h^k, *the relation* $\ll$ *is the intersection of all the weight orders* $<_{(\log p_0,\ldots,\log p_k)}$, *restricted to* D_h^k, *such that* $p_0 < \cdots < p_k$ *are primes.*

The above result will be derived by the following proposition, whose basic number-theoretic ingredient is *Bertrand's postulate* (see, for example, [6]):

$$\forall x \in \mathbf{N} \setminus \{0, 1\} \exists p \text{ prime} \ : \ x < p < 2x.$$

PROPOSITION 6.2. *If there exists a sequence of real numbers* $1 < x_0 < \cdots < x_n$ *which satisfies finitely many inequalities of the form*

$$\prod_{i=0}^{n} x_i^{a_i} < \prod_{i=0}^{n} x_i^{b_i}$$

for some nonnegative real numbers $\{a_i\}, \{b_i\}$, *then there exist prime numbers* $q_0 < \cdots < q_n$ *such that the same inequalities hold when each* x_i *is replaced by* q_i *throughout.*

Proof. A fixed inequality can be rewritten as $\prod_i x_i^{b_i - a_i} > 1$, whence some number N large enough can be chosen in such a way that

$$\left(\prod_{i=0}^{n} x_i^{b_i - a_i} \right)^N > 3^{(n+1)\sum_i a_i}.$$

Let M be the maximal value of N among all the inequalities; we can assume that $x_0^M \geq 2$. Let us set $r_0 := \lceil x_0^M \rceil$. By Bertrand's postulate, there exists a prime $q_0 \in (r_0, 2r_0)$. Then,

$$x_0^M \leq r_0 < q_0 < 2r_0 < 3x_0^M.$$

We choose q_0 as the first prime of the sequence. The construction of the other primes is done by induction: suppose that q_i has been defined for some $i < n$. Set $r_{i+1} := \lceil q_i (x_{i+1}/x_i)^M \rceil$. Again by Bertrand's postulate, we can find $r_{i+1} < q_{i+1} < 2r_{i+1}$ so as to get

$$\frac{q_i x_{i+1}^M}{x_i^M} \leq r_{i+1} < q_{i+1} < 2r_{i+1} < 3\frac{q_i x_{i+1}^M}{x_i^M}.$$

Thus, the primes $\{q_i\}$ are increasing and verify

$$x_0^M < q_0 < 3x_0^M; \qquad \left(\frac{x_{i+1}}{x_i}\right)^M < \frac{q_{i+1}}{q_i} < 3\left(\frac{x_{i+1}}{x_i}\right)^M, \quad 0 \leq i \leq n-1.$$

Now the following calculation, performed for each fixed inequality, leads to the conclusion.

$$
\begin{aligned}
\prod_{i=0}^{n} q_i^{a_i} &= q_0^{\sum_{i=0}^{n} a_i} \prod_{i=0}^{n-1}\left(\left(\frac{q_{i+1}}{q_i}\right)^{\sum_{j=i+1}^{n} a_j}\right) \\
&< 3^{\sum_{i=0}^{n} a_i} x_0^{M\sum_{i=0}^{n} a_i} \prod_{i=0}^{n-1}\left(3^{\sum_{j=i+1}^{n} a_j} \cdot \left(\frac{x_{i+1}}{x_i}\right)^{M\sum_{j=i+1}^{n} a_j}\right) \\
&\leq 3^{(n+1)\sum_{i=0}^{n} a_i} x_0^{M\sum_{i=0}^{n} a_i} \prod_{i=0}^{n-1}\left(\frac{x_{i+1}}{x_i}\right)^{M\sum_{j=i+1}^{n} a_j} = 3^{(n+1)\sum_{i=0}^{n} a_i}\left(\prod_{i=0}^{n} x_i^{a_i}\right)^M \\
&< \left(\prod_{i=0}^{n} x_i^{b_i}\right)^M = x_0^{M\sum_{i=0}^{n} b_i} \prod_{i=0}^{n-1}\left(\frac{x_{i+1}}{x_i}\right)^{M\sum_{j=i+1}^{n} b_j} \\
&< q_0^{\sum_{i=0}^{n} b_i} \prod_{i=0}^{n-1}\left(\frac{q_{i+1}}{q_i}\right)^{\sum_{j=i+1}^{n} b_j} = \prod_{i=0}^{n} q_i^{b_i}.
\end{aligned}
$$

□

We are now ready for the

Proof of Theorem 6.1. We prove the nontrivial implication. Assume that $\underline{u} \not\ll \underline{v}$; then, by Theorem 2.3 there exists an increasing sequence of real numbers $\{x_i\}$ which does not satisfy (1). We can suppose that $x_0 > 1$. Now Proposition 6.2 ensures the existence of a prime sequence which contradicts (1). The last assert is proved in the same fashion of Corollary 2.4. □

As a further consequence of Proposition 6.2, we can define the classes of β-linearizations by replacing real numbers with primes in Definition 2.5, according to the straightforward

COROLLARY 6.3 (to Proposition 6.2). *Every β-linearization is induced by some sequence of primes.*

Therefore, we also obtain a new proof of the first part of Property 2.6. Finally, the proof of Theorem 4.3 can be adapted to obtain the

THEOREM 6.4. *Let P denote the set of primes. Then $\beta_M \Rightarrow_P \beta_L$ if and only if $L <^w M$.*

Proof. We proceed analogously to the basic case, using Proposition 6.2 in the end of Lemma 4.6 to guarantee a prime sequence $\{x_i\}$ satisfying either (6) or the system with reversed inequalities. By doing so, we obtain the 'prime number' version of Lemma 4.6. Subsequently, we follow the argumentation of the basic case. □

The above results show that prime sequences may provide a valid tool for relating the β-linearizations to arithmetical questions. We will give some more details in the next section.

7. Colored β-Linearizations

This section is devoted to the initial motivation which led us to the current analysis. Admittedly we have not been able, so far, to find connections worth mentioning between the coloring properties here described and the concept of β-linearization.

DEFINITION 7.1. Let $(x_n x_{n-1} \dots x_0)_2$, $(y_n y_{n-1} \dots y_0)_2$ be the binary representations of the nonnegative integers x, y for some suitable n large enough. The *exclusive or* between x and y is the nonnegative integer $x \oplus y = (z_n z_{n-1} \dots z_0)_2$ such that for all i, $z_i := x_i \oplus y_i$, recalling that $\oplus$ is the exclusive or (*XOR*) between two binary digits.

DEFINITION 7.2. For a fixed $\underline{u} = (u_0, \dots, u_k) \in D_h^k$, let $I \subseteq \{0, 1, \dots, k\}$ be such that $i \in I \Leftrightarrow u_i$ is odd. By definition, the map c_h^k sends $\underline{u}$ to $\bigoplus_{i \in I} i$ if $I \neq \emptyset$, to 0 otherwise, e.g.

$$c_2^3(1001) = 3 = c_2^3(0110), \qquad c_6^3(4200) = 0 = c_5^3(0131).$$

DEFINITION 7.3. Let $\{q_i\}_{0 \le i \le k}$ be an increasing sequence of primes and $n \in A_h^k(q_0, \dots q_k)$. The *color* of n is defined as $c(n) := c_h^k(\underline{u})$, where $\underline{u} \in D_h^k$ and u_i is the power of q_i in the factorization of n.

If $k = 3$ and $\{q_i\} = \{2, 3, 5, 7\}$, following the above examples we get $c(14) = c(15) = 3$, $c(144) = c(2625) = 0$. Now we expose the two symmetries mentioned in the Introduction.

FACT 7.4. Let $\{\nu_i\}_{1 \le i \le 35}$ be the enumeration of the elements of $A_4^3(2, 3, 5, 7)$ such that $i < j \Rightarrow \nu_i < \nu_j$. Then, for any index i, $c(\nu_i) = c(\nu_{36-i})$. That is, the string consisting of the colors of $A_4^3(2, 3, 5, 7)$, ordered with respect to the usual ordering in $\mathbf{N}$, is palindrome.

FACT 7.5. Let $\{\eta_i\}_{1 \le i \le 20}$ be the enumeration of the elements of $A_3^3(2, 3, 5, 7)$ such that $i < j \Rightarrow \eta_i < \eta_j$. Then, for any index i, $c(\eta_i) = 3 - c(\nu_{21-i})$.

The latter fact can be regarded as a 'reversed palindromy', by associating the colors as follows: $0 \leftrightarrow 3$, $1 \leftrightarrow 2$. The reader may check that the the two strings are

0 1 0 2 1 3 3 0 2 2 0 3 3 1 1 2 0 0 0 2 1 1 3 3 0 2 2 0 3 3 1 2 0 1 0,

0 1 0 2 1 3 3 2 2 0 3 1 1 0 0 2 1 3 2 3.

These two symmetries are not a full consequence of the given combinatorial definitions; one can easily see this by changing the prime numbers involved and checking

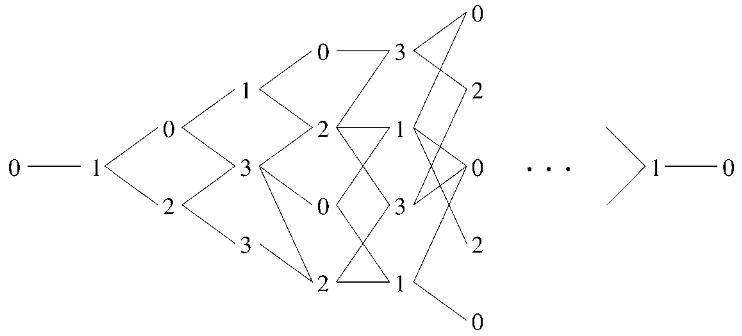

Figure 6. $(D^3_4, \ll)$ colored.

the strings obtained. We wish to understand whether the information carried by the coloring is a proper tool for investigating the β-linearizations. In Figure 6 we have depicted $(D^3_4, \ll)$ with colors in place of vectors (the coloring of any D^3_h is easily seen to be symmetric in the sense that $c(s, t, u, v) = c(v, u, t, s)$).

By comparing the partial order in Figure 6 to the palindrome string obtained for $A^3_4(2, 3, 5, 7)$ one notices that all the elements of the same level are grouped together, up to permutations, in the related β-linearization. For clearness, we write the beginning of the palindrome string using separation marks:

$$0 - 1 - 02 - 133 - 0220 - 3311 - 20002 - \cdots.$$

It might be interesting to understand whether the above phenomenon occurs in every β-linearization of D^3_4 and more generally in every D^k_h. Which are the possible colored linearizations or β-linearizations of a fixed poset? Which other sequences of primes behave like 2, 3, 5, 7? Can palindromy provide a way for characterizing a certain class of prime numbers? In the end of Section 3 we have shortly considered arithmetical progressions with only three numbers. This kind of investigation might involve arbitrarily long (finite) arithmetical progressions, by choosing D^k_h with k large enough. As in the case $h = 3, k = 2$, it is desirable to show that only certain β-linearizations may arise. Coloring these linearizations might provide some valid information. We think that a satisfactory knowledge of the admitted linearizations might enable us to describe certain properties of **N** using a combinatorial language.

To conclude, we remark that some work is in progress in order to rephrase the above coloring inside a graph-theoretical environment. More precisely, the coloring of D^k_h has been interpreted as the greedy edge-coloring of a suitable linear $(k+1)$-uniform hypergraph $(B^k_h, \prec)$, where $\prec$ is a total order given to the edges (for the basic definitions related to greedy colorings, see, for example, [4]; for details concerning the hypergraphs B^k_h, see [9]).

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