Multidimensional Systems and Signal Processing, 12, 365–376, 2001

Gröbner Bases for Problem Solving in Multidimensional Systems*

C. CHAROENLARPNOPPARUT†
121 Electrical Engineering East, The Pennsylvania State University, University Park, PA 16802, USA

N. K. BOSE† nkb@stspbkn.ee.psu.edu
121 Electrical Engineering East, The Pennsylvania State University, University Park, PA 16802, USA

Received February 2, 2001; Revised April 30, 2001

Abstract. The objective here is to underscore recent usage of the algorithmic theory of Gröbner bases in multidimensional systems since that possibility was highlighted about fifteen years back. The main contribution here focuses on the constructive aspects of the solution, known to exist, of the two-band multidimensional IIR perfect reconstruction problem using Gröbner bases. Other recent research results on the subject with future prospects are also briefly cited.

Key Words: Gröbner bases, multidimensional systems, IIR perfect reconstruction filter banks, multivariate polynomial matrix factorization

1. Introduction

The topic of Gröbner bases provides an algorithmic method for solving a number of computability and decidability problems concerning the ideal; for example, given a multivariate polynomial $f(\mathbf{z})$ and an ideal $\boldsymbol{I}$ specified by a finite number of generators, one can perform computations in the original polynomial coefficient field to decide constructively whether or not $f(\mathbf{z})$ belongs to $\boldsymbol{I}$. Many computational problems that are extremely difficult for polynomial ideals generated by arbitrary bases are very easy for polynomial ideals generated by *Gröbner bases*. One instance of particular interest in multidimensional systems theory is a formula expressing the greatest common divisor of a set of polynomials of several variables in terms of a *Gröbner basis* of the ideal generated by them [1].

The appropriateness of Gröbner bases over polynomial rings and modules for tackling problems in multidimensional systems, substantiated in 1985 by J. P. Guiver and N. K. Bose [2], continues to be exploited in the development of computational methods for output feedback stabilizability and stabilization of 2-D systems [3], matrix-fraction descriptions of multidimensional systems (when such descriptions exist) with tests for

* This research was supported by NSF Grant ECS-0080499.
† The authors are affiliated with the Department of Electrical Engineering, The Spatial and Temporal Signal Processing Center, The Pennsylvania State University, University Park, PA 16802, USA.

various primeness or coprimeness issues [4], construction of strong stabilizers of multidimensional systems and related problems [5]. The current status of the extent to which multidimensional multiband filter banks incorporating constraints of perfect reconstruction and linear phase can actually be constructed using algorithmic algebra is provided in [6]. Potential applications of interest involves the analysis, processing, coding and compression for transmission and reconstruction of multidimensional multimedia signals over bandwidth constrained communication channels. Scopes for further applications of the developed theory of multivariate polynomial matrix factorization and the constructive techniques in the theory of Gröbner bases over polynomial rings and modules, when the coefficients belong to a finite field, include multidimensional convolutional code construction [7] with intriguing possibilities for influencing compression and transmission of image, video, and multimedia signals over bandlimited channels, wired as well as wireless.

The two-band multidimensional infinite impulse response (IIR) perfect reconstruction filter bank problem by using the constructive theory of Gröbner bases is considered in the next Section. Subsequently, a recent result on the use of Gröbner bases in the classical problems of multidimensional filter stability test and stability margin determination is highlighted. This paper concludes with an open research problem of interest to multidimensional system theorists, on the concept of determinantal left and right joint factorization introduced in this paper.

2. Perfect Reconstruction n-D IIR Filter Bank Design

The problem of multidimensional (n-D), perfect reconstruction (PR), FIR filter bank design using Gröbner bases has been discussed in detail in [6]. In this section, its counterpart, the n-D PR, IIR filter bank design problem is tackled by using Gröbner bases. The final design method in this section is universally applicable to all sampling schemes (separable as well as nonseparable).

Let $\mathcal{S}$ be the set of all (structurally) stable proper rational functions (in reduced form i.e. relatively prime numerator and denominator polynomials) in n variables $z_1, z_2, \ldots, z_n$, having real coefficients. Structural stability requires that the denominator polynomial of the rational function be devoid of zeros in the closed unit polydisc $\bar{U}^n$. Therefore, structural stability implies bounded-input bounded-output (BIBO) stability but not vice versa [8]. This set $\mathcal{S}$ is known to form a commutative ring [9].

Definition 1. Two or more elements of $\mathcal{S}$ are said to be zero-coprime, if they are devoid of any common zero in the closed unit polydisc $\bar{U}^n$.

Definition 2. A minimum-phase rational function is a stable rational function with a stable inverse.

Definition 3. A square matrix with elements in $\mathcal{S}$ is said to be unimodular if its determinant is a minimum-phase rational function. The determinant of a unimodular matrix is a unit of the ring $\mathcal{S}$.

Similar to the FIR case, an m-band n-D PR IIR analysis (synthesis) filter bank can be described by an $m \times m$ polyphase matrix whose elements belong to the ring $\mathcal{S}$. For a pair of synthesis and analysis filter banks to satisfy a perfect reconstruction property, the polyphase matrix associated with the synthesis and thus, also analysis filter banks must be unimodular, or in more general algebraic terms, their determinants must be units in the ring. In the context of PR subband coding, the framework of IIR filter bank design then reduces to the construction of a polyphase matrix, associated with the analysis filter bank, whose determinant is a minimum-phase rational function.

Consider the two-band, two-dimensional case when the sampling matrix M is of the quincunx type.

$$M = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

The reason for considering the quincunx case is because it is the simplest *nonseparable* multidimensional sampling structure. Since the magnitude of the determinant of M is 2, the two-channel filter bank is critically sampled.

Let $H_0(z_1, z_2)$ be a stable proper transfer function of one of the two filters in the analysis filter bank, which can be designed by using various 2-D IIR filter design techniques such as Prony's method [10]. By applying the polyphase decomposition [9], [11, pp. 577–589], this prototype filter can be expressed in the polyphase form, using the technique demonstrated in Example 1 below for the needed clarification, as

$$H_0(z_1, z_2) = H_{00}(z_1 z_2, z_1 z_2^{-1}) + z_1 H_{01}(z_1 z_2, z_1 z_2^{-1}), \tag{1}$$

where $H_{00}(z_1, z_2)$ and $H_{01}(z_1, z_2)$ are the polyphase components of $H_0(z_1, z_2)$ with respect to the quincunx sampling raster. Assume that the necessarily stable polyphase components $H_{00}(z_1, z_2)$ and $H_{01}(z_1, z_2)$ are also zero-coprime on $\mathcal{S}$ i.e. by Definition 1 are devoid of zeros in the closed unit bidisc $\bar{U}^2$. This assumption is imposed so that the existence of the other filter $H_1(z_1, z_2)$ is guaranteed. It is necessary to point out that choosing the prototype filter $H_0(z_1, z_2)$ to be minimum-phase is not enough to ensure the satisfaction of the above assumption.

Example 1. Polyphase decomposition of 2-D IIR filter with respect to the quincunx sampling raster.

Consider a 2-D lowpass filter whose transfer function is given by

$$H_0(z_1, z_2) = \frac{\frac{4}{5} + \frac{2}{5}z_1 + \frac{1}{2}z_2 + \frac{1}{8}z_1 z_2}{1 + \frac{1}{5}z_1 + \frac{1}{3}z_2 + \frac{1}{4}z_1 z_2}. \tag{2}$$

Multiply the right-hand side of Eq. (2) by the identity,

$$\frac{\left(1+\frac{1}{4}z_1z_2\right)-\left(\frac{1}{5}z_1+\frac{1}{3}z_2\right)}{\left(1+\frac{1}{4}z_1z_2\right)-\left(\frac{1}{5}z_1+\frac{1}{3}z_2\right)},$$

involving the conjugate factor of the denominator polynomial to obtain,

$$H_0(z_1,z_2)=\frac{\frac{4}{5}+\frac{6}{25}z_1+\frac{7}{30}z_2+\frac{11}{120}z_1z_2-\frac{2}{25}z_1^2+\frac{3}{40}z_1^2z_2-\frac{1}{6}z_2^2+\frac{1}{12}z_1z_2^2+\frac{1}{32}z_1^2z_2^2}{1+\frac{11}{30}z_1z_2-\frac{1}{25}z_1^2-\frac{1}{9}z_2^2+\frac{1}{16}z_1^2z_2^2}. \tag{3}$$

Observe that each monomial in the denominator of $H_0(z_1,z_2)$ in Eq. (3) can be expressed as $(z_1z_2)^a(z_1z_2^{-1})^b$, where a and b are integers. Similarly, each monomial in the numerator can be expressed as either $(z_1z_2)^a(z_1z_2^{-1})^b$, or $z_1(z_1z_2)^a(z_1z_2^{-1})_b$, where a and b are again integers. Then, by inspection,

$$H_{00}(z_1,z_2)=\frac{\frac{4}{5}+\frac{11}{120}z_1-\frac{2}{25}z_1z_2-\frac{1}{6}z_1z_2^{-1}+\frac{1}{32}z_1^2}{1+\frac{11}{30}z_1-\frac{1}{25}z_1z_2-\frac{1}{9}z_1z_2^{-1}+\frac{1}{16}z_1^2}$$

$$H_{01}(z_1,z_2)=\frac{\frac{6}{25}+\frac{7}{30}z_2^{-1}+\frac{3}{40}z_1+\frac{1}{12}z_1z_2^{-1}}{1+\frac{11}{30}z_1-\frac{1}{25}z_1z_2-\frac{1}{9}z_1z_2^{-1}+\frac{1}{16}z_1^2}.$$

It can be verified that

$$H_0(z_1,z_2)=H_{00}(z_1z_2,z_1z_2^{-1})+z_1H_{01}(z_1z_2,z_1z_2^{-1}).$$

The goal here is to determine the other filter $H_1(z_1,z_2)$ such that its polyphase components $H_{10}(z_1,z_2)$ and $H_{11}(z_1,z_2)$ along with those of $H_0(z_1,z_2)$ can be used to form a unimodular matrix

$$\mathbf{H}(z_1,z_2)=\begin{bmatrix}H_{00}(z_1,z_2) & H_{01}(z_1,z_2)\\ H_{10}(z_1,z_2) & H_{11}(z_1,z_2)\end{bmatrix}.$$

Since the determinant of a unimodular matrix whose elements are in $\mathcal{S}$, must be a minimum-phase rational function, therefore

$$H_{00}(z_1,z_2)H_{11}(z_1,z_2)-H_{01}(z_1,z_2)H_{10}(z_1,z_2)=\frac{n(z_1,z_2)}{r(z_1,z_2)}, \tag{4}$$

where both polynomials $n(z_1,z_2)$ and $r(z_1,z_2)$ have no zeros in the closed unit bidisc.

Let the components of the first row of the polyphase matrix $\mathbf{H}(z_1,z_2)$ be $H_{0i}(z_1,z_2) = \frac{n_{0i}(z_1,z_2)}{d_{0i}(z_1,z_2)}$, $i=0,\ 1$. Eq. (4) can then be expressed as

$$\frac{n_{00}(z_1,z_2)}{d_{00}(z_1,z_2)}H_{11}(z_1,z_2) - \frac{n_{01}(z_1,z_2)}{d_{01}(z_1,z_2)}H_{10}(z_1,z_2) = \frac{n(z_1,z_2)}{r(z_1,z_2)}, \tag{5}$$

Since $n_{00}(z_1,z_2)$ and $n_{01}(z_1,z_2)$ are zero-coprime in $\bar{U}^2$ and $d_{00}(z_1,z_2)$, $d_{01}(z_1,z_2)$ are each devoid of zero in $\bar{U}^2$, consider the possibility of finding a polynomial $h(z_1,z_2)$ that vanishes when $n_{00}(z_1,z_2)d_{01}(z_1,z_2)$ and $n_{01}(z_1,z_2)d_{00}(z_1,z_2)$ do. Since the number of common zeros of the relatively prime polynomials $n_{00}(z_1,z_2)d_{01}(z_1,z_2)$ and $n_{01}(z_1,z_2)d_{00}(z_1,z_2)$ is necessarily finite, the desired $h(z_1,z_2)$ can be constructed by the procedure of J.P. Guiver and N.K. Bose [2, pp. 64–66]. It is noted that $h(z_1,z_2)$ can be constructed without finding all the common zeros of polynomials $n_{00}(z_1,z_2)d_{01}(z_1,z_2)$ and $n_{01}(z_1,z_2)d_{00}(z_1,z_2)$ [3]. By Hilbert's Nullstellensatz, it follows that there exists an integer N and polynomials $p(z_1,z_2)$ and $q(z_1,z_2)$ such that Eq. (6) below holds.

$$p(z_1,z_2)n_{00}(z_1,z_2)d_{01}(z_1,z_2) + q(z_1,z_2)n_{01}(z_1,z_2)d_{00}(z_1,z_2) = h^N(z_1,z_2). \tag{6}$$

The polynomials $p(z_1,z_2)$ and $q(z_1,z_2)$ can then be constructed by applying B. Buchberger's algorithm for construction of Gröbner basis. First, the reduced Gröbner basis G of the ideal generated by $n_{00}(z_1,z_2)d_{01}(z_1,z_2)$ and $n_{01}(z_1,z_2)d_{00}(z_1,z_2)$ is constructed (see [13, pp. 53–57] for an illustrative example). Then, the normal form [13, pp. 57] of the polynomial $h^i(z_1,z_2)$ for $i=1,2,\ldots$ ($h(z_1,z_2)$ has already been computed as in [2, pp. 64–66]) is computed with respect to G until the first time this normal form is zero. The least value of the power i in $h^i(z_1,z_2)$ for which the normal form is zero may be set to N in Eq. (6), and $p(z_1,z_2)$, $q(z_1,z_2)$ in Eq. (6) are obtained by retracing the steps in the algorithm implementing the ideal membership problem. It is noted that the standard Rabinowitsch's trick to deal with each inequation by introducing a slack variable may also be applied to calculate N as explicitly pointed out in [3, pp. 48–49] by using Gröbner basis and used in several other contexts [8, pp. 88–90], besides optimization theory, that require conversion of inequalities to equalities.

Divide both sides of Eq. (6) by $m(z_1,z_2)d_{00}(z_1,z_2)d_{01}(z_1,z_2)$ where $m(z_1,z_2)$ is an arbitrary polynomial that does not contain any zero in the closed unit bidisc to obtain,

$$\frac{p(z_1,z_2)}{m(z_1,z_2)}\frac{n_{00}(z_1,z_2)}{d_{00}(z_1,z_2)} + \frac{q(z_1,z_2)}{m(z_1,z_2)}\frac{n_{01}(z_1,z_2)}{d_{01}(z_1,z_2)} = \frac{h^N(z_1,z_2)}{m(z_1,z_2)d_{00}(z_1,z_2)d_{01}(z_1,z_2)}, \tag{7}$$

Comparing Eq. (5) and Eq. (7) yields

$$H_{11}(z_1,z_2) = \frac{p(z_1,z_2)}{m(z_1,z_2)}, \quad H_{10}(z_1,z_2) = -\frac{q(z_1,z_2)}{m(z_1,z_2)} \tag{8}$$

Since both $h^N(z_1,z_2)$ and $m(z_1,z_2)d_{00}(z_1,z_2)d_{01}(z_1,z_2)$ do not have any zero in the closed unit bidisc, the matrix $\mathbf{H}(z_1, z_2)$ in Eq. (1), formed by choosing $H_{10}(z_1, z_2)$ and $H_{11}(z_1, z_2)$ as in Eq. (8), is then unimodular. If desired, $H_1(z_1, z_2)$ can be designed to be FIR by choosing $m(z_1, z_2)$ to be a nonzero constant.

Consequently, the corresponding synthesis polyphase matrix $F(z_1, z_2)$ can be computed by

$$\mathbf{F}(z_1, z_2) = [\mathbf{H}(z_1, z_2)]^{-1}.$$

Comment: The fact that the number of common zeros of two relatively prime bivariate polynomials is finite plays a decisive role in the construction of $h(z_1, z_2)$ in Eq. (6). This procedure to do this, described in [2, pp. 64–66], generalizes, in principle, to the n-variate ($n > 2$) case provided the number of common zeros of the two relatively prime n-variate polynomials is finite. If the number of common zeros is finite, a constructive algorithm to construct $h(z_1, z_2, \ldots, z_n)$ has been supplied in [21]. When this is not the case, a constructive procedure for obtaining the n-variate counterpart of the right-hand side of Eq. (6) has not, yet, been advanced in general. Some promising special cases, however, can be tackled using Gröbner basis. One of the approaches involves the construction of a Gröbner basis of the ideal generated by the two generic polynomials followed by a search for an element of the Gröbner basis for absence of zeros in $\bar{U}^n$. If such an element is found, then it can be set to $h(z_1, z_2, \ldots, z_n)$ with $N=1$. Other approaches are also pursued by researchers (Lin/Lam/Galkowski/Xu).

A detailed algorithm for constructing the class of unimodular matrix $\mathbf{H}(z_1, z_2, \ldots, z_n) = \mathbf{H}(\mathbf{z})$ with a prespecified first row is then summarized next. Step 2 of the algorithm is dependent on the comment above.

2.1. Algorithm for Designing a Two-Band n-D PR IIR Filter Bank

- **Step 1.** Given two zero-coprime elements to form the first row of the analysis polyphase matrix $H_{00}(\mathrm{z}) = \frac{n_{00}(\mathrm{z})}{d_{00}(\mathrm{z})}$ and $H_{01}(\mathrm{z}) = \frac{n_{01}(\mathrm{z})}{d_{01}(\mathrm{z})}$, with respect to a suitable ordering, compute a Gröbner basis of an ideal generated by the polynomials $\boldsymbol{n}_{00}(\mathbf{z})\boldsymbol{d}_{01}(\mathbf{z})$ and $\boldsymbol{n}_{01}(\mathbf{z})\boldsymbol{d}_{00}(\mathbf{z})$, by using Buchberger's algorithm [13]. Let the Gröbner basis be $G = \{g_1(\mathbf{z}), \boldsymbol{g}_2(\mathbf{z}), \ldots, \boldsymbol{g}_s(\mathbf{z})\}$. Without loss of generality, $H_{00}(\mathbf{z})$ and $H_{01}(\mathbf{z})$ are assumed to be devoid of common factor in their denominator polynomials as well as in their numerator polynomials.
- **Step 2.** The existence of $h(\mathbf{z})$ is guaranteed because the elements of G are devoid of common zero in $\bar{U}^n$ [14]. Construct a polynomial $h(\mathbf{z})$ that is devoid of zeros in $\bar{U}^n$ and which vanishes when $n_{00}(z_1, z_2)d_{01}(z_1, z_2)$ and $n_{01}(z_1, z_2)d_{00}(z_1, z_2)$ do.
- **Step 3.** First find N by reducing $h^i(\mathbf{z})$, $i = 1,2,\ldots$ with respect to G until the normal form with respect to G is zero for the first time. Then by retracing the steps in Buchberger's algorithm [13, pp. 53–57], find the polynomials $p(\mathbf{z})$ and $q(\mathbf{z})$ such that

$$p(\mathbf{z})n_{00}(\mathbf{z})d_{01}(\mathbf{z}) + q(\mathbf{z})n_{01}(\mathbf{z})d_{00}(\mathbf{z}) = h^N(\mathbf{z})$$

• **Step 4.** Set

$$H_{11}(\mathbf{z}) = \frac{p(\mathbf{z})}{m(\mathbf{z})}, \quad H_{10}(\mathbf{z}) = \frac{q(\mathbf{z})}{m(\mathbf{z})}, \tag{9}$$

where $m(\mathbf{z})$ is an arbitrary polynomial that have no zeros in the closed unit polydisc.

3. Stability of Multidimensional Systems

Consider a multidimensional system whose input-output relationship is described by a rational transfer function

$$H(z_1, z_2, \ldots, z_n) = \frac{N(z_1, z_2, \ldots, z_n)}{D(z_1, z_2, \ldots, z_n)},$$

where the multivariate polynomials $N(z_1, z_2, \ldots, z_n)$ and $D(z_1, z_2, \ldots, z_n)$ are assumed to be mutually prime. The multidimensional system will be called robustly stable if and only if $D(z_1, z_2, \ldots, z_n)$ is zero-free on the unit polydisc, $\bar{U}^n = \{(z_1, z_2, \ldots, z_n) | |z_1| \le 1, |z_2| \le 1, \ldots, |z_n| \le 1\}$.

Recently, Curtin and Saba [15] applied Gröbner basis technique for determining robust stability as well as stability margin of a linear shifted-invariant multidimensional system. By using Buchberger's algorithm to generate a Gröbner basis with respect to lexicographical ordering, the triangular system of polynomial equations formed from the elements of the reduced Gröbner basis are solved recursively by back-substitution [13],[16]. Given $D(z_1, z_2, \ldots, z_n)$, they compute the reduced Gröbner basis with respect to lexicographical ordering, $z_1 > z_2 > \ldots > z_n > r$, of the ideal generated by the polynomials in $z_1, z_2, \ldots, z_n$ and $r = |z_1| = |z_2| = \ldots = |z_n|$ that result following multiplication of the left-hand sides of the system of equations

$$D(z_1, z_2, \ldots z_n) = 0,$$

$$\overline{D(z_1, z_2, \ldots, z_n)} = 0,$$

$$z_1 \frac{\partial D}{\partial z_1} \left(\overline{z_j \frac{\partial D}{\partial z_j}} \right) - \left(\overline{z_1 \frac{\partial D}{\partial z_1}} \right) z_j \frac{\partial D}{\partial z_j} = 0, \quad \text{for } j = 2, 3, \ldots, n,$$

by appropriate monomials. The procedure is nicely illustrated on the polynomial $D(z_1, z_2, z_3, z_4) = 5 + z_2 + z_4 + z_1 z_4 + z_3 z_4$ (devoid of zeros on $\bar{U}^4$ because the magnitude of one coefficient is larger than the sum of the magnitudes of the remaining coefficients). To facilitate reading, it is pointed out that the first polynomial of the Gröbner basis should be

$$r^6 (z_4 + 5)^2 (2r - 5)(2r + 5)(2r^2 + 2r - 5)(2r^2 - 2r - 5)(2r^2 - 5)$$
$$\times (2r^2 + 5)(2r^2 - 2r + 5)(2r^2 + 2r + 5).$$

The missing factor $(2r^2+2r-5)$ in [15, pp. 308] gives the least positive root $r = r_0 = \frac{\sqrt{11}-1}{2}$ of the preceding polynomial. All other calculation relevant to the example in [15] were verified to be correct using the software package SINGULAR [17] instead of Mathematica.

4. Determinantal Factorization and Related Issues

With the technical background and recent developments summarized above, the research problem proposed is described below in sufficient detail so that the theoretical challenges are simultaneously manifest with the approaches, background, tools for solution and the present as well as emerging applications in multidimensional systems theory that reap the benefits from such solutions. Let $K[\mathbf{z}]=K[z_1,\ldots,z_n]$ denote the set of polynomials in n variables $z_1,\ldots,z_n$ with coefficients in a specified field K. First the well-known determinantal factorization (left or right) of a multivariate polynomial matrix is defined, and then the concept of determinantal left and right joint factorization is formally introduced.

4.1. Determinantal Factorization

Let, for brevity, D (like $K[z_1,z_2,\ldots,z_n]$) be a Noetherian unique factorization domain (UFD). Consider a $p\times q$ matrix $F\in D^{p\times q}$, where $p\leq q$ $(p\geq q)$. Let $g\in D$ be a common divisor of the $p\times p$ minors ($q\times q$ minors) of F. A left (right) determinantal factorization of F with respect to g is $F=GF_1$ $(F=F_1G)$, where $G\in D^{p\times p}$, $F_1\in D^{p\times q}$, and $\det G=g$ $(G\in D^{q\times q}$, $(G\in D^{q\times q}$, $F_1\in D^{p\times q}$, and $\det G=g)$.

When $D=K[z_1,z_2]$, determinantal factorization is always possible for any matrix whose elements are in $K[z_1,z_2]$ and, of course, in this case factor primeness implies minor primeness [12]. Tests for zero primeness and minor primeness can be implemented by algorithm for construction of Gröbner basis [4] and the greatest common divisor may also be extracted in the determinantal factorization domain (DFD) $K[z_1,z_2]$ [4]. The fact that $K[z_1,z_2]$ is DFD is of fundamental importance in analysis and synthesis problems of 2-D systems. It is now well-known that $D=K[z_1,z_2,\ldots,z_n]$ is not a DFD, when $n\geq 3$, because here factor primeness of a matrix does not, necessarily, implies minor primeness. The joint left and right factorization, considered next, may be possible when left or right determinantal factorization are not possible.

4.2. Determinantal Left and Right Joint Factorization

Let D be a Noetherian UFD that is not a DFD. Consider $F\in D^{p\times q}$, where $p\leq q$. Suppose that the greatest common divisor of the $p\times p$ minors of F is not irreducible. Let $g_1\in D$ and $g_2\in D$ be two common divisors of the $p\times p$ minors of F. A determinantal left and right factorization of F with respect to g_1 and g_2 is $F=G_1F_1G_2$ where $G_1\in D^{p\times p}$, $F\in D^{p\times q}$, $G_2\in D^{q\times q}$, det $G_1=g_1$, and det $G_2=g_2$. The case when $p\geq q$ can be similarly defined.

The determinantal left and right joint factorization problem explained below is not only of mathematical interest but, conceivably, could be useful in multidimensional system analysis and synthesis. For example, the resulting decomposition may facilitate the cascade synthesis of a multi-input multi-output (MIMO or multivariable) multivariate polynomial transfer matrix, where a block realizing a particular matrix may be associated with a particular property not readily discernible in the composite matrix.

It is possible that a multivariate polynomial matrix with elements in $K[\mathbf{z}] = K[z_1, \ldots, z_n]$ might not yield a left or a right determinantal factorization but a left and right joint factorization may be possible as illustrated next. Consider the 4×5 matrix A shown below whose elements belong to the ring $K[\mathbf{z}] = K[z_1, z_2, z_3]$. This particular matrix over a binary field was, possibly, first used by Paul Wiener in his study of multidimensional convolutional codes.

$$A = \begin{bmatrix} z_1 & z_2 & z_3 & 0 & z_2 \\ z_2 & z_1 & z_3 & z_1 & 0 \\ z_1 & z_2 & 0 & z_2 & z_3 \\ z_1 & z_1 & 0 & 0 & z_1 + z_3 \end{bmatrix}$$

It is easily shown by routine manipulation that the 5 reduced minors of the 5 major determinants of A have a common zero at (0,0,0). From a recently proved result [18], a necessary and sufficient condition for a determinantal (left) factorization to hold in the special case of a $m \times (m+1)$ matrix (i.e. the number of columns exceed the number of rows by exactly one) is that its reduced minors be zero coprime. Therefore, the above matrix A does not have a determinantal (left) factorization. A joint left and right factorization is exhibited below both when K is the finite binary field $\mathcal{Z}_2$ and also when it is the field of real numbers $\Re$. The computations are easily implemented by using the computer algebra system *SINGULAR* which is suitable for calculating Gröbner bases of ideals in $K[z_1, z_2, \ldots, z_n]$ as well as modules whose elements belong to $K[z_1, z_2, \ldots, z_n]$.

The factorizations in fields $\mathcal{Z}_2$ and $\Re$ are given below.

$$A = \begin{bmatrix} z_1 & 0 & 0 & 1 \\ z_2 & 1 & 1 & 1 \\ z_1 & 1 & 0 & 1 \\ z_1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & z_2 & z_2 + z_3 \\ 0 & z_1 + z_2 & 1 & 0 & z_1 + z_2 + z_3 \\ 0 & z_2 & 1 & z_1 & z_1 + z_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & z_3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} z_1 - z_2 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 & -1 & -1 \\ z_2 & z_1 & 1 & z_1 & 0 \\ z_1 & z_2 & 0 & z_2 & z_3 \\ z_1 & z_1 & 0 & 0 & z_1 + z_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & z_3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The problem is to investigate necessary and sufficient conditions for a left and right joint multivariate polynomial factorization to hold and explore algorithmic approaches to construct such a factorization. The case when the specified matrix is not of full normal rank needs to be addressed subsequently. The limitations of Gröbner basis theory for the purpose need to be fully delineated.

5. Summary

The multidimensional filter bank design problem considered in [6] is limited to finite impulse response (FIR) filters. In many applications, infinite impulse response (IIR) filters are more desirable because of their computational efficiency and lower storage requirement. Satisfying the perfect reconstruction (PR) condition is equivalent to having the analysis and synthesis filter banks that are unimodular or units in the ring. For this IIR case, the units of the ring are matrices whose determinants are minimum phase transfer functions. The problem of designing a PR IIR filter bank is then reduced to the finding of a polyphase matrix $\mathbf{H}(\mathbf{z})$, associated with the analysis filter bank, that is a stable proper rational matrix with minimum phase determinant. The polyphase matrix $\mathbf{F}(\mathbf{z})$, associated with the synthesis filter bank, can be obtained by directly computing the inverse of the polyphase matrix of the analysis filter bank. Since the determinant of the polyphase matrix $\mathbf{H}(\mathbf{z})$ is minimum phase, the synthesis filter bank associated with the polyphase matrix $\mathbf{F}(\mathbf{z})$ is guaranteed to be stable. If one of the filters in the analysis IIR filter bank is specified and its polyphase components are devoid of common zeros, then similar to the FIR case, it is always possible to obtain the complete analysis-synthesis IIR filter bank by unimodular completion [9]. The solution to this type of unimodular completion problem is feasible because a set of all proper stable rational multivariate functions is known to form a type of commutative ring for which, the Quillen-Suslin theorem holds [9, p. 1405], [19]. In the multiband IIR PR problem, the common zeros in the elements of the first row of the analysis polyphase matrix are permitted as long as the common zeros do not fall in $\bar{U}^n$. This prevents the direct application of the constructive methods [19, 20] known for unimodular completion of a row of zero-coprime elements belonging to a ring. This paper shows the use of Gröbner basis theory in algorithmic algebra for unimodular completion over the ring of stable rational multivariate functions when the elements of the first row of a 2×2 polyphase matrix are not necessarily zero-coprime as long as common zeros in $\bar{U}^n$ are absent. Other recent as well as future uses of Gröbner basis theory in multidimensional systems are cited.

Acknowledgments

The authors thank Dr. Z. Lin, Dr. L. Xu and the reviewers for their careful reading of the paper and valuable comments. This research was supported by NSF Grant ECS-0080499.

References

1. S. V. Duzhin and S. V. Chmutov, "Gaydar's formula for the greatest common divisor of several polynomials," *Communications of the Moscow Mathematical Society*, 1993, pp. 171–172.
2. N. K. Bose, *Multidimensional Systems Theory: Progress, Directions, and Open Problems*, Dordrecht, Holland: D. Reidel Publishing Co., 1985.
3. Li Xu, O. Saito and K. Abe, "Output feedback stabilizability and stabilization algorithms for 2-D systems," *Multi Systems and Signal Processing*, vol. 5, Jan. 1994, pp. 41–60.
4. N.K. Bose and C. Charoenlarpnopparut, "Multivariate matrix factorization: new results," Mathematical Theory of Networks and Systems, Proceedings of MTNS-98, Padova, Italy, 1998, pp. 97–100.
5. J. Q. Ying, Li Xu and Z. Lin, "A computational method for determining strong stabilizability of n-D systems," *J. Symbolic Computation*, vol. 27, 1999, pp. 479–499.
6. C. Charoenlarpnopparut and N.K. Bose, "Multidimensional FIR filter bank design using Gröbner bases," *IEEE Transactions on Circuits and Systems II: Analog and Digital Signal Processing*, vol. 46, Dec. 1999, pp. 1475–1486.
7. E. Fornasini and M. E. Valcher, "Algebraic aspects of two-dimensional convolutional codes," *IEEE Trans. Inform. Theory*, vol. 40, July 1994, pp. 1068–1082.
8. N. K. Bose, *Applied Multidimensional Systems Theory*, New York: Van Nostrand Reinhold, 1982.
9. S. Basu, "Multi-dimensional filter banks and wavelets–a system theoretic perspective," *J Franklin Inst.*, 335B, 8, 1998, pp. 1367–1409.
10. J.S. Lim, "*Two-dimensional Signal and Image Processing*," Prentice Hall, Englewood Cliffs, New Jersey, 1990.
11. P.P. Vaidyanathan, "*Multirate Systems and Filter Banks*," Prentice Hall, Englewood Cliffs, New Jersey, 1993, pp. 577–589.
12. J.P. Guiver and N.K. Bose, "Polynomial matrix primitive factorization over arbitrary coefficient field and related results," *IEEE Transactions on Circuits and Systems*, CAS-29, no. 10, October 1982, pp. 649–657.
13. W.W. Adams and P. Loustaunau, "*An Introduction to Gröbner Bases*," Graduate Studies in Mathematics, vol. 3, American Mathematical Society, 1994.
14. S. Shanker and V.R. Sule, "Algebraic geometric aspects of feedback stabilization," *SIAM Journal on Control and Optimization*, vol. 30, no. 1, 1992, pp. 11–13.
15. E. Curtin and S. Saba, "Stability and margin of stability tests for multidimensional filters," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 46, Jul. 1999, pp. 806–809.
16. D. Cox, J. Little and D. O'Shea, "*Ideals, Varities, and Algorithm: An Introduction to Computational Algebraic Geometry and Commutative Algebra*," 2^{nd} edition, Springer-Verlag, New York, 1996.
17. G.-M. Greuel, G. Pfister, and H. Schoenemann. Singular version 1.2 User Manual, In: Reports On Computer Algebra, number 21. Centre for Computer Algebra, University of Kaiserslautern, June 1998, http://www.mathematik.uni-kl.de/~zca/Singular.
18. Z. Lin, "Notes on n-D polynomial matrix factorization," *Multidimensional Systems and Signal Processing*, vol. 10, no. 4, October 1999, pp. 379–393.

19. A. Logar and B. Sturmfels, "Algorithms for the Quillen–Suslin theorem," *J. Algebra*, vol. 145, 1992, pp. 231–239.
20. H. Park and C. Woodburn, "An algorithmic proof of Suslin's stability theorem for polynomial rings," *Journal of Algebra*, vol. 176, 1995, pp. 277–298.
21. L. Xu, J.Q. Ying and O. Saito, "Feedback stabilization for a class of MIMO n-D systems by Gröbner basis approach," In *Abstract of First International Workshop on Multidimensional Systems*, Lagow, Poland, 1998, pp. 88–90.