Monomial Orderings, Young Diagrams and Gröbner Bases

Nikolay N.Vasiliev

Steklov Institute of Mathematics at St.Petersburg,
Russian Academy of Sciences, Nab. Fontanki, 27, St. Petersburg, 191011, Russia;
vasiliev@pdmi.ras.ru

Abstract. We prove a generalization of a classic Dickson lemma for Young diagrams defined in this paper and related to the polynomials from an ideal in $K[x_1, x_2, x_3, ..., x_n]$. We define an universal base of an ideal using all minimal elements in the partially ordered set of Young diagrams. This universal base is different from the universal Gröbner base in the sense of Robbiano and Mora [1]. Nevertheless we prove that their universal Gröbner base $UGB(I)$ is a subset of the universal base. We give also a new proof of finiteness of elements of all Gröbner bases using a generalization of Dickson's lemma for the Young diagrams of polynomials introduced in this paper.

1 Dickson Sets

Definition 1. *Let $(T, \prec)$ be the set with partial ordering $\prec$. We say that T satisfies the decreasing chains breaking condition, if for any sequence $(x_i)_{i=1}^{\infty}$ where $x_i \in T$ and $x_1 \succeq x_2 \succeq x_3 \succeq x_4 ...$ $(\exists n)(\forall m \geq n)(x_n = x_m)$.*

In other words any decreasing chains of elements of T is stabilized.

Definition 2. *The partially ordered set $(T, \prec)$ is a Dickson set if any infinite subset $S \subset T$ contains a pair of elements x, y with $x \prec y$ and $x \neq y$.*

Let $\max(T)$ and $\min(T)$ be the sets of all maximal and minimal elements of T, respectively

$\max(T) \stackrel{\text{def}}{=} \{x \in T \mid (y \succeq x) \Rightarrow (y = x)\}$

$\min(T) \stackrel{\text{def}}{=} \{x \in T \mid (y \preceq x) \Rightarrow (y = x)\}$.

Corollary 1. Let T be a Dickson set. Then $\forall S \subset T$ the sets $\max(S)$ and $\min(S)$ are finite.

Proof. These sets contain pairwise incomparable elements only.

Definition 3. *Given a partially ordered set T a subset $S \subset T$ is increasing ideal in T (decreasing ideal in T, respectively) if*

$(s \in S) \wedge (x \succeq s) \Rightarrow (x \in S)$ for increasing ideal and

$(s \in S) \wedge (x \preceq s) \Rightarrow (x \in S)$ for decreasing ideal, respectively.

The sets of all increasing ideals and all decreasing ideals in T are partially ordered sets too if we use the inclusion as an ordering relation.

$(A \preceq B) \Leftrightarrow (A \subset B)$.

Definition 4. *A partially ordered set $(T, \prec)$ is Noetherian if the set of increasing ideals of T satisfies the increasing chains breaking condition.*

It is easy to see that if T itself satisfies the decreasing chains breaking condition then T is Noetherian.

The classic Dickson lemma [2] used for the proof of finiteness of elements in any Gröbner base of an ideal shows that the set of all monomials M of n variables $x_1, x_2, ..., x_n$ or what is equivalent the set Z_+^n is the Dickson set. This set is also Noetherian.

The main result of this chapter is the following:

Theorem 1. *Let $(T, \prec)$ be a Noetherian Dickson partially ordered set. Then the set YT of all finitely generated increasing ideals of T is also a Dickson set.*

Proof. Let $S_1, S_2, S_3, ...$ be an infinite sequence of finitely generated ideals of T. We shall prove that $(\exists i, j)$ such that $S_i \subset S_j$. Each ideal S_i is generated by a finite set of elements. Let $\{g_{i,k}\}_{k=1}^{d_i}$ be the generators of S_i. Suppose $S_i \not\subset S_j$ $\forall (i,j)$ we have $\forall i > 1 \quad S_1 \not\subset S_i$.

This implies that $\forall i > 1 (\exists k_i)(1 < k_i \leq d_1)$ such that $g_{1,k_i} \notin S_i$. We now have the function $\phi : i \to k_i$. This function is defined for all naturals not equal to 1 and acts on the finite set $\{1, 2, ..., d_1\}$. Therefore, $(\exists k)(1 \leq k \leq d_1)$ that $\phi^{-1}(k)$ is infinite. We have for any $i \in \phi^{-1}(k)$ $f_1 = g_{1,k} \notin S_i$. Thus, all generators of S_i for an infinite set of $i \in \phi^{-1}(k)$ are outside the ideal $\{f_1\}$ generated by f_1. Now we can renumber our infinite set of ideals $S_i \quad i \in \phi^{-1}(k)$ as $\tilde{S}_1, \tilde{S}_2, \tilde{S}_3, ...$ to apply our previous construction to the new set of ideals $\tilde{S}_1, \tilde{S}_2, \tilde{S}_3, ...$ Making this we shall have a new element f_2 and a new infinite sequence of ideals, which all are outside $\{f_1, f_2\}$. Repeating the process further we could build the infinite sequence of strongly increasing ideals $\{f_1\}, \{f_1, f_2\}, \{f_1, f_2, f_3\}, ...$, which contradicts the Noetherity of the set T.

2 Young Diagrams of Polynomials

Consider from this viewpoint the set T of monomials of n variables $x_1, x_2, ..., x_n$ with a natural partial ordering $m_1 \prec m_2 \stackrel{\text{def}}{=} m_1 \mid m_2$. We identify T and Z_+^n.

Definition 5. *We define the n-dimensional Young diagrams as finitely generated decreasing ideal in T in the sense of theory of partially ordered sets.*

As a motivation of this definition we could say that the two-dimensional diagrams in this sense strongly correspond to the conventional Young diagrams used in the groups representations theory for the classification of representations of symmetric groups S_n [3]. Monomials as elements of the diagrams in our sense correspond to the cells of classic Young diagrams. It is shown in the diagram of Fig. 1.

Definition 6. *The degree of Young diagram Y is the number of elements in Y.*

As a consequence of Theorem 1 we have

Theorem 2. *The set of all n-dimensional Young diagrams is the Dickson set.*

Proof. It follows from Theorem 1 because the set Z_+^n is the Noetherian set and the Dickson set.

Corollary 2.1 For any infinite set of n-dimensional Young diagrams $Y_1, Y_2, Y_3, ...$ there exist two of them Y_i and Y_j such that $Y_i \subset Y_j$.

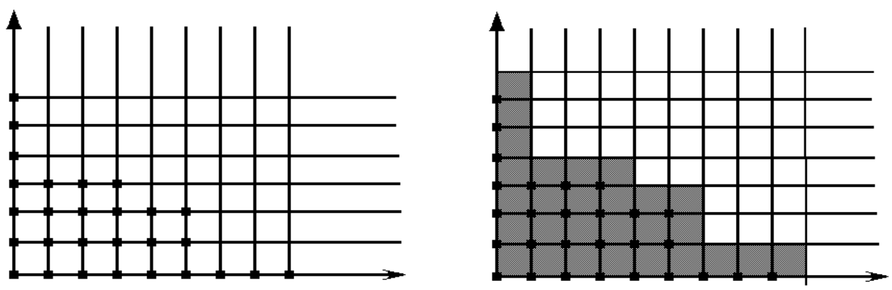

Fig. 1. Two-dimensional Young diagram of the polynomial $x^7 + x^8 + x^4y + x^5y + x^4y^2 + x^5y^2 + x^3y^3 + y^5 + y^6$.

Now we start to study ideals in the ring of polynomials $K[x_1, x_2, ..., x_n]$. From this moment the word "ideal" will be used in the sense of commutative algebra as an ideal of a commutative ring.

In the theory of Gröbner bases the following standard construction is used. If some admissible ordering $\prec Z_+^n$ is fixed then each $f \in K[x_1, x_2, x_3, ...x_n]$ defines its leading term $lead(f)$. All monomials $\{lterm(f)\}_{f\in I}$ generate a monomial ideal $Mon(I)$. Then the generators of the ideal $Mon(I)$ corresponds exactly to all leading terms of Gröbner base of ideal I with respect to the fixed admissible ordering. Instead of this we will not fix any admissible ordering but we define Young diagrams corresponded to polynomials in $K[x_1, x_2, x_3, ...x_n]$ using the natural partial ordering of monomials only.

Definition 7. *For $f \in K[x_1, x_2, x_3, ..., x_n]$ $supp(f)$ is the set of all nonzero monomials of polynomial f.*

Definition 8. *For $f \in K[x_1, x_2, x_3, ..., x_n]$ we define a Young diagram $Y(f)$ as* $Y(f) = \{m \in T = Z_+^n \mid (\exists \tilde{m} \in supp(f)) \quad m \mid \tilde{m}\}$.

It is evident that $Y(f)$ is a Young diagram.

According to Theorem 2 there exists in the set $\{Y(f) \mid f \in I\}$ only a finite subset of minimal diagrams.

We emphasize this subset of minimal diagrams corresponds to the polynomials from I as $UY(I)$. According to the minimality of elements of $UY(I)$ we have $\forall Y \in UY(I) \quad (\exists f \in I)$ such that supp$(f) \subset Y$ and $(Y(g) \subset Y \in UY(I)) \wedge (Y(g) \neq Y) \Rightarrow (g \not\subset I)$.

This property is very close to the irreducibility in the GB theory.

3 Universal Base of an Ideal

We start this part with the following Lemma.

Lemma 1. *Let $f_1, f_2 \in I$ such that $Y(f_1) = Y(f_2) \in UY(I)$. Then $\exists (c \neq 0)(c \in K)$ that $f_1 = cf_2$.*

Proof. If $f_1 \neq cf_2$ then there exists a linear combination $g = af_1 + bf_2$ that $(Y(g) \subset Y) \wedge (Y(g) \neq Y)$ in contradiction to the minimality of the diagram Y.

According to this lemma, if we normalize all the polynomials f making the sum of coefficients of all maximal monomials equal to 1, then all normalized polynomials $f \in I$ such that $Y(f) \in UY(I)$ are defined in unique way.

Definition 9. *Universal base $U(I)$ of an ideal $I \in K[x_1, x_2, x_3, ..., x_n]$ is the set of all normalized polynomials $f \in I$ such that $Y(f) \in UY(I)$.*

According to Theorem 2 the universal base of I is finite. The next theorem shows why we have used the word "universal".

Theorem 3. *Let $\prec$ be any admissible monomial ordering. Assume that $g \in Gb(I, \prec)$ is an element of Gröbner base of ideal I with respect to the ordering prec. Then $Y(g) \in UY(I)$ and $\exists c \in K$ that $cg \in U(I)$.*

Therefore, the universal base contains all elements of all Gröbner bases of ideal I, which correspond to all admissible monomial orderings if we neglect the multiplicative constants.

Proof. It is evident that $lterm_\prec(g) \in \max(g)$, where $\max(g)$ is the set of maximal monomials of the polynomial g. In the case if $Y(g) \notin UY(I)$ its Young diagram $Y(g)$ is not minimal and $\exists g_1 \in I$ that $Y(g_1) \subset Y(g)$.

Therefore, $lterm(g_1) \mid lterm(g)$. Since $g \in Gb(I, \prec)$ we have $lterm(g_1) = lterm(g)$. It shows that element g could be reduced to the $g1$ in contradiction to the irreducibility of the Gröbner base $Gb(I, \prec)$.

As a consequence of this theorem we obtain the well known theorem by Robbiano about finiteness of elements of all Gröbner bases.

4 Characterization of Elements of Universal Base

We prove in this part a theorem, which describes all elements of universal base $U(I)$ in terms of linear algebra. In particular, this theorem allows us to get a criterion for membership problem for universal Gröbner bases introduced by Robbiano and Mora.

Definition 10. *Let* $f \in K[x_1, x_2, x_3, ..., x_n]$. $L(f)$ *is a linear vector space over* K *generated by all monomials* $m \in Y(f)$.

Theorem 4. *The next conditions are equivalent.*

1. $Y(f) \in UY(I)$
2. $dim_K(I \bigcap L(f))) = 1$

Proof. $1 \Rightarrow 2$ follows from Lemma 1 since any two polynomials from I with the same Young diagram are proportional.

We now show that $2 \Rightarrow 1$. Let $dim_K(I \bigcap L(f)) = 1$. Since $f \neq 0$ and $f \in I \bigcap L(f))$ we have for any $a \in K$ $af \in I \bigcap L(f)$. Therefore, the element f is defined up to multiplicative constant. In the case if its diagram Y(f) is not minimal $Y(f) \notin UY(I)$ we have that $\exists$ some diagram Y_1 $Y_1 \subset Y$ such that $Y1 \in UY(I)$. Let $f_1 \in I$ such that $\text{supp}(f_1) \subset Y_1$. Evidently $f_1 \in L(f)$ and is not proportional to the polynomial f because their diagrams are not equivalent. We have found two linear independent elements in $I \bigcap L(f)$ that contradicts $dim_K(I \bigcap L(f)) = 1$.

Remark 1. Condition 2 of Theorem 4 can be checked effectively if we have any arbitrary Gröbner base of the ideal I. It could be done in the following way. Let Y be a Young diagram. Let $Y = \{m_i\}_{i=1}^{deg(Y)}$. We can write $m_i = \sum_{j=1}^{k} \alpha_{ij} l_j$, where all images $\tilde{l}_j$ of elements l_j in factor algebra $K[x_1, x_2, x_3, ..., x_n]$ are linearly independent.

Condition 2 of Theorem 4 in terms of elements of the matrix $A = (\alpha_{ij})$ will be equivalent to the condition $\text{rank}(A) = N - 1$, where N is the degree of Young diagram Y. Using any Gröbner base of the ideal I we define a canonical monomial basis of factor algebra $K[x_1, x_2, x_3, ..., x_n]/I$ and only finite number of elements of the basis are represented in the expansion $m_i = \sum_{j=1}^{k} \alpha_{ij} l_j$ even in the case if $dimK[x_1, x_2, x_3, ..., x_n]/I = \infty$!

Remark 2. The universal base $U(I)$ could contain more elements than the universal Gröbner base UGb(I) by Robbiano and Mora. Other elements of the base $U(I)$ which are not the elements of any Gröbner base can also be described in this context. If we generalize the definition of Gröbner base in an appropriate way for inadmissible orderings, but for weak admissible ones we can get all elements of generalized Gröbner bases for all weak admissible orderings. All elements of the universal base $U(I)$ indeed represent the elements of generalized Gröbner bases for all weak admissible orderings but it is out of the scope of the present paper.

5 Conclusion

We have described an universal base of ideal using the minimality of Young diagrams related to the polynomials from an ideal. This approach does not need to fix any admissible monomial ordering but uses the natural partial monomial ordering only. We have also given an algorithmic characterization of all elements of the universal base of an ideal and proved that the universal Gröbner base is the subset of our universal base.

Acknowledgements

A part of this work was done during my stay in IRMA (Strasbourg) in October–December 2002. I would like to thank CNRS for supporting this visit.

References

1. Mora, T., Robbiano, L.: The Grobner fan of an ideal. *J. Symb. Comput.* **6** (1988) 183–208
2. Cox, D., Little, J., O'Shea, D.: *Ideals, Varietes, and Algorithms. An Introduction to Computational Alebraic Geometry and Commutative Algebra.* Springer-Verlag, New York (1997,1992)
3. Young, A.: On the quantitative substitutional analysis. *Proc.Lond. Math. Soc.* **33** (1900), bf 34 (1902)