Optimized parameters of multilevel algorithms: conforming FEM

Svetozar Margenov

margenov@parallel.bas.bg

Institute for Parallel Processing, Bulgarian Academy of Sciences, Acad. G. Bonchev Str. Bl. 25-A, 1113 Sofia, Bulgaria

1. Introduction

The AMLI method reads as follows:

$C^{(1)} = A^{(1)}$;

for $k = 1, 2, \ldots, \ell - 1$

$$C^{(k+1)} = \begin{bmatrix} A_{11}^{(k+1)} & 0 \\ A_{21}^{(k+1)} & \tilde{A}^{(k)} \end{bmatrix} \begin{bmatrix} I & {A_{11}^{(k+1)}}^{-1} A_{12}^{(k+1)} \\ 0 & I \end{bmatrix},$$

where the Schur complement approximation is stabilized by

$$ {\tilde{A}^{(k)}}^{-1} = \left[I - p_\beta \left({C^{(k)}}^{-1} A^{(k)} \right) \right] {A^{(k)}}^{-1}.$$

The acceleration polynomial is explicitly defined by

$$p_\beta(t) = \frac{1 + T_\beta\left(\dfrac{1+\alpha-2t}{1-\alpha}\right)}{1 + T_\beta\left(\dfrac{1+\alpha}{1-\alpha}\right)},$$

were $\alpha \in (0,1)$ is a properly chosen parameter, and T_β stands for the Chebyshev polynomial of degree β.

Theorem

There exists $\alpha \in (0,1)$, such that the AMLI preconditioner $C = C^{(\ell)}$ has optimal condition number $\kappa\left(C^{-1}A\right) = O(1)$, and the total computational complexity is $O(N)$, if β satisfies the condition

$$4 > \beta > \frac{1}{\sqrt{1-\gamma^2}}.$$

The following estimates hold uniformly with respect to both mesh and coefficient anisotropy:

- $\gamma^2 < 3/4 \Rightarrow \beta \in \{2, 3\}$
- $\kappa(B_{11}^{-1} A_{11}^{(k+1)}) = O(1) \Rightarrow \mathcal{N}_{AMLI} = O(N)$

This seminar is devoted to further consideration of:

- Improved estimates of the CBS constant and the relative condition number $\kappa(B_{11}^{-1} A_{11}^{(k+1)})$ for particular classes of anisotropic elliptic problems.
- Construction of new locally optimized preconditioners of the first pivot block $A_{11}^{(k+1)}$.

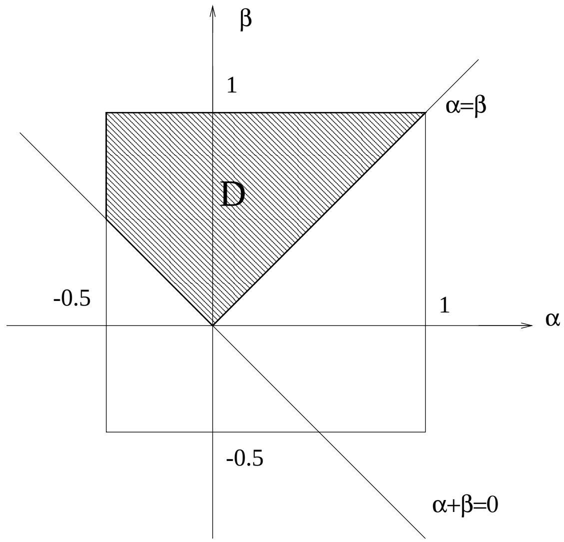
Remark: The improved estimate of the CBS constant means a better setting of the constant α in the acceleration polynomial $p_\beta(t)$ which directly leads to a faster convergence of the so constructed AMLI preconditioner.

Problems

Problem 1. **The following rapid estimates hold uniformly $\forall(\alpha,\beta)\in D$:**

$$\gamma^2 < \frac{3}{4}\,, \quad \kappa_A\left({B_{11}}^{-1}A_{11}\right) < 5.31\,, \quad \kappa_M\left({B_{11}}^{-1}A_{11}\right) < 1.88$$

Here, the domain D includes any case of mesh and/or coefficient anisotropy.



Polygonal approximation D of the domain $\widehat{D}$: $(\alpha,\beta)\in\widehat{D}\subset D$.

Derive improved estimates for γ, κ_A and κ_M if $(\alpha,\beta)\in\widetilde{D}\subset\widehat{D}\subset D$.

Problem 2. **The additive preconditioner of A_{11} was introduced as $B_{11} = \sum B_{11:E}$, where**

$$B_{11:E} = 2\,r\,c \begin{bmatrix} \alpha+\beta+1 & -1 & 0 \\ -1 & \alpha+\beta+1 & 0 \\ 0 & 0 & \alpha+\beta+1 \end{bmatrix}.$$

Find the optimal SPD matrix $B_{11:E}^{(opt)} = 2\,r\,c \begin{bmatrix} t_1 & t_4 & 0 \\ t_4 & t_2 & 0 \\ 0 & 0 & t_3 \end{bmatrix}$ such that

$$\kappa\left({B_{11:E}^{(opt)}}^{-1} A_{11:E} \right) = \min_{t_i} \kappa\left(B_{11:E}^{-1} A_{11:E} \right)\ , \forall (\alpha, \beta) \in D.$$

Then, the locally optimized additive preconditioner $B_{11}^{(lopt)} = \sum B_{11:E}^{(opt)}$.

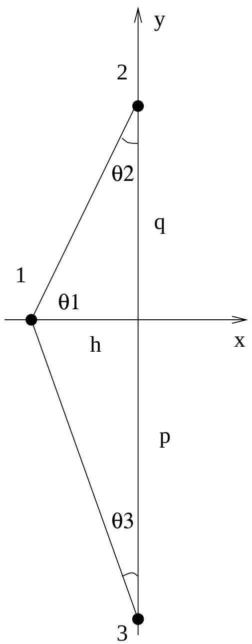
Exercises

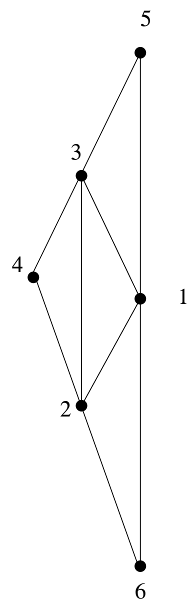
Exercise 1. **Derive the element stiffness matrix corresponding to the isotropic bilinear form**

$$a_e(u,v) = \int_e (u_x v_x + u_y v_y) de$$

for a given arbitrary non-degenerate triangle e.

Solution.





Let $\theta_1, \theta_2, \theta_3$ be the angles of the triangle, and let $a = \cot\theta_1$, $b = \cot\theta_2$, $c = \cot(\theta_3)$. The following relations are readily seen:

$$b = \frac{q}{h}\,, \quad c = \frac{p}{h}\,, \quad a = \cot(\pi - (\theta_2 + \theta_3)) = -\frac{\cot\theta_2\cot\theta_3 - 1}{\cot\theta_2 + \cot\theta_3} = \frac{h^2 - pq}{h(p+q)}.$$

For the element basis functions and their derivatives we have

$$\begin{aligned}
\phi_1 &= -\frac{x}{h}, & \phi_2 &= \frac{px + h(p+y)}{h(p+q)}, & \phi_3 &= \frac{qx + h(q-y)}{h(p+q)},\\
\partial_x\phi_1 &= -\frac{1}{h}, & \partial_x\phi_2 &= \frac{p}{h(p+q)}, & \partial_x\phi_3 &= \frac{q}{h(p+q)},\\
\partial_y\phi_1 &= 0, & \partial_y\phi_2 &= \frac{1}{p+q}, & \partial_y\phi_3 &= -\frac{1}{p+q}.
\end{aligned}$$

Taking into account that the derivatives of the element basis functions are constants and that

$$|e| = \int_e de = \frac{h(p+q)}{2},$$

we get the element stiffness matrix in the form

$$A_e = \frac{1}{2}\begin{pmatrix} b+c & -b & -c \\ -b & a+b & -a \\ -c & -a & a+c \end{pmatrix}.$$

Exercise 2. Let us consider the isotropic elliptic problem associated with the bilinear form

$$a(u,v) = \sum_{e \in \mathcal{T}_1} \int_e a(e)(u_x v_x + u_y v_y) de.$$

Derive improved estimates for κ_A and κ_M if the minimal angle of the initial triangulation $\mathcal{T}_1$ is restricted by a given parameter $\pi/3 \geq \tau > 0$.

Solution. Consider two consecutive meshes $\mathcal{T}_H \subset \mathcal{T}_h$, and let $E \in \mathcal{T}_h$ be a given arbitrary macro-element, where

$$\theta_1 \geq \theta_2 \geq \theta_3 \geq \tau \ ,$$

and respectively (see Lemma 2.1)

$$|a| \leq \ b \leq \ c \leq \ t = \cot \tau \ .$$

Now, we estimate $\alpha = \dfrac{a}{c}$, $\beta = \dfrac{b}{c}$ in terms of the parameter t, namely

$$\alpha = \frac{a}{c} = \frac{1-bc}{c(b+c)} \geq \frac{1-c^2}{2c^2} \geq \frac{1-t^2}{2t^2} \, ,$$

$$\beta = \frac{b}{c} = \frac{1-ac}{c(a+c)} = \frac{1-\alpha c^2}{c^2(\alpha+1)} \geq \frac{1-c^2}{2c^2} \geq \frac{1-\alpha t^2}{2t^2(\alpha+1)} \, ,$$

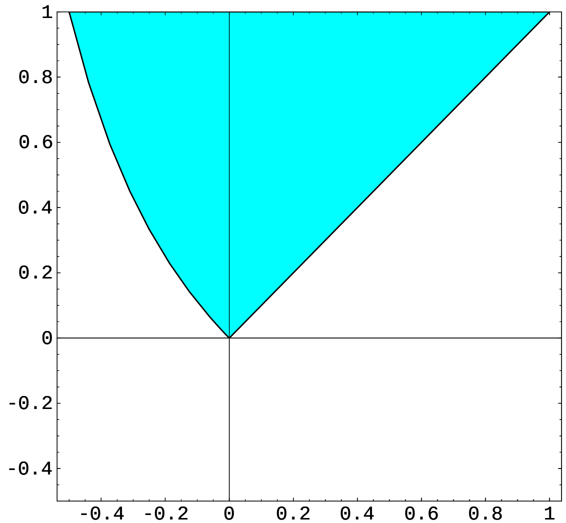
or

$$1 \geq \alpha \geq \frac{1-t^2}{2t^2} \, , \quad 1 \geq \beta \geq \max\left\{|\alpha|, \frac{1-\alpha t^2}{2t^2(\alpha+1)}\right\} \, .$$

In such a way we determine the subdomain $\widetilde{D}: \ (\alpha, \beta) \in \widetilde{D} \subset \widehat{D} \subset D$.

The exact domain $\widehat{D}$ is obtained when $\tau \to 0$ ($t \to \infty$), i.e.

$$1 \geq \alpha \geq -\frac{1}{2}\,, \quad 1 \geq \beta \geq -\frac{\alpha}{2(\alpha+1)}.$$



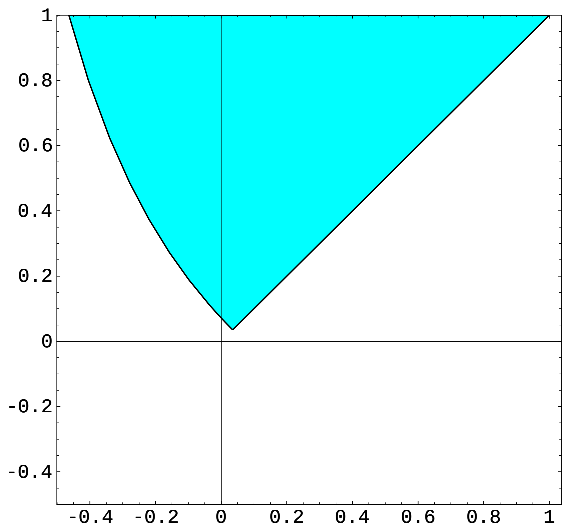
Domain $\widehat{D}$ of the parameters (α, β).

What we need to solve the problem, see Theorems 2.2, 2.3, is to improve the estimate of Lemma 2.2 in $\widetilde{D}$, i.e. to get a better $\mathcal{R}(\tau)$ such that

$$\frac{\alpha\beta+\alpha+\beta+1}{(\alpha+\beta+1)(\alpha+\beta+2)} \geq \mathcal{R}(\tau)\,, \quad \forall(\alpha,\beta) \in \widetilde{D}.$$

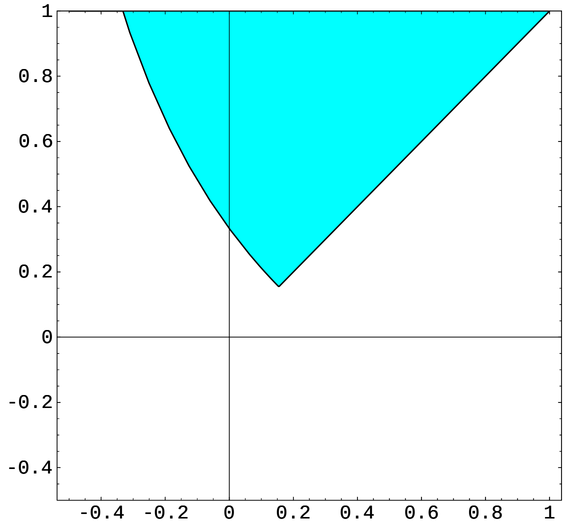
We consider three particular cases to illustrate the behavior of the related improved estimates. To the related analysis we have used symbolic computations with MATHEMATICA.

Case 1. $\tau = \frac{\pi}{12}$



$$\mathcal{R}(\frac{\pi}{12}) = \frac{52+30\sqrt{3}}{189+109\sqrt{3}}\,, \quad \kappa_A < 5.071\,, \quad \kappa_M < 1.820\,.$$

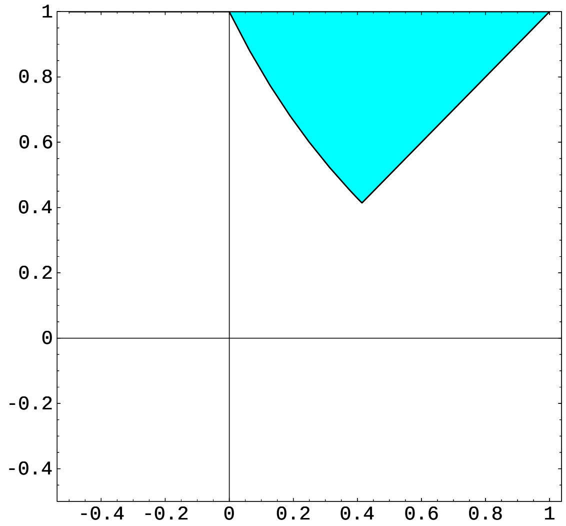
Case 2. $\tau = \frac{\pi}{6}$



Domain $\widetilde{D}$ **of the parameters** (α, β)**.**

$$\mathcal{R}(\frac{\pi}{12}) = \frac{3}{10}\,, \quad \kappa_A < 4.442\,, \quad \kappa_M < 1.667\,.$$

Case 3. $\tau = \frac{\pi}{4}$



Domain $\widetilde{D}$ of the parameters (α, β).

$$\mathcal{R}(\frac{\pi}{12}) = \frac{1}{3}\,, \quad \kappa_A < 3.732\,, \quad \kappa_M < 1.5\,.$$

τ	κ_A	κ_M
$\pi/12$	**5.071**	**1.820**
$\pi/6$	**4.442**	**1.667**
$\pi/4$	**3.732**	**1.500**

- **The presented results show that $\kappa_A(\tau_1) > \kappa_M(\tau_2)$ for all (τ_1, τ_2), i.e. the multiplicative preconditioner has always smaller condition number then the additive one.**
- **In the case $\tau = \pi/4$, the off diagonal entries of all element stiffness matrices are non positive, i.e., the global stiffness matrix is an M-matrix.**
- **The same estimates of κ_A and κ_M hold when $\tau \in [\pi/4, \pi/6]$, including the limit case $\theta_1 = \theta_2 = \theta_3 = \pi/6$.**