Parameter Identification in Partial Differential Equations - Tutorial

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We start presenting some examples which represent typical phenomena which occur while solving inverse problems. They are all concerned with the heat conduction, where the classical direct (well-posed) problem is to determine the temperature evolution in a body with known thermal parameters, given the initial temperature and the temperature or the heat flux on the whole boundary. Several types of inverse problems arise, we have a look at the following: The examples are taken from the book H.W. Engl, M. Hanke, A. Neubauer, *Regularization of Inverse Problems*, Kluwer, Dordrecht, 1996.

1 Sideways heat equation

This is the problem of determining the temperature on an inaccessible part of the boundary from heat flux and temperature measurements on other parts of the boundary. We will consider this task for the one dimensional linear heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \tag{1}$$

where we pose the problem of determining

$$f(t) := u(1,t) \quad \text{for } t \in \mathbb{R} \tag{2}$$

from measurements

$$g(t) := u(0,t) \quad \text{for } t \in \mathbb{R} \tag{3}$$

and the information that the left-hand boundary is of $\Omega = [0,1]$ is insulated, i.e. that

$$\frac{\partial u}{\partial x}(0,t) = 0 \quad \text{for } t \in \mathbb{R} \tag{4}$$

Assuming that for all $x \in [0,1], u(x,\cdot) \in \mathcal{L}^2(\mathbb{R})$, we can treat this problem by taking Fourier transforms: For $v = v(x,t)$, we denote by $\hat{v}$ its Fourier transform with respect to t, i.e.

$$\hat{v}(x,w) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iwt} v(x,t)dt, \quad w \in \mathbb{R} \tag{5}$$

Since $\hat{\frac{\partial v}{\partial t}} = iw\hat{v}$, (1) becomes

$$\frac{\partial^2 \hat{u}}{\partial x^2} = iw\hat{u} \quad \in [0,1] \times \mathbb{R}, \tag{6}$$

which implies together with the boundary values that

$$\hat{u}(x,w) = \frac{\cosh(x\sqrt{iw})\hat{f}(w)}{\cosh(\sqrt{iw})}, \quad x \in [0,1], w \in \mathbb{R} \tag{7}$$

and hence

$$\hat{f}(w) = \cosh(\sqrt{iw})\hat{g}(w), \quad w \in \mathbb{R}. \tag{8}$$

A formal solution for the problem of determining f from g can thus be obtained using the inverse Fourier transform

$$f(t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iwt} \cosh(\sqrt{iw})\hat{g}(w)dw, \quad t \in \mathbb{R} \tag{9}$$

However (9) makes sense only if the integral exists in some reasonable sense, which is the case if $\cosh(\sqrt{iw})\hat{g}(w)$ is an $\mathcal{L}^2$-function. But since $|\cosh\sqrt{iw}| = (\sinh^2\sqrt{w/2} + \cos^2\sqrt{w/2})^{1/2}$ goes to infinity exponentially as $|w| \to \infty$, this can only be the case if $\hat{g}$ decays very rapidly as $|w| \to \infty$, i.e if $\hat{g}$ is very smooth. Even then, arbitrarily small errors in the data g can lead to arbitrarily large errors in the result f: If g^δ is such that $||g - g^\delta||_{\mathcal{L}^2(\mathbb{R})} = \delta$, then also $||\hat{g} - \hat{g}^\delta||_{\mathcal{L}^2(\mathbb{R})} = \delta$. The result corresponding to the perturbed data g^δ is then determined by

$$\hat{f}^\delta(w) = \cosh\sqrt{iw}\hat{g}^\delta(w), \quad w \in \mathbb{R}. \tag{10}$$

Now let $w_0 \in \mathbb{R}$ be arbitrary and define g^δ via its Fourier transform by

$$\hat{g}^\delta(w) = \hat{g}^\delta(w) \text{ if } w \notin [w_0, w_0+1] \quad \text{and} \quad \hat{g}^\delta(w) = \hat{g}^\delta(w) + \delta \text{ if } w \in [w_0, w_0+1]. \tag{11}$$

Then, $||g - g^\delta||_{\mathcal{L}(\mathbb{R})} = ||\hat{g} - \hat{g}^\delta||_{\mathcal{L}(\mathbb{R})} = \delta$. For the corresponding error in the result, we then have that

$$\begin{aligned} ||f - f^\delta||^2_{\mathcal{L}(\mathbb{R})} &= ||\hat{f} - \hat{f}^\delta||^2_{\mathcal{L}(\mathbb{R})} = \delta^2 \int_{w_0}^{w_0+1} |\cosh\sqrt{iw}|^2 dw & (12) \\ &\geq \delta^2 \int_{w_0}^{w_0+1} \sinh^2\sqrt{\frac{w}{2}}dw \geq \delta^2 \sinh^2\sqrt{\frac{w_0}{2}}, & (13) \end{aligned}$$

so that for $w_0 > 0$

$$||f - f^\delta||_{\mathcal{L}^2(\mathbb{R})} \geq \frac{\delta}{2} \exp\sqrt{\frac{w_0}{2}} \tag{14}$$

holds. Thus, the data error δ is amplified essentially with the factor $\exp\sqrt{w_0/2}$ if it is concentrated around frequencies of w_0. High frequency components of the error are magnified very much. Thus, if we cannot exclude hight frequency error in our data, arbitrarily small data errors can lead to arbitrarily large errors in the result.

Remark 1 *A-priori information about the unknown solution f can stabilize the problem*

See. e.g. H. W. Engl, P. Manselli, *Stability estimates and regularization for an inverse heat conduction problem in semi-infinite and finite time intervals*, Numer. Funct. Anal. Optim. 10, 1989.

2 Backwards heat equation

A classical problem in connection with the heat conduction is the "backwards heat equation"

$$\triangle u = \frac{\partial u}{\partial t} \quad \text{in } \Omega \times [0,T] \tag{15}$$

with Dirichlet boundary condition

$$u = 0 \quad \text{in } \partial\Omega \times [0,T]. \tag{16}$$

The task (the inverse problem) is now to determine the initial temperature distribution $u(x,0),\ x \in \Omega$ from measurements of the final temperature distribution

$$f(x) = u(x,T) \quad \text{for } x \in \Omega. \tag{17}$$

The operator representing the "direct problem" mapping $u(x,.)$ onto f has strong smoothing properties and therefore as we will see the inverse problem does not depend continuously on the data as the following construction will show: Let $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ...$ be the eigenvalues for the Dirichlet problem in (15), $\phi_i, \phi_2, ... \in L^2(\Omega)$ denote the corresponding normalized eigenvalues, i.e. for all $k \in \mathbb{N}$, $||\phi_k|| = 1$ and

$$\begin{aligned} \triangle\phi_k + \lambda_k\phi_k &= 0 \quad \text{in } \Omega & (18) \\ \phi_k &= 0 \quad \text{on } \partial\Omega. & (19) \end{aligned}$$

Let

$$u_k(x,t) = \frac{1}{\lambda_k}\phi_k(x)\exp(\lambda(T-t)) \quad (x \in \Omega,\ t \in [0,T]), \tag{20}$$

then u_k fulfills (15) and (16). With $f_k := \frac{\phi_k}{\lambda_k}$, u_k solves further (17). Since $(\lambda_k) \to \infty$,

$$\lim_{k\to\infty} ||f_k||_{\mathcal{L}^2(\Omega)} = 0,$$

but

$$\lim_{k\to\infty} ||u_k(\cdot,0)||_{\mathcal{L}^2(\Omega)} = \lim_{k\to\infty} ||\frac{1}{\lambda}\exp(\lambda_k T)||_{\mathcal{L}^2(\Omega)} = +\infty.$$

Thus, if we consider f_k as perturbations of $f = 0$ with L^2 - error $\frac{1}{\lambda_k}$, the corresponding error in the solution of the inverse problem is amplified exponentially, namely by a factor $\exp(\lambda_k T)$. So, the solution of the inverse problem does not depend continuously on the data. The exponential amplification of the error already appears for arbitrarily small short time $T > 0$ but gets worse as T increases.

This kind of inverse problem can also be represented as an integral equation of the first kind. In the 1D case for an unbounded domain $\Omega = \mathbb{R}$ the finite temperature $u(\cdot,T)$ is related to the initial temperature $u(\cdot,0)$ via

$$\frac{1}{2\sqrt{\pi T}} \int_{-\infty}^{\infty} u(s,0)\exp\left(-\frac{(x-s)^2}{4T}\right) ds = u(x,T) \tag{21}$$

which is a "convolution equation" with a very smooth kernel.