An a priori error analysis of the local discontinuous Galerkin method for elliptic problems

Presentation in the lecture of DG method
in Radon Special Semester at Linz

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Reference:

P.Castillo, B.Cockburn, I.Perugia, and D. Schötzau,
An a priori error analysis of the local discontinuous Galerkin method for elliptic problems, SIAM J, Numer.Anal., 38(5)(2000),1676-1706.

Outline

Outline

1 Introduction

We consider the following classical model elliptic problem:

$$
\begin{aligned}
-\Delta u &= f \text{ in } \Omega, \\
u &= g_{\mathcal{D}} \text{ on } \Gamma_{\mathcal{D}}, \\
\frac{\partial u}{\partial n} &= \mathbf{g}_{\mathcal{N}} \cdot \mathbf{n} \text{ on } \Gamma_{\mathcal{N}},
\end{aligned}
\tag{1.1}
$$

where Ω is a bounded domain of $\mathbb{R}^d$ and $\mathbf{n}$ is the outward unit normal to its boundary $\overline{\Gamma}_{\mathcal{D}} \cup \overline{\Gamma}_{\mathcal{N}}$.

Outline

Introduce the auxiliary variable $\mathbf{q} = \nabla u$ and obtain the equations

$$\begin{aligned}
\mathbf{q} &= \nabla u \text{ in } \Omega, && (2.1)\\
-\nabla \cdot \mathbf{q} &= f \text{ in } \Omega, && (2.2)\\
u &= g_{\mathcal{D}} \text{ on } \Gamma_{\mathcal{D}}, && (2.3)\\
\mathbf{q} \cdot \mathbf{n} &= \mathbf{g}_{\mathcal{N}} \cdot \mathbf{n} \text{ on } \Gamma_{\mathcal{N}}, && (2.4)
\end{aligned}$$

We multiply (2.1) and (2.2) by arbitrary, smooth test functions $\mathbf{r}$ and v, respectively, and integrate by parts over the element $K \in \mathcal{T}$ to obtain,

$$\begin{aligned} \int_K \mathbf{q} \cdot \mathbf{r} &= -\int_K u \nabla \cdot \mathbf{r} d\mathbf{x} + \int_{\partial K} u \mathbf{r} \cdot \mathbf{n}_K ds, \\ \int_K \mathbf{q} \cdot \nabla v &= -\int_K f v d\mathbf{x} + \int_{\partial K} v \mathbf{q} \cdot \mathbf{n}_K ds. \end{aligned}$$

The above equations are well defined when $(\mathbf{q}, u)$ and $(\mathbf{r}, v)$ belong to $\mathbf{M} \times V$, where

$$\begin{aligned} \mathbf{M} &:= \{\mathbf{q} \in (L^2(\Omega))^d : \mathbf{q}|_K \in H^1(K)^d, \forall K \in \mathcal{T}\}, \\ V &:= \{u \in L^2(\Omega) : u|_K \in H^1(K), \forall K \in \mathcal{T}\}. \end{aligned}$$

Next we seek to approximate the exact solution $(\mathbf{q}, u)$ with functions $(\mathbf{q}_N, u_N)$ in space $\mathbf{M}_N \times V_N \subset \mathbf{M} \times V$, where

$$\begin{aligned} \mathbf{M}_N &:= \{\mathbf{q} \in (L^2(\Omega))^d : \mathbf{q}|_K \in \mathcal{S}(K)^d, \forall K \in \mathcal{T}\}, \\ V_N &:= \{u \in L^2(\Omega) : u|_K \in \mathcal{S}(K), \forall K \in \mathcal{T}\}, \end{aligned}$$

and $\mathcal{S}(K)$ typically consists of polynomials.

The approximate solution $(\mathbf{q}_N, u_N)$ is then defined by using the above weak formulation; for all $(\mathbf{r}, v) \in \mathcal{S}(K)^d \times \mathcal{S}(K)$,

$$\int_K \mathbf{q}_N \cdot \mathbf{r} = -\int_K u_N \nabla \cdot \mathbf{r} d\mathbf{x} + \int_{\partial K} \hat{u}_N \mathbf{r} \cdot \mathbf{n}_K ds, \quad (2.5)$$

$$\int_K \mathbf{q}_N \cdot \nabla v = -\int_K f v d\mathbf{x} + \int_{\partial K} v \hat{\mathbf{q}}_N \cdot \mathbf{n}_K ds, \quad (2.6)$$

where $\hat{u}_N$ and $\hat{\mathbf{q}}_N$ are the numerical fluxes.

Let K^+ and K^- be two adjacent elements of $\mathcal{T}$; set $e = \partial K^+ \cap \partial K^-$. Then define averages $\{\!\{\cdot\}\!\}$ and jumps $[\![\cdot]\!]$ by

$$\{\!\{u\}\!\} := (u^+ + u^-)/2, \qquad \{\!\{\mathbf{q}\}\!\} := (\mathbf{q}^+ + \mathbf{q}^-)/2,$$
$$[\![u]\!] := u^+\mathbf{n}^+ + u^-\mathbf{n}^-, \qquad [\![\mathbf{q}]\!] := \mathbf{q}^+ \cdot \mathbf{n}^+ + \mathbf{q}^- \cdot \mathbf{n}^-.$$

If the set e is *inside* the domain Ω, we take

$$\begin{aligned} \hat{\mathbf{q}} &:= \{\!\{\mathbf{q}\}\!\} - C_{11}\,[\![u]\!] - \mathbf{C_{12}}\,[\![\mathbf{q}]\!], \\ \hat{u} &:= \{\!\{u\}\!\} + \mathbf{C_{12}} \cdot [\![u]\!] - C_{22}\,[\![\mathbf{q}]\!], \end{aligned} \tag{2.7}$$

where C_{11}, $\mathbf{C}_{12}$ and C_{22} depend on $\mathbf{x} \in e$. On the *boundary*

$$\hat{\mathbf{q}} := \begin{cases} \mathbf{q}^+ - C_{11}(u^+\mathbf{n}^+ + g_{\mathcal{D}}\mathbf{n}^-) & \text{on} \quad \Gamma_{\mathcal{D}}, \\ \mathbf{g}_{\mathcal{N}} & \text{on} \quad \Gamma_{\mathcal{N}}, \end{cases}$$

$$\hat{u} := \begin{cases} g_{\mathcal{D}}, & \text{on} \quad \Gamma_{\mathcal{D}}, \\ u^+ - C_{22}(\mathbf{q}^+ \cdot \mathbf{n}^+ + \mathbf{g}_{\mathcal{N}} \cdot \mathbf{n}^-) & \text{on} \quad \Gamma_{\mathcal{N}}, \end{cases}$$

The exterior traces are taken to be

$$(\mathbf{q}^-, u^-) = (\mathbf{q}^+, g_{\mathcal{D}}) \text{ on } \Gamma_{\mathcal{D}}, \quad (\mathbf{q}^-, u^-) = (\mathbf{g}_{\mathcal{N}}, u^+) \text{ on } \Gamma_{\mathcal{N}}. \quad (2.8)$$

$\mathbf{C}_{12}$: $\mathbf{C}_{12} \cdot \mathbf{n}^- = 1/2$ on $\Gamma_{\mathcal{D}}$ and $\mathbf{C}_{12} \cdot \mathbf{n}^+ = 1/2$ on $\Gamma_{\mathcal{N}}$.

" Selection principle" of numerical fluxes

- The numerical fluxes are consistent,
- they lead to a stable scheme,
- the scheme has an optimal order of convergence.

The LDG method

The LDG method is obtained when $C_{22} \equiv 0$. I.e. when the numerical flux $\hat{u}_N$ does not depend on $\mathbf{q}_N$. In this case, the auxiliary variable $\mathbf{q}_N$ can be locally solved in terms of u_N by (2.5) and then easily eliminated from (2.6); the resulting problem has only u_N as unknown. This unusual *local solvability* property gives its name to the LDG method.

The classical mixed setting.

Denote by $\mathcal{E}_0$ the union of all interior faces, by $\mathcal{E}_\mathcal{D}$ the union of faces on $\Gamma_\mathcal{D}$, and by $\mathcal{E}_\mathcal{N}$ the union of faces on Γ_N.

We sum (2.5) and (2.6) over all elements and obtain

$$\int_\Omega \mathbf{q}_N \cdot \mathbf{r} d\mathbf{x} + \sum_{K\in\mathcal{T}} \int_K u_N \nabla \cdot \mathbf{r} d\mathbf{x} - \int_{\mathcal{E}_0} \hat{u}_N [\![\mathbf{r}]\!] ds - \int_{\partial\Omega} \hat{u}_N \mathbf{r} \cdot \mathbf{n} ds = 0$$

and

$$\sum_{K\in\mathcal{T}} \int_K \mathbf{q}_N \cdot \nabla v d\mathbf{x} - \int_{\mathcal{E}_0} \hat{\mathbf{q}}_N [\![v]\!] ds - \int_{\partial\Omega} \hat{\mathbf{q}}_N \cdot \mathbf{n} ds = \int_\Omega f v d\mathbf{x}.$$

Insert the definition of the numerical fluxes,

$$\left(\int_\Omega \mathbf{q}_N \cdot \mathbf{r} d\mathbf{x} + \int_{\mathcal{E}_0} C_{22} [\![\mathbf{q}_N]\!] [\![\mathbf{r}]\!] ds + \int_{\mathcal{E}_\mathcal{N}} C_{22} (\mathbf{q}_N \cdot \mathbf{n})(\mathbf{r} \cdot \mathbf{n}) ds\right)$$
$$+ \left(\sum_{K \in \mathcal{T}} \int_K u_N \nabla \cdot \mathbf{r} d\mathbf{x} - \int_{\mathcal{E}_0} (\{\!\{u_N\}\!\} + \mathbf{C}_{12} \cdot [\![u_N]\!]) [\![\mathbf{r}]\!] ds\right.$$
$$\left. - \int_{\mathcal{E}_\mathcal{N}} u_N \mathbf{r} \cdot \mathbf{n} ds\right) = \int_{\mathcal{E}_\mathcal{D}} g_\mathcal{D} \mathbf{r} \cdot \mathbf{n} ds + \int_{\mathcal{E}_\mathcal{N}} C_{22} (\mathbf{g}_\mathcal{N} \cdot \mathbf{n})(\mathbf{r} \cdot \mathbf{n}) ds$$

Insert the definition of the numerical fluxes,

$$\underbrace{\left(\int_\Omega \mathbf{q}_N \cdot \mathbf{r} d\mathbf{x} + \int_{\mathcal{E}_0} C_{22} [\![\mathbf{q}_N]\!] [\![\mathbf{r}]\!] ds + \int_{\mathcal{E}_\mathcal{N}} C_{22}(\mathbf{q}_N \cdot \mathbf{n})(\mathbf{r} \cdot \mathbf{n}) ds\right)}_{a(\mathbf{q}_N, \mathbf{r})}$$

$$+ \underbrace{\left(\sum_{K \in \mathcal{T}} \int_K u_N \nabla \cdot \mathbf{r} d\mathbf{x} - \int_{\mathcal{E}_0} (\{\!\{u_N\}\!\} + \mathbf{C}_{12} \cdot [\![u_N]\!]) [\![\mathbf{r}]\!] ds\right.}_{b(u_N, \mathbf{r})}$$

$$\underbrace{\left. - \int_{\mathcal{E}_\mathcal{N}} u_N \mathbf{r} \cdot \mathbf{n} ds \right)} = \underbrace{\int_{\mathcal{E}_\mathcal{D}} g_\mathcal{D} \mathbf{r} \cdot \mathbf{n} ds + \int_{\mathcal{E}_\mathcal{N}} C_{22}(\mathbf{g}_\mathcal{N} \cdot \mathbf{n})(\mathbf{r} \cdot \mathbf{n}) ds}_{F(\mathbf{r})}$$

and

$$
\begin{aligned}
&-\left(\sum_{K\in\mathcal{T}}\int_K \nabla v\cdot\mathbf{q}_N d\mathbf{x} - \int_{\mathcal{E}_0}\left(\{\!\{\mathbf{q}_N\}\!\} - \mathbf{C}_{12}\,[\![\mathbf{q}_N]\!]\right)[\![v]\!]\,ds\right.\\
&\left.-\int_{\mathcal{E}_{\mathcal{D}}} v\mathbf{q}_N\cdot\mathbf{n}ds\right) + \left(\int_{\mathcal{E}_0} C_{11}\,[\![u_N]\!]\cdot[\![v]\!]\,ds + \int_{\mathcal{E}_{\mathcal{D}}} u_N v ds\right)\\
&=\int_\Omega fv d\mathbf{x} + \int_{\mathcal{E}_{\mathcal{D}}} C_{11} g_{\mathcal{D}} v ds + \int_{\mathcal{E}_{\mathcal{N}}} v\mathbf{g}_{\mathcal{N}}\cdot\mathbf{n}ds.
\end{aligned}
$$

and

$$-\underbrace{\left(\sum_{K\in\mathcal{T}}\int_K \nabla v\cdot\mathbf{q}_N d\mathbf{x} - \int_{\mathcal{E}_0}(\{\!\{\mathbf{q}_N\}\!\} - \mathbf{C}_{12}[\![\mathbf{q}_N]\!])[\![v]\!]\,ds\right.}_{b(v,\mathbf{q}_N)}$$

$$\underbrace{\left.-\int_{\mathcal{E}_\mathcal{D}} v\mathbf{q}_N\cdot\mathbf{n}ds\right)} + \underbrace{\left(\int_{\mathcal{E}_0} C_{11}[\![u_N]\!]\cdot[\![v]\!]\,ds + \int_{\mathcal{E}_\mathcal{D}} u_N v ds\right)}_{c(u_N,v)}$$

$$= \underbrace{\int_\Omega f v d\mathbf{x} + \int_{\mathcal{E}_\mathcal{D}} C_{11} g_\mathcal{D} v ds + \int_{\mathcal{E}_\mathcal{N}} v\mathbf{g}_\mathcal{N}\cdot\mathbf{n}ds}_{G(v)}.$$

Define the following the bilinear forms:

$$a(\mathbf{q},\mathbf{r}) := \int_\Omega \mathbf{q}\cdot\mathbf{r}d\mathbf{x} + \int_{\mathcal{E}_0} C_{22}\,[\![\mathbf{q}]\!]\,[\![\mathbf{r}]\!]\,ds + \int_{\mathcal{E}_\mathcal{N}} C_{22}(\mathbf{q}\cdot\mathbf{n})(\mathbf{r}\cdot\mathbf{n})ds$$

$$b(u,\mathbf{r}) := \sum_{K\in\mathcal{T}}\int_K u\nabla\cdot\mathbf{r}d\mathbf{x} - \int_{\mathcal{E}_0}(\{\!\{u\}\!\} + \mathbf{C}_{12}\cdot[\![u]\!])\,[\![\mathbf{r}]\!]\,ds$$

$$-\int_{\mathcal{E}_\mathcal{N}} u\mathbf{r}\cdot\mathbf{n}ds$$

$$c(u,v) := \int_{\mathcal{E}_0} C_{11}\,[\![u]\!]\cdot[\![v]\!]\,ds + \int_{\mathcal{E}_\mathcal{D}} uvds$$

and the linear forms:

$$F(\mathbf{r}) := \int_{\mathcal{E}_\mathcal{D}} g_\mathcal{D}\mathbf{r}\cdot\mathbf{n}ds + \int_{\mathcal{E}_\mathcal{N}} C_{22}(\mathbf{g}_\mathcal{N}\cdot\mathbf{n})(\mathbf{r}\cdot\mathbf{n})ds$$

$$G(v) := \int_\Omega fvd\mathbf{x} + \int_{\mathcal{E}_\mathcal{D}} C_{11}g_\mathcal{D}vds + \int_{\mathcal{E}_\mathcal{N}} v\mathbf{g}_\mathcal{N}\cdot\mathbf{n}ds.$$

Thus, the DG approximation $(\mathbf{q}_N, u_N)$ can be characterized as the unique solution of the following problem:
Find $(\mathbf{q}_N, u_N) \in \mathbf{M}_N \times V_N$ such that

$$\begin{aligned} a(\mathbf{q}_N, \mathbf{r}) + b(u_N, \mathbf{r}) &= F(\mathbf{r}), \\ -b(v, \mathbf{q}_N), +c(u_N, v) &= G(v), \end{aligned}$$

which can be rewritten as:

$$\mathcal{A}(\mathbf{q}_N, u_N; \mathbf{r}, v) = \mathcal{F}(\mathbf{r}, v) \text{ for all } (\mathbf{r}, v) \in \mathbf{M}_N \times V_N,$$

where

$$\begin{aligned} \mathcal{A}(\mathbf{q}, u; \mathbf{r}, v) &:= a(\mathbf{q}, \mathbf{r}) + b(u, \mathbf{r}) - b(v, \mathbf{q}) + c(u, v), \\ \mathcal{F}(\mathbf{r}, v) &:= F(\mathbf{r}) + G(v). \end{aligned}$$

The existence of the approximate solution of DG method is guaranteed if the following condition is satisfied:

$$v \in \mathcal{S}(K) : \int_K \nabla v \cdot \mathbf{r} dx = 0, \forall \mathbf{r} \in \mathcal{S}(K)^d \Longrightarrow \nabla v \equiv 0 \text{ on } K. \quad (2.9)$$

This condition is satisfied whenever

$$\nabla \mathcal{S}(K) \subset \mathcal{S}(K)^d.$$

Proposition 2.1 (well posedness of the DG method)

Consider the DG method defined by the weak formulation (2.5) and (2.6), and by the numerical fluxes in (2.7) and (2.8). If the coefficients $C_{11} > 0$ and $C_{22} \geq 0$, the DG method defines a unique approximate solution $(\mathbf{q}_N, u_N) \in \mathbf{M}_N \times V_N$.

Outline

For each $K \in \mathcal{T}$, denote by h_K its size and further $h = \max_{K\in\mathcal{T}} h_K$. For local space $\mathcal{S}(K)$, we require

$$\nabla\mathcal{S}(K) \subseteq \mathcal{S}(K)^d, \;\; \mathcal{S}(K) \supset \mathcal{P}^k(K). \tag{3.1}$$

We define $|(\mathbf{q}, u)|^2_{\mathcal{A}} := \mathcal{A}(\mathbf{q}, u; \mathbf{q}, u)$, that is,

$$|(\mathbf{q}, u)|^2_{\mathcal{A}} = \|\mathbf{q}\|^2_0 + \Theta^2(\mathbf{q}, u), \tag{3.2}$$

where

$$\begin{aligned}\Theta^2(\mathbf{q}, u) &= \int_{\mathcal{E}_0} C_{22} [\![\mathbf{q}]\!]^2 \, ds + \int_{\mathcal{E}_\mathcal{N}} C_{22}(\mathbf{q}\cdot\mathbf{n})^2 ds + \int_{\mathcal{E}_0} C_{11} [\![u]\!]^2 \, ds \\ &+ \int_{\mathcal{E}_\mathcal{D}} C_{11} u^2 ds.\end{aligned}$$

Assume that C_{11} and C_{22} defining the numerical fluxes in (2.7) are defined as follows:

$$C_{11}(\mathbf{x}) = \begin{cases} \zeta \min\{h_{K^+}^{\alpha}, h_{K^-}^{\alpha}\} & \text{if } \mathbf{x} \in \langle K^+, K^- \rangle, \\ \zeta h_{K^+}^{\alpha} & \text{if } \mathbf{x} \in \partial K^+ \cap \Gamma_{\mathcal{D}}, \end{cases} \tag{3.3}$$

$$C_{22}(\mathbf{x}) = \begin{cases} \tau \min\{h_{K^+}^{\beta}, h_{K^-}^{\beta}\} & \text{if } \mathbf{x} \in \langle K^+, K^- \rangle, \\ \tau h_{K^+}^{\beta} & \text{if } \mathbf{x} \in \partial K^+ \cap \Gamma_{\mathcal{N}}, \end{cases} \tag{3.4}$$

with $\zeta > 0$, $\tau \geq 0$, $-1 \leq \alpha \leq 0 \leq \beta \leq 1$ independent of the mesh-size and $|\mathbf{C}_{12}|$ of order one.

Define

$$\mu^* = \max\{-\alpha, \hat{\beta}\}, \quad \mu_* = \min\{-\alpha, \hat{\beta}\},$$

where $\hat{\beta} = 1$ if $\tau = 0$ and $\hat{\beta} = \beta$ otherwise.

⁎ *Theorem* ⁎

Let $(\mathbf{q}, u)$ *be the solution of (2.1)-(2.4) and let* $(\mathbf{q}_N, u_N)$ *be the approximate solution given by the DG method (2.5) and (2.6). Assume that the inclusion properties (3.1) of the local spaces hold and that the stabilization parameters are as described above. Then we have that, for* $(\mathbf{q}, u) \in H^{s+1}(\Omega)^d \times H^{s+2}(\Omega)$ *with* $s \geq 0$*,*

$$\|u - u_N\|_0 + h^D |(\mathbf{q} - \mathbf{q}_N, u - u_N)|_{\mathcal{A}} \leq Ch^{P+D} \|u\|_{s+2},$$

where C *solely depends on* Ω*,* s *and triangulation (not when* $\alpha = \hat{\beta} = 0$*),* ζ*,* τ*,* k *and* d*; and*

$$P = \min\left\{s + \frac{1}{2}(1+\mu_*), k + \frac{1}{2}(1-\mu^*)\right\}, D = \frac{1}{2}(1+\mu_*) \text{ if } k \geq 1.$$

When $k = 0$*, we have* $P = D = \frac{1}{2}(1 - \mu^*)$*.*

Brief analysis

- The same orders of convergence are obtained with either $C_{22} = 0$ or C_{22} of order h.
- In general, the same orders of convergence are obtained by taking $(\alpha, \beta) = (-a, b)$ or by taking $(\alpha, \beta) = (-b, a)$.
- The most remarkable cases occur when $-\alpha, \beta \in \{0, 1\}$.

Table 1
Orders of convergence for $u \in H^{s+2}(\Omega)$ for $s \geq 0$ and $k \geq 1$.

C_{22}	C_{11}	$\lvert(\mathbf{q} - \mathbf{q}_N, u - u_N)\rvert_{\mathcal{A}}$	$\lVert u - u_N \rVert_0$
$0, \mathcal{O}(h)$	$\mathcal{O}(1)$	$\min\{s + 1/2, k\}$	$\min\{s + 1/2, k\} + 1/2$
$0, \mathcal{O}(h)$	$\mathcal{O}(1/h)$	$\min\{s + 1, k\}$	$\min\{s + 1/2, k\} + 1$
$\mathcal{O}(1)$	$\mathcal{O}(1)$	$\min\{s, k\} + 1/2$	$\min\{s, k\} + 1$
$\mathcal{O}(1)$	$\mathcal{O}(1/h)$	$\min\{s + 1/2, k\}$	$\min\{s + 1/2, k\} + 1/2$

Table 2

Orders of convergence for $u \in H^{s+2}(\Omega)$ for $s \geq k$ and $k \geq 1$.

C_{22}	C_{11}	$\lvert(\mathbf{q}-\mathbf{q}_N, u-u_N)\rvert_{\mathcal{A}}$	$\lVert u-u_N\rVert_0$
$0, \mathcal{O}(h)$	$\mathcal{O}(1)$	k	$k+1/2$
$0, \mathcal{O}(h)$	$\mathcal{O}(1/h)$	k	$k+1$
$\mathcal{O}(1)$	$\mathcal{O}(1)$	$k+1/2$	$k+1$
$\mathcal{O}(1)$	$\mathcal{O}(1/h)$	k	$k+1$

Table 3

Orders of convergence for $u \in H^{s+2}(\Omega)$ for $s \geq 0$ and $k \geq s+1$.

C_{22}	C_{11}	$\lvert(\mathbf{q}-\mathbf{q}_N, u-u_N)\rvert_{\mathcal{A}}$	$\lVert u-u_N\rVert_0$
$0, \mathcal{O}(h)$	$\mathcal{O}(1)$	$s+1/2$	$s+1$
$0, \mathcal{O}(h)$	$\mathcal{O}(1/h)$	$s+1$	$s+2$
$\mathcal{O}(1)$	$\mathcal{O}(1)$	$s+1/2$	$s+1$
$\mathcal{O}(1)$	$\mathcal{O}(1/h)$	$s+1/2$	$s+1$

Express the error $(\mathbf{e_q}, e_u) = (\mathbf{q} - \mathbf{q}_N, u - u_N)$ as follows:

$$(\mathbf{e_q}, e_u) = (\mathbf{q} - \vec{\Pi}\mathbf{q}, u - \Pi u) + (\vec{\Pi}\mathbf{e_q}, \Pi e_u),$$

where $\vec{\Pi}$ and Π are projections from $\mathbf{M}$ and V onto $\mathbf{M}_N$ and V_N.

a. The basic ingredients

- Galerkin orthogonality:

$$\mathcal{A}(\mathbf{e_q}, e_u; \mathbf{r}, v) = 0 \ \ \forall(\mathbf{r}, v) \in \mathbf{M}_N \times V_N. \tag{3.5}$$

- Approximation of the projections $\vec{\Pi}, \Pi$:

$$|\mathcal{A}(\mathbf{q} - \vec{\Pi}\mathbf{q}, u - \Pi u; \vec{\Phi} - \vec{\Pi}\vec{\Phi}, \phi - \Pi\varphi)| \leq K_{\mathcal{A}}(\mathbf{q}, u, \vec{\Phi}, \varphi) \tag{3.6}$$

for any $(\mathbf{q}, u), (\vec{\Phi}, \varphi) \in \mathbf{M} \times V$ and,

$$|\mathcal{A}(\mathbf{r}, v; \mathbf{q} - \vec{\Pi}\mathbf{q}, u - \Pi u) \leq |(\mathbf{r}, \nu)|_{\mathcal{A}} K_{\mathcal{B}}(\mathbf{q}, u) \tag{3.7}$$

for any $(\mathbf{r}, v) \in \mathbf{M}_N \times V_N$ and $(\mathbf{q}, u) \in \mathbf{M} \times V$

The idea of the proof of Theorem

1. By assumptions (3.6) and (3.7), show that the error estimates we are interested in can be obtained solely in terms of functionals $K_{\mathcal{A}}$ and $K_{\mathcal{B}}$.
2. Determine $K_{\mathcal{A}}$ and $K_{\mathcal{B}}$ such that (3.6) and (3.7) hold true respectively.
3. Prove Theorem by results above.

b.The estimate of the error in the $\mathcal{A}$-seminorm.

Lemma 3.1 We have

$$|(\mathbf{e_q}, e_u)|_{\mathcal{A}} \le K_{\mathcal{A}}^{1/2}(\mathbf{q}, u; \mathbf{q}, u) + K_{\mathcal{B}}(\mathbf{q}, u).$$

Proof. Since $|(\cdot,\cdot)|_{\mathcal{A}}$ is a seminorm,

$$|(\mathbf{e_q}, e_u)|_{\mathcal{A}} \le |(\mathbf{q} - \vec{\Pi}\mathbf{q}, u - \Pi u)|_{\mathcal{A}} + |(\vec{\Pi}\mathbf{e_q}, \Pi e_u)|_{\mathcal{A}},$$

and since

$$\begin{aligned} |(\vec{\Pi}\mathbf{e_q}, \Pi e_u)|_{\mathcal{A}}^2 &= \mathcal{A}(\vec{\Pi}\mathbf{e_q}, \Pi e_u; \vec{\Pi}\mathbf{e_q}, \Pi e_u) \\ \text{(Galerkin orthogonality)} &= \mathcal{A}(\vec{\Pi}\mathbf{q} - \mathbf{q}, \Pi u - u; \vec{\Pi}\mathbf{e_q}, \Pi e_u) \\ \text{(definition of } \mathcal{A}\text{)} &= \mathcal{A}(-\vec{\Pi}\mathbf{e_q}, \Pi e_u; \vec{\Pi}\mathbf{q} - \mathbf{q}, \Pi u - u) \\ \text{(Assumption (3.7))} &\le |(\vec{\Pi}\mathbf{e_q}, \Pi e_u)|_{\mathcal{A}} K_{\mathcal{B}}(\mathbf{q}, u). \end{aligned}$$

This implies that

$$|(\mathbf{e_q}, e_u)|_{\mathcal{A}} \le |(\mathbf{q} - \vec{\Pi}\mathbf{q}, u - \Pi u)|_{\mathcal{A}} + K_{\mathcal{B}}(\mathbf{q}, u). \quad \square$$

c. Estimate of the error in u in nonpositive order norms

Let φ be the solution to adjoint problem, (with elliptic regularity $\|\varphi\|_2 \leq \|\lambda\|_0$)

$$-\Delta\varphi = \lambda \qquad \text{in } \Omega, \tag{3.8}$$

$$\varphi = 0 \qquad \text{on } \Gamma_{\mathcal{D}}, \tag{3.9}$$

$$\frac{\partial\varphi}{\partial n} = 0 \qquad \text{on } \Gamma_{\mathcal{N}}. \tag{3.10}$$

Lemma 3.2 Let t be a natural number. Then we have

$$\|e_u\|_{-t,D} \leq \sup_{\lambda\in C_0^\infty(D)} \frac{K_{\mathcal{A}}(\mathbf{q}, u, \vec{\Phi}, \varphi)}{\|\lambda\|_{t,D}} + K_{\mathcal{B}}(\mathbf{q}, u) \sup_{\lambda\in C_0^\infty(D)} \frac{K_{\mathcal{B}}(\vec{\Phi}, \varphi)}{\|\lambda\|_{t,D}} \tag{3.11}$$

with φ denoting the solution of (3.8)-(3.10) and $\vec{\Phi} = -\nabla\varphi$.

Proofs

Lemma 3.3 Let $w \in H^{r+1}(K)$, $r \geq 0$. Let Π be a linear continuous operator from $H^{r+1}(K)$ onto $\mathcal{S}(K)$ such that $\Pi\omega = \omega$ for all $\omega \in \mathcal{P}^k(K)$. Then for $m = 0, 1$ we have

$$|\omega - \Pi\omega|_{m,K} \leq C h_K^{\min\{r,k\}+1-m} \|\omega\|_{r+1,K},$$

$$|\omega - \Pi\omega|_{0,\partial K} \leq C h_K^{\min\{r,k\}+\frac{1}{2}} \|\omega\|_{r+1,K},$$

from some constant C that solely depends on k and r.

Lemma 3.4 There exists a positive constant C_{inv} that solely depends on k and d, such that for all $\mathbf{s} \in \mathcal{S}(K)^d$ we have

$$\|\mathbf{s}\|_{0,\partial K} \leq C_{inv} h_K^{-\frac{1}{2}} \|\mathbf{s}\|_{0,K}, \tag{3.12}$$

for all $K \in \mathcal{T}$.

The functional $K_{\mathcal{A}}$.

Lemma 3.5 Assume $(\mathbf{q}, u) \in H^{s+1}(\Omega)^2 \times H^{s+2}(\Omega)$ and $(\vec{\Phi}, \varphi) \in H^{t+1}(\Omega)^2 \times H^{t+2}(\Omega)$, $s, t \geq 0$. Then the approximation property (3.6) holds true with

$$K_{\mathcal{A}}(\mathbf{q}, u, \vec{\Phi}, \varphi) = \sum_{i=1}^{5} S_i(\mathbf{q}, u; \vec{\Phi}, \varphi), \quad \text{where}$$

$$S_1 = C\left(\sum_{K\in\mathcal{T}} h_K^{2\min\{s,k\}+2}\|\mathbf{q}\|_{s+1,K}^2\right)^{\frac{1}{2}}\left(\sum_{K\in\mathcal{T}} h_K^{2\min\{t,k\}+2}\|\vec{\Phi}\|_{t+1,K}^2\right)^{\frac{1}{2}},$$

$$S_2 = C\left(\sum_{K\in\mathcal{T}} C_{22}^{\partial K} h_K^{2\min\{s,k\}+1}\|\mathbf{q}\|_{s+1,K}^2\right)^{\frac{1}{2}}\left(\sum_{K\in\mathcal{T}} C_{22}^{\partial K} h_K^{2\min\{t,k\}+1}\|\vec{\Phi}\|_{t+1,K}^2\right)$$

$$S_3 = C\left(\sum_{K\in\mathcal{T}} h_K^{2\min\{s+1,k\}}\|u\|_{s+2,K}^2\right)^{\frac{1}{2}}\left(\sum_{K\in\mathcal{T}} h_K^{2\min\{t,k\}+2}\|\vec{\Phi}\|_{t+1,K}^2\right)^{\frac{1}{2}},$$

$$S_4 = C\left(\sum_{K\in\mathcal{T}} \hat{h}_K^{2\min\{s,k\}+2}\|\mathbf{q}\|_{s+1,K}^2\right)^{\frac{1}{2}}\left(\sum_{K\in\mathcal{T}} h_K^{2\min\{t+1,k\}}\|\varphi\|_{t+2,K}^2\right)^{\frac{1}{2}}$$

$$S_5 = C \left(\sum_{K \in \mathcal{T}} C_{11}^{\partial K} h_K^{2\min\{s+1,k\}+1} \|u\|_{s+1,K}^2 \right)^{\frac{1}{2}}$$

$$\left(\sum_{K \in \mathcal{T}} C_{11}^{\partial K} h_K^{2\min\{t+1,k\}+1} \|\varphi\|_{t+2,K}^2 \right)^{\frac{1}{2}},$$

$\hat{h}_K := \sup\{h_{K'} : \langle K, K' \rangle \neq \emptyset\}$, $C_{ii}^{\partial K} := \sup\{C_{ii}(\mathbf{x}) : \mathbf{x} \in \partial K\}$, $i = 1, 2$. Furthermore, when $(\vec{\phi}, \varphi) = (\mathbf{q}, u)$

$$K_{\mathcal{A}}(\mathbf{q}, u; \mathbf{q}, u) = S_1(\mathbf{q}, u; \mathbf{q}, u) + S_2(\mathbf{q}, u; \mathbf{q}, u) + S_5(\mathbf{q}, u; \mathbf{q}, u).$$

Sketch of the proof Lemma 3.5.

Set $\vec{\xi}_{\mathbf{q}} := \mathbf{q} - \vec{\Pi}\mathbf{q}$, $\xi_u = u - \Pi u$, $\vec{\xi}_{\vec{\Phi}} := \vec{\Phi} - \vec{\Pi}\vec{\Phi}$, $\xi_\varphi = \varphi - \Pi\varphi$. We write

$$\mathcal{A}(\vec{\xi}_{\mathbf{q}}, \xi_u; \vec{\xi}_{\vec{\Phi}}, \xi_\varphi) = a(\vec{\xi}_{\mathbf{q}}, \vec{\xi}_{\vec{\Phi}}) + b(\xi_u, \vec{\xi}_{\vec{\Phi}}) - b(\xi_\varphi, \vec{\xi}_{\mathbf{q}}) + c(\xi_u, \xi_\varphi).$$

$$a(\vec{\xi}_{\mathbf{q}}, \vec{\xi}_{\vec{\Phi}}) = \sum_{K\in\mathcal{T}} \Big(\int_K \vec{\xi}_{\mathbf{q}} \cdot \vec{\xi}_{\vec{\Phi}} d\mathbf{x} + \int_{\partial K \cap \Gamma_{\mathcal{N}}} C_{22}(\vec{\xi}_{\mathbf{q}} \cdot \mathbf{n})(\vec{\xi}_{\vec{\Phi}} \cdot \mathbf{n}) ds$$
$$+ \int_{\partial K \setminus \partial\Omega} C_{22}(\vec{\xi}_{\mathbf{q}} \cdot \mathbf{n}_K)(\vec{\xi}_{\vec{\Phi}} \cdot \mathbf{n}_K) ds - \int_{\partial K \setminus \partial\Omega} C_{22}(\vec{\xi}^{out}_{\mathbf{q}} \cdot \mathbf{n}_K)(\vec{\xi}_{\vec{\Phi}} \cdot \mathbf{n}_K) ds \Big)$$
$$\leq \sum_{K\in\mathcal{T}_1} \Big(\|\vec{\xi}_{\mathbf{q}}\|_{0,K} \|\vec{\xi}_{\vec{\Phi}}\|_{0,K} + \|C_{22}^{\frac{1}{2}} \vec{\xi}_{\mathbf{q}} \cdot \mathbf{n}\|_{0,\partial K \cap \Gamma_{\mathcal{N}}} \|C_{22}^{\frac{1}{2}} \vec{\xi}_{\vec{\Phi}} \cdot \mathbf{n}\|_{0,\partial K \cap \Gamma_{\mathcal{N}}}$$
$$+ (\|C_{22}^{\frac{1}{2}} \vec{\xi}_{\mathbf{q}} \cdot \mathbf{n}_K\|_{0,\partial K \setminus \partial\Omega} + \|C_{22}^{\frac{1}{2}} \vec{\xi}^{out}_{\mathbf{q}} \cdot \mathbf{n}_K\|_{0,\partial K \setminus \partial\Omega}) \|C_{22}^{\frac{1}{2}} \vec{\xi}_{\vec{\Phi}} \cdot \mathbf{n}_K\|_{0,\partial K \setminus \partial\Omega} \Big)$$
$$\leq \Big(\sum_{K\in\mathcal{T}} \|\vec{\xi}_{\mathbf{q}}\|^2_{0,K} \Big)^{\frac{1}{2}} \Big(\sum_{K\in\mathcal{T}} \|\vec{\xi}_{\vec{\Phi}}\|^2_{0,K} \Big)^{\frac{1}{2}}$$
$$+2 \Big(\sum_{K\in\mathcal{T}} C_{22}^{\partial K} \|\vec{\xi}_{\mathbf{q}} \cdot \mathbf{n}_K\|^2_{0,\partial K} \Big)^{\frac{1}{2}} \Big(\sum_{K\in\mathcal{T}} C_{22}^{\partial K} \|\vec{\xi}_{\vec{\Phi}} \cdot \mathbf{n}_K\|^2_{0,\partial K} \Big)^{\frac{1}{2}}.$$

Combining Lemma 3.3, we have

$$a(\vec{\xi}_{\mathbf{q}}, \vec{\xi}_{\vec{\Phi}}) \leq S_1(\mathbf{q}, u; \vec{\Phi}, \varphi) + S_2(\mathbf{q}, u; \vec{\Phi}, \varphi).$$

Corollary 3.6. Let $(\mathbf{q}, u) \in H^{s+1}(\Omega)^2 \times H^{s+2}(\Omega)$, be the solution of equation (2.1)-(2.4); let $\varphi \in H^{t+2}(\Omega)$ be the solution of the dual problem (3.8)-(3.10), and $\vec{\Phi} = -\nabla\varphi$. Assume that C_{11} and C_{22} satisfy (3.3) and (3.4). Then there exists a constant C such that

$$K_{\mathcal{A}(\mathbf{q},u;\vec{\Phi},\varphi)} = Ch^{Q_{\mathcal{A}}}\|u\|_{s+2}\|\varphi\|_{t+2},$$

where $Q_{\mathcal{A}} = \min\{s+1+\min\{t+\hat{\beta},k\}, k+1+\min\{t,k+\alpha\}\}$, which reduces to $Q_{\mathcal{A}} = 1+\alpha$ for $k=0$. Moreover,

$$K_{\mathcal{A}}(\mathbf{q},u;\mathbf{q},u) = Ch^{2P_{\mathcal{A}}}\|u\|_{s+2},$$

where $Q_{\mathcal{A}} = \min\{s+\frac{1}{2}(1+\hat{\beta}), k+\frac{1}{2}(1+\alpha)\}$ for $k \geq 1$ and $P_{\mathcal{A}} = \frac{1}{2}(1+\alpha)$ for $k=0$.

The functional $K_{\mathcal{B}}$

We will show that there exists a form $|(\cdot,\cdot)|_{\mathcal{B}}$ such that for any $(\mathbf{r}, v) \in \mathbf{M}_N \times V_N$ and $(\mathbf{q}, u) \in \mathbf{M} \times V$,

$$|\mathcal{A}(\mathbf{r}, v; , u - \Pi u)| \leq C|(\mathbf{r}, v)|_{\mathcal{A}}|(\mathbf{q} - \vec{\Pi}\mathbf{q}; u - \Pi u)|_{\mathcal{B}} \qquad (3.13)$$

with C independent of the mesh-size. Then it is enough to determine $K_{\mathcal{B}}$ such that

$$|(\mathbf{q} - \vec{\Pi}\mathbf{q}; u - \Pi u)|_{\mathcal{B}} \leq K_{\mathcal{B}}(\mathbf{q}, u) \qquad (3.14)$$

for any $(\mathbf{q}, u) \in \mathbf{M} \times V$. We will show that (3.13) is satisfied when

$$|(\mathbf{q}, u)|^2_{\mathcal{B}} := \int_{\mathcal{E}_{\mathcal{D}}} \left(\frac{1}{C_{11}} (\mathbf{q} \cdot n)^2 + C_{11} u^2 \right) ds + \int_{\mathcal{E}_{\mathcal{N}}} \left(C_{22} (\mathbf{q} \cdot n)^2 + \frac{u^2}{\chi} \right) ds \qquad (3.15)$$

$$+ \int_{\mathcal{E}_I} \left(C_{22} [\![\mathbf{q}]\!]^2 + \frac{1}{C_{11}} |\{\!\{\mathbf{q}\}\!\} - C_{12} [\![\mathbf{q}]\!]|^2 + \frac{1}{\chi} (\{\!\{u\}\!\} + \mathbf{C}_{12} [\![u]\!])^2 + C_{11} [\![u]\!]^2 \right) ds$$

where for each internal or Neumann boundary face e,

$$\chi(\mathbf{x}) := \begin{cases} \min\{h_K, h_{K'}\} \text{ for } \mathbf{x} \in \langle K, K' \rangle, h_K \text{ for } \mathbf{x} \in \Gamma_{\mathcal{N}} & \text{if } C_{22}(\mathbf{x}) = 0, \\ C_{22}(\mathbf{x}) & \text{otherwise.} \end{cases}$$

Lemma 3.7 Define $|(\cdot,\cdot)|_{\mathcal{B}}$ as (3.15), then (3.13) holds true.
Lemma 3.8 For any $(\mathbf{q}, u) \in H^{s+1}(\Omega)^2 \times H^{s+2}(\Omega)$, the approximation property (3.14) holds true with

$$K_{\mathcal{B}}^2(\mathbf{q}, u) = C \sum_{K \in \mathcal{T}} \left(h_K^{2\min\{s,k\}+1} \left(\frac{1}{\tilde{C}_{11}^{\partial K}} + C_{22}^{\partial K} \right) \|\mathbf{q}\|_{s+1,K}^2 \right) + C \sum_{K \in \mathcal{T}} \left(h_K^{2\min\{s+1,k\}+1} \left(C_{11}^{\partial K} + \frac{1}{\tilde{\chi}^{\partial K}} \right) \|u\|_{s+2,K}^2 \right)$$

where $\tilde{C}_{11}^{\partial K} = \inf\{C_{11}(\mathbf{x}) : \mathbf{x} \in \partial K\}$, $\tilde{\chi}^{\partial K} = \inf\{\chi(\mathbf{x}) : \mathbf{x} \in \partial K\}$.

Corollary 3.9

Let $(\mathbf{q}, u) \in H^{s+1}(\Omega)^2 \times H^{s+2}(\Omega)$. Assume that C_{11} and C_{22} are chosen as above, then there exists a constant C such that

$$K_{\mathcal{B}}^2(\mathbf{q}, u) = Ch^{2P} \|u\|_{s+2}^2, \tag{3.16}$$

where $P = \frac{1}{2}(1 - \mu^*)$ if $k = 0$ and $P = \min\{s + \frac{1}{2}(1 + \mu_*), k + \frac{1}{2}(1 - \mu^*)\}$ if $k \geq 1$.

The proof of the Theorem

From Lemma 3.1 and Corollaries 3.6 and 3.9, we get

$$|(\mathbf{q} - \mathbf{q}_N, u - u_N)|_{\mathcal{A}} \leq Ch^{\min\{P_{\mathcal{A}}, P\}} \|u\|_{s+2},$$

and since $\min\{P_{\mathcal{A}}, P\} = P$ the estimate

$$|(\mathbf{q} - \mathbf{q}_N, u - u_N)|_{\mathcal{A}} \leq Ch^{P} \|u\|_{s+2}$$

follows.

Next take $t = 0$ and $D = \Omega$ in Lemma 3.2. From the elliptic regularity of the adjoint problem (3.8)-(3.10), we have $\|\varphi\|_2 \leq C\|\lambda\|_0$. The estimates of $\|u - u_N\|_0$ directly follow from the substituting the expression of $K_{\mathcal{A}}(\mathbf{q}, u, \vec{\Phi}, \varphi)$ given by Corollary 3.6 and the expression $K_{\mathcal{B}}(\vec{\Phi}, \varphi)$ given by Corollary 3.9 in (3.11). Finally, we get

$$\|u - u_N\|_0 \leq Ch^{\min\{Q_{\mathcal{A}}|_{t=0}, P+P|_{s=0}\}}\|u\|_{s+2},$$

and since $\min\{Q_{\mathcal{A}}|_{t=0}, P + P|_{s=0}\} = P + P|_{s=0}$, the estimate $\|u - u_N\|_0 \leq Ch^{P+D}$ follows with $D = P|_{s=0}$.
This completes the proof of Theorem.

Outline

4 Concluding remarks

The advantages

- The LDG method is well suited for *hp*-adaptivity.
- The coding of the LDG method is simpler.

Conclusions

- The orders of convergence of approximation given by LDG method with the stabilization parameter C_{11} of order h^{-1} are optimal.
- The effect of taking nonzero stabilization parameters C_{22} does not significantly improve the orders of convergence.