

# The Method of Baumann and Oden

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# Problem

Find  $u : \Omega \rightarrow \mathbb{R}$  such that

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

where

- $\Omega \subset \mathbb{R}^2$  convex, polygonal Lipschitz domain
- $f \in L^2(\Omega)$

Elliptic regularity gives

$$u \in H^2(\Omega)$$

# Definitions

Triangulation  $\mathcal{T}_h = \{K\}$  of  $\Omega$

Set of all edges  $\mathcal{E}_h = \{E\}$

$\mathcal{T}_h$  shape-regular  $\Rightarrow |E| \sim h$

Define the “broken Sobolev space”

$$V = V(\mathcal{T}_h) = \{v \in L^2(\Omega) : v|_K \in H^2(K) \text{ for all } K \in \mathcal{T}_h\}$$

and the finite element space

$$V_h = \{v_h \in V : v|_K \in P^p \text{ for all } K \in \mathcal{T}_h\}$$

for  $p \geq 1$ .

## Definitions

For the edge  $E = \bar{K}_1 \cap \bar{K}_2$ , scalar function  $v \in V$ , vector function  $\mu \in V^2$ , define the

- jump

$$[[v]] = \begin{cases} v_1 n_1 + v_2 n_2 & E \subset \text{int}(\Omega) \\ vn & E \subset \partial\Omega \end{cases}$$

$$[[\mu]] = \begin{cases} \mu_1 \cdot n_1 + \mu_2 \cdot n_2 & E \subset \text{int}(\Omega) \\ \mu \cdot n & E \subset \partial\Omega \end{cases}$$

- average

$$\{v\} = \begin{cases} \frac{1}{2}(v_1 + v_2) & E \subset \text{int}(\Omega) \\ v & E \subset \partial\Omega \end{cases}$$

$$\{\mu\} = \begin{cases} \frac{1}{2}(\mu_1 + \mu_2) & E \subset \text{int}(\Omega) \\ \mu & E \subset \partial\Omega \end{cases}$$

# Definitions

Norm on  $V$  (DG norm):

$$\|v\|^2 = \sum_K \int_K |\nabla v|^2 dx + h^{-1} \sum_E \int_E \llbracket v \rrbracket^2 ds$$

Seminorm on  $V$ :

$$|v|_{1,h}^2 = \sum_K \int_K |\nabla v|^2 dx$$

For  $|\cdot|_{1,h}$  there holds

$$|v|_{1,h} = 0 \quad \text{for } v \text{ piecewise constant}$$

# Variational Formulation

For  $K \in \mathcal{T}_h$  the solution  $u$  satisfies

$$-\int_K \Delta u v \, dx = \int_K f v \, dx \quad \forall v \in H^2(K).$$

Integration by parts:

$$\int_K \nabla u \cdot \nabla v \, dx - \int_{\partial K} v \nabla u \cdot n \, ds = \int_K f v \, dx \quad \forall v \in H^2(K)$$

Summation over all elements  $K \in \mathcal{T}_h$ :

$$\sum_K \int_K \nabla u \cdot \nabla v \, dx - \sum_E \int_E \llbracket v \nabla u \rrbracket ds = \int_\Omega f v \, dx \quad \forall v \in V$$

## Variational Formulation

As  $\nabla u \in H^1(\Omega)^2$ , the trace of  $\nabla u$  on edge  $E$  exists:

$$\begin{aligned} \int_E \llbracket v \nabla u \rrbracket ds &= \int_E \llbracket v \rrbracket \cdot \nabla u ds \\ &= \int_E \llbracket v \rrbracket \cdot \{\nabla u\} ds \end{aligned}$$

The solution  $u \in H^2(\Omega)$  is continuous, all jumps vanish

$$\int_E \llbracket u \rrbracket \cdot \mu ds = 0 \quad \forall E \in \mathcal{E}_h, \mu \in L^2(E).$$

Set  $\mu = \{\nabla v\}$ , sum up equations

$$\begin{aligned} \sum_K \int_K \nabla u \cdot \nabla v dx - \sum_E \int_E (\llbracket v \rrbracket \cdot \{\nabla u\} - \llbracket u \rrbracket \cdot \{\nabla v\}) ds \\ = \int_{\Omega} f v dx \quad \forall v \in V \end{aligned}$$

# Method of Baumann and Oden

Find  $u_h \in V_h$  such that

$$a_h(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h$$

where

$$a_h(u, v) = \sum_K \int_K \nabla u \cdot \nabla v \, dx - \sum_E \int_E ([v] \cdot \{\nabla u\} - [u] \cdot \{\nabla v\}) \, ds$$

The method is

- consistent, for  $u$  there holds  $a_h(u, v_h) = (f, v_h)$  for  $v_h \in V_h$ .
- not adjoint consistent, non-symmetric bilinear form
- not stable w. r. t. the DG norm  $\|\cdot\|$

# Missing adjoint consistency

Let  $w$  satisfy adjoint equation,

$$\begin{aligned} -\Delta w &= g && \text{in } \Omega \\ w &= 0 && \text{on } \partial\Omega. \end{aligned}$$

For  $g \in L^2(\Omega)$ , we have  $w \in H^2(\Omega)$ .

The method is not adjoint consistent:

Solution  $w$  does not satisfy discrete adjoint equation

$$a_h(v_h, w) = (g, v_h) \quad \forall v_h \in V_h.$$

Proof: Let  $v_h \in V_h$ :

$$\begin{aligned}
 a_h(v_h, w) &= \sum_K \int_K \nabla v_h \cdot \nabla w \, dx \\
 &\quad - \sum_E \int_E \underbrace{([w])}_{=0} \cdot \{\nabla v_h\} - [v_h] \cdot \{\nabla w\} \, ds \\
 &= - \sum_K \int_K v_h \cdot \Delta w \, dx + \sum_k \int_{\partial K} v_h \nabla w \cdot n \, ds \\
 &\quad + \sum_E \int_E [v_h] \cdot \{\nabla w\} \, ds \\
 &= \int_{\Omega} g v_h \, dx + 2 \sum_E \int_E [v_h] \cdot \{\nabla w\} \, ds \neq (g, v_h)
 \end{aligned}$$

□

# Stability

$a_h$  is bounded w.r.t. DG norm:

$$|a_h(u, v)| \leq \alpha_2 \|u\| \|v\| \quad \forall u, v \in V$$

$a_h$  is not coercive in  $V$ , for  $v$  piecewise constant we have

$$a_h(v, v) = \sum_K \int_K \nabla v \cdot \nabla v \, dx = 0.$$

Possible method of stabilization: Choose bilinear form

$$a_h(u, v) + \alpha \sum_E \int_E \llbracket u \rrbracket \cdot \llbracket v \rrbracket \, ds$$

for  $\alpha \sim h^{-1}$  ( $\rightarrow$  NIPG method).

## $H^1$ estimate

For  $v \in V$  there holds

$$a_h(v, v) = |v|_{1,h}^2$$

Therefore, we try to bound  $|u - u_h|_{1,h}$ .

Choose an interpolant  $u_I \in V_h$  of  $u$  such that

$$\|u - u_I\| \leq ch^p |u|_{p+1}$$

and

$$\int_E \{\nabla_h(u - u_I)\} ds = 0 \quad \forall E \in \mathcal{E}_h.$$

This is possible for  $p \geq 2$ .

Triangle inequality:

$$|u - u_h|_{1,h} \leq |u - u_I|_{1,h} + |u_I - u_h|_{1,h}$$

# $H^1$ estimate

## Lemma

There holds

$$a_h(u - u_I, v_h) \leq \alpha \|u - u_I\| \|v_h\|_{1,h} \quad \forall v_h \in V_h.$$

*Proof.* Let  $v_0 = P_0 v_h$  be the  $L^2$  projection of  $v_h$  onto the space of piecewise constant functions. The interpolate  $u_I$  satisfies

$$\sum_E \int_E \{\nabla(u - u_I)\} \llbracket v_0 \rrbracket ds = 0.$$

and therefore

$$\begin{aligned} a_h(u - u_I, v_0) &= \sum_K \int_K \nabla(u - u_I) \cdot \underbrace{\nabla v_0}_{=0} dx \\ &\quad - \sum_E \int_E (\llbracket v_0 \rrbracket \cdot \{\nabla(u - u_I)\} - \llbracket u - u_I \rrbracket \underbrace{\{\nabla v_0\}}_{=0}) ds = 0 \end{aligned}$$

# $H^1$ estimate

The seminorm  $|\cdot|_{1,h}$  is a norm on

$$\{v_0 \in V_h : P_0 v_0 = 0\}.$$

Therefore, there exists a constant  $C$  such that

$$\|v_h - P_0 v_h\| \leq C |v_h|_{1,h}$$

and further

$$\begin{aligned} a_h(u - u_I, v) &= a_h(u - u_I, v_h - v_0) \\ &\leq \alpha_2 \|u - u_I\| \|v_h - v_0\| \\ &\leq \alpha_2 C \|u - u_I\| |v_h|_{1,h} \end{aligned}$$



# $H^1$ estimate

We can now prove an estimate for  $|u_h - u_I|_{1,h}$ .

The method is consistent: There holds

$$\begin{aligned} |u_h - u_I|_{1,h}^2 &= a_h(u_h - u_I, \underbrace{u_h - u_I}_{\in V_h}) \\ &= a_h(u - u_I, u_h - u_I). \end{aligned}$$

Lemma 1 gives

$$a_h(u - u_I, u_h - u_I) \leq C \|u - u_I\| |u_h - u_I|_{1,h}.$$

# $H^1$ estimate

We have shown

$$|u_h - u_I|_{1,h}^2 \leq C \|u - u_I\| \|u_h - u_I\|_{1,h} \leq Ch^p |u|_{p+1} |u_h - u_I|_{1,h}.$$

Therefore we have an optimal order estimate

$$|u - u_h|_{1,h} \leq qCh^p |u|_{p+1}.$$

Note that the estimate is weak, it does not imply convergence in  $V$ . We need an additional  $L^2$  estimate.

## $L^2$ error

Method is not adjoint consistent  $\Rightarrow$  no optimal order  $L^2$  error.  
We will prove suboptimal order estimate:

$$\|u - u_h\|_0 \leq ch^p |u|_{p+1}$$

Let  $w$  be the solution to the adjoint problem

$$\begin{aligned} -\Delta w &= u - u_h && \text{in } \Omega \\ w &= 0 && \text{on } \partial\Omega \end{aligned}$$

Elliptic regularity:  $w \in H^2(\Omega)$ ,  $\|w\|_2 \leq c \|u - u_h\|_0$ .

## $L^2$ error

There holds

$$\begin{aligned}
 \|u - u_h\|_0^2 &= (-\Delta w, u - u_h) = a_h(w, u - u_h) \\
 &= a_h(w, u - u_h) + a_h(u - u_h, w) - a_h(u - u_h, w) \\
 &= 2 \sum_K \int_K \nabla w \cdot \nabla (u - u_h) dx - a_h(u - u_h, w) \\
 &= 2 \sum_K \int_K \nabla w \cdot \nabla (u - u_h) dx - a_h(u - u_h, w - w_I)
 \end{aligned}$$

1st term:

$$\begin{aligned}
 \sum_K \int_K \nabla w \cdot \nabla (u - u_h) dx &\leq |w|_1 |u - u_h|_{1,h} \\
 &\leq C \|w\|_2 |u - u_h|_{1,h} \leq C \|u - u_h\|_0 h^p |u|_{p+1}
 \end{aligned}$$

# $L^2$ error

2nd term

$$\begin{aligned} a_h(u - u_h, w - w_I) &= a_h(u - u_h - P_0(u - u_h), w - w_I) \\ &\leq \alpha_2 \|u - u_h - P_0(u - u_h)\| \|w - w_I\| \\ &\leq c |u - u_h|_{1,h} h \|w\|_2 \\ &\leq ch^{p+1} |u|_{p+1} \|u - u_h\|_0 \end{aligned}$$

Collecting all results, we get

$$\|u - u_h\|_0 \leq ch^p |u|_{p+1}.$$

# Numerical Results

Problem: Find  $u : \Omega \rightarrow \mathbb{R}$  such that

$$\begin{aligned} -\Delta u &= \sin(\pi x) \sin(\pi y) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

with  $\Omega = [0, 1]^2$ .

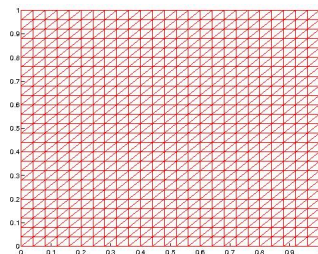


Figure: Triangulation

# Solution

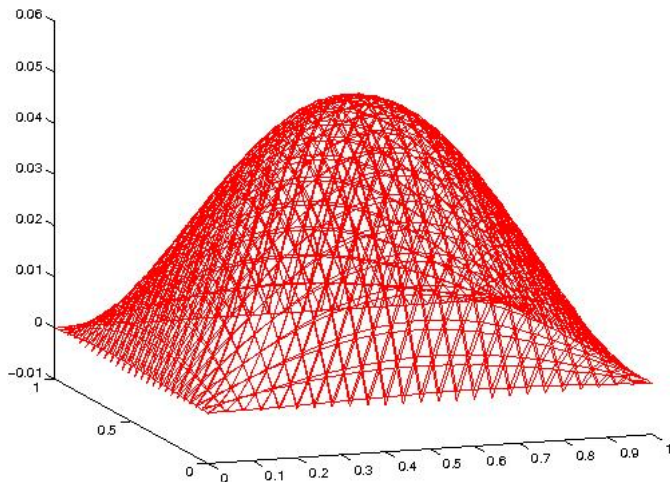


Figure: DG Solution,  $p = 1$ ,  $h = 0.04$

# Solution

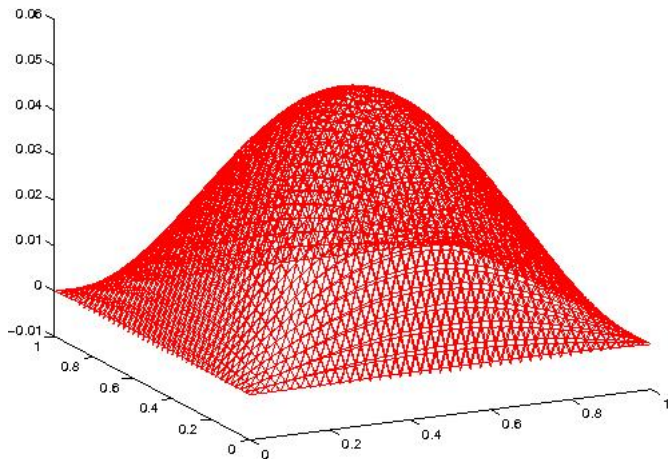
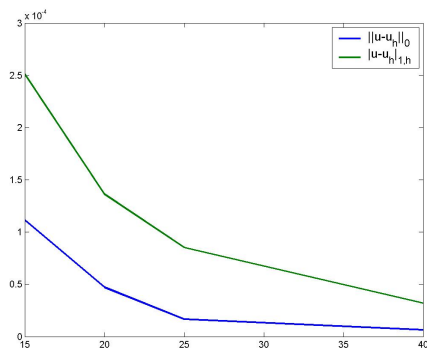


Figure: DG Solution,  $p = 1$ ,  $h = 0.025$

## Error



	$h$	ndof	$\ u_h - u\ _0$	$ u_h - u _{1,h}$
Mesh 1	0.07	1350	$1.1142 \cdot 10^{-4}$	$2.5120 \cdot 10^{-4}$
Mesh 2	0.05	2400	$4.7224 \cdot 10^{-5}$	$1.3616 \cdot 10^{-4}$
Mesh 3	0.04	3750	$1.6755 \cdot 10^{-5}$	$8.5133 \cdot 10^{-5}$
Mesh 4	0.025	9600	$6.5425 \cdot 10^{-6}$	$3.2029 \cdot 10^{-5}$

# Conclusion

We have seen that the method of Baumann and Oden is

- consistent
- not adjoint consistent
- not stable, but can be stabilized by penalty terms ( $\rightarrow$  NIPG method)
- easy to implement

We have bounded the error  $u - u_h$  in both  $L^2$  and the broken  $H^1$  space.