# Loewy Decomposition of Partial Differential Operators

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## Outline of the Talk

→ Loewy decomposition of linear second- and third order ode's.

→ Loewy's theorem. Types of Loewy decompositions.

→ Blumberg's example.

→ The structure of ideal intersections.

→ Factoring and solving second order pde's.

→ Factoring and solving certain third-order pde's.

→ Summary and outlook.

## Loewy Decomposition of Second Order Linear ODE's

Example 1.

$$D^2 - \frac{6}{x^2} = Lclm(D + \frac{2}{x}, D - \frac{3}{x})$$

$$= Lclm(D + \frac{2}{x} - \frac{5x^4}{x^5 + 1}, D + \frac{2}{x} - \frac{5x^4}{x^5 + 2})$$

$$= Lclm(D + \frac{2}{x} - \frac{5x^4}{x^5 + C_i})_{i=1,2}$$

Fundamental System: $\{x^3, \frac{1}{x^2}\}$ or $\{x^3 + \frac{1}{x^2}, x^3 + \frac{2}{x^2}\}$

Example 2.

$D^2 - D - \frac{1}{x} = (D + \frac{1}{x})(D - 1 - \frac{1}{x})$

Fundamental System: $\{xe^x, 1 + xe^x Ei(-x)\}$

## Basic Concepts for Loewy Decomposition of Linear ODE's

→ An operator $P$ is called *reducible* if it may be written as the product $P = Q \cdot R$ of two lower order operators, otherwise it is called *irreducible.*

→ An operator $P$ is called *completeley reducible* if it may be written as the least common left multiple $P = Lclm(Q, R)$ of two operators $Q$ and $R$.

→ An irreducible operator is completely reducible.

## Loewy Decomposition of Third Order Linear ODE's

Example 1.

$$D^3+\left(\frac{1}{x+1}+\frac{2}{x}\right)D^2-\left(\frac{6}{x+1}+\frac{6}{x}+\frac{4}{x^2}\right)D-\frac{8}{x+1}+\frac{8}{x}+\frac{8}{x^2}+\frac{4}{x^3}$$
$$=\left(D-\frac{2}{x+1}+\frac{1}{x}\right)\left(D-\frac{1}{x}\right)\left(D-\frac{2}{x}\right)$$

Fundamental System: $\{x^2, x^2\log x, x+x^2+(x\log x)^2\}$

Example 2.

$$D^3-\frac{8x}{2x-1}D+\frac{8}{2x-1}$$
$$=\left(D+2+\frac{1}{x-\frac{1}{2}}\right)\left(D^2-\frac{2x}{x-\frac{1}{2}}D+\frac{2}{2x-1}\right)$$
$$=\left(D+2+\frac{1}{x-\frac{1}{2}}\right)Lclm\left(D-2, D-\frac{1}{x}\right)$$

Fundamental System: $\{x, e^{2x}, \frac{1}{4}e^{2x}\int\frac{xe^{-4x}dx}{x^2-x+\frac{1}{4}}-\frac{1}{4}x\int\frac{e^{-2x}dx}{x^2-x+\frac{1}{4}}\}$

**Theorem 1** *(Loewy 1906) Let* $D = \frac{d}{dx}$ *a derivative and* $a_i \in \mathcal{F}$ *some differential field with derivative* $D$. *A differential operator*

$$L \equiv D^n + a_1 D^{n-1} + \ldots + a_{n-1} D + a_n$$

*of order* $n$ *may be written uniquely as the product of largest* <u>*completely reducible*</u> *factors* $L_k^{(d_k)}$ *of order* $d_k$ *over the same base field in the form*

$$L = L_m^{(d_m)} L_{m-1}^{(d_{m-1})} \ldots L_1^{(d_1)}$$

*with* $d_1 + \ldots + d_m = n$. *The factors* $L_k^{(d_k)}$ *are unique.*

**Corollary 1** *Any factor* $L_k^{(d_k)}$, $k = 1, \ldots, m$ *in Theorem 1 may be written as*

$$L_k^{(d_k)} = Lclm(l_{j_1}^{(e_1)}, l_{j_2}^{(e_2)}, \ldots, l_{j_k}^{(e_k)})$$

*with* $e_1 + e_2 + \ldots + e_k = d_k$; $l_{j_i}^{(e_i)}$ *for* $i = 1, \ldots, k$, *denote* <u>*irreducible operators*</u> *of order* $e_i$.

## Loewy Decomposition of ODE's: 4

Solutions of Reducible Equation

**Corollary 2** *Let a linear differential operator $P$ of order $n$ factor into $P = QR$ with $R$ of order $m$ and $Q$ of order $n-m$. Further let $y_1, \ldots, y_m$ be a fundamental system for $R(y) = 0$, and $\bar{y}_1, \ldots, \bar{y}_{n-m}$ a fundamental system for $Q(y) = 0$. Then a fundamental system for $P(y) = 0$ is given by the union of $y_1, \ldots, y_m$ and special solutions of $R(y) = \bar{y}_i$ for $i = 1, \ldots, n-m$.*

### General Solution Scheme

1. Solving reducible equations.

   $\mathcal{L}_2^2:\ (D + a_2)(D + a_1)y = 0,$
   $y_1 = \exp\left(-\int a_1 dx\right),\ y_2 = y_1 \int \exp\left(\int (a_1 - a_2)dx\right)dx.$

   $\mathcal{L}_3^2$: $Lclm(D + a_2, D + a_1)y = 0,\ \ y_i = \exp\left(-\int a_i dx\right),\ \ i = 1, 2,$

   $\mathcal{L}_4^2$: $Lclm(D + a(C))y = 0,\ \ y_i = \exp\left(-\int a(C_i)dx\right),\ \ i = 1, 2.$
2. Algebraic solutions of irreducible type $\mathcal{L}_1^2$ equations with non-trivial Galois group.
3. Equivalence to special functions equation.

## Blumberg's Example

$$L \equiv \partial_{xxx} + x\partial_{xxy} + 2\partial_{xx} + 2(x+1)\partial_{xy} + \partial_x + (x+2)\partial_y$$

Factorization into completely reducible factors:

$$L = \begin{cases} (\partial_{xx} + x\partial_{xy} + \partial_x + (x+2)\partial_y)(\partial_x + 1) \\ (\partial_{xx} + 2\partial_x + 1)(\partial_x + x\partial_y). \end{cases}$$

Factorization into irreducible factors:

$$L = \begin{cases} (\partial_{xx} + x\partial_{xy} + \partial_x + (x+2)\partial_y)(\partial_x + 1) \\ Lclm(\partial_x + 1, \partial_x + 1 - \frac{1}{x})(\partial_x + x\partial_y). \end{cases}$$

$$\begin{aligned} &Lclm(\partial_x + 1, \partial_x + x\partial_y) \\ &=< \partial_{xxy} + x\partial_{xyy} - \tfrac{1}{x}\partial_{xy} + x\partial_{yy} - \tfrac{1}{x}\partial_x - \frac{x+1}{x}\partial_y, \\ &\qquad \partial_{xxx} - x^2\partial_{xyy} + 3\partial_{xx} + (2x+3)\partial_{xy} - x^2\partial_{yy} \\ &\qquad\qquad +2\partial_x + (2x+3)\partial_y > . \end{aligned}$$

General Solution:

$$z = F(y - \frac{1}{2}x^2) + \int G(\bar{y} - \frac{1}{2}x^2)e^{-x}dx \,|_{\bar{y}=\phi} + \int H(\bar{y} - \frac{1}{2}x^2)e^{-x}dx \,|_{\bar{y}=\phi} \,. \tag{1}$$

**The Structure of Ideal Intersections 1**

**Theorem 2** *Let the ideals* $I_i =< \partial_x + a_i\partial_y + b_i >$ *for* $i = 1, 2$ *with* $I_1 \neq I_2$ *be given. There are three different cases for their intersection* $I_1 \cap I_2$*.*

*i) Separable case* $a_1 \neq a_2$*. If* $\left(\frac{b_1 - b_2}{a_1 - a_2}\right)_x = \left(\frac{a_1b_2 - a_2b_1}{a_1 - a_2}\right)_y$ *there holds* $I_1 \cap I_2 = <\partial_{xx}>_{LT}$ *and*

$$I_1 + I_2 =< \partial_x + \frac{a_1b_2 - a_2b_1}{a_1 - a_2}, \partial_y + \frac{b_1 - b_2}{a_1 - a_2} > .$$

*ii) Double root* $a_1 = a_2 = a$, $b_1 \neq b_2$*. There holds* $I_1 \cap I_2 =< \partial_{xx} >_{LT}$ *and* $I_1 + I_2 =< 1 >$*.*

*iii) If the preceding two cases do not apply there holds* $I_1 \cap I_2 =< \partial_{xxy}, \partial_{xxx} >_{LT}$ *and* $I_1 + I_2 =< 1 >$*.*

## The Structure of Ideal Intersections 2

*Proof.* Parameter $u$, define

$u(\partial_x + a_1\partial_y + b_1)$ and $(1-u)(\partial_x + a_2\partial_y + b_2)$.

New indeterminate $w = uz$, $w > z$

Consider differential polynomials

$w_x + a_1 w_y + b_1 w$ and $w_x + a_2 w_y + b_2 w - z_x - a_2 z_y - b_2 z$.

$$
\begin{aligned}
&w_y + \frac{b_1 - b_2}{a_1 - a_2} w + \frac{1}{a_1 - a_2}(z_x + a_2 z_y + b_2 z), \\
&w_x + \frac{a_1 b_2 - a_2 b_1}{a_1 - a_2} w - \frac{a_1}{a_1 - a_2}(z_x + a_2 z_y + b_2 z).
\end{aligned} \tag{2}
$$

Integrability condition:

$$\frac{b_1-b_2}{a_1-a_2} w_x - \frac{a_1 b_2 - a_2 b_1}{a_1 - a_2} w_y + \left( \left(\frac{b_1-b_2}{a_1-a_2}\right)_x - \left(\frac{a_1 b_2 - a_2 b_1}{a_1-a_2}\right)_y \right) w + 0(z_{xx})$$

## The Structure of Ideal Intersections 3

**Theorem 3** *Let the ideals $I_i =< \partial_x + a_i\partial_y + b_i >$, $i = 1, 2, 3$ be given with $I_i \neq I_j$ for $i \neq j$. There are four different cases for their intersection $I_1 \cap I_2 \cap I_3$.*

*i) Separable case $a_1 \neq a_2 \neq a_3$. If the three constraints given below are satisfied there holds $I_1 \cap I_2 \cap I_3 =< \partial_{xxx} >_{LT}$.*

*a) If in addition there holds $\dfrac{b_1 - b_2}{a_1 - a_2} = \dfrac{b_2 - b_3}{a_2 - a_3}$, there is a nontrivial sum ideal*

$$Gcrd(I_1, I_2, I_3) =< \partial_x + \frac{a_2b_3 - a_3b_2}{a_2 - a_3}, \partial_y + \frac{b_2 - b_3}{a_2 - a_3} >.$$

*b) In general $I_1 + I_2 + I_3 =< 1 >$.*

*ii) Double root $a_1 = a_2 = a \neq a_3$, $b_1 \neq b_2$. If the two constraints given below are satisfied, there holds*

*$I_1 \cap I_2 \cap I_3 =< \partial_{xxx} >_{LT}$ and $I_1 + I_2 =< 1 >$*

*iii) Triple root $a_1 = a_2 = a_3 = a$, $b_1 \neq b_2 \neq b_3$. If the condition given below is satisfied, the intersection ideal is $< \partial_{xx} >_{LT}$. If the latter condition is not satisfied, the intersection ideal is $< \partial_{xxx} >_{LT}$. In either case there holds $I_1 + I_2 =< 1 >$.*

*iv) If the preceding three cases do not apply, the intersection ideal is generated by two operators. It may be $< \partial_{xxxx}, \partial_{xxxy} >_{LT}$, $< \partial_{xxxxy}, \partial_{xxxyy} >_{LT}$ or $< \partial_{xxxxyy}, \partial_{xxxyyy} >_{LT}$.*

## Rational Solutions of Riccati Equations

**Lemma 1** *If a first order Riccati equation $z' + z^2 + az + b = 0$ with $a, b \in Q(x)$ has rational solutions, one of the following cases applies:*

*i) The general solution is rational and has the form*

$$z = \frac{r'}{r + C} + p \tag{3}$$

*where $p, r \in \bar{Q}(x)$, $\bar{Q}$ is a suitable algebraic extension of $Q$, and $C$ is a constant.*

*ii) There is only one, or there are two inequivalent special rational solutions.*

**Lemma 2** *If the partial Riccati equation*

$$z_x + a_1 z_y + a_2 z^2 + a_3 z + a_4 = 0 \tag{4}$$

*where $a_k \in Q(x, y)$ for all $k$ has rational solutions, two cases may occur.*

*i) The general solution is rational and has the form*

$$z = \frac{1}{a_1} \left[ \, \frac{r_x(x, \bar{y})}{r(x, \bar{y}) + \Phi(\bar{y})} + p(x, \bar{y}) \, \right]\Big|_{\bar{y} = \phi(x,y)} \tag{5}$$

*where $\phi(x, y)$ is the rational first integral of $\frac{dy}{dx} = a_1(x, y)$, $r$ and $p$ are rational functions of its arguments and $\Phi$ is an undetermined function.*

*ii) There is a single rational solution, or there are two inequivalent rational solutions which do not contain undetermined elements.*

# Decomposition of Second Order Operators: 1

**Proposition 1** *Let the second-order partial differential operator*

$$D_{xx} \equiv \partial_{xx} + A_1\partial_{xy} + A_2\partial_{yy} + A_3\partial_x + A_4\partial_y + A_5 \tag{6}$$

*be given with $A_k \in \mathbf{Q}(x,y)$ for all $k$. Its factorization is determined by the roots $a_1$ and $a_2$ of $a^2 - A_1 a + A_2 = 0$. The following alternatives occur.*

*i) If $a_1 \neq a_2$ are two different rational solutions and $b_1$ and $b_2$ are determined by (8), a factor $l_i = \partial_x + a_i\partial_y + b_i$ exists if $a_i$ and $b_i$ satisfy (9). If there are two factors $l_1$ and $l_2$, $D_{xx}$ is completely reducible, there holds $D_{xx} = Lclm(l_1, l_2)$.*

*ii) If $a_1 = a_2$ is a double root that obeys the constraint $a_{1,x} + (A_1 - a_1)a_{1,y} + A_3 a_1 = A_4$, the factorization depends on the rational solutions of the partial Riccati equation (9) for $b$ with $a = a_1$.*

*a) A right factor $l(\Phi) \equiv \partial_x + \frac{1}{2}A_1\partial_y + R(x,y,\Phi(\phi))$ exists if (9) has the general solution $R(x,y,\Phi(\phi))$; $\phi(x,y)$ is the first integral of $\frac{dy}{dx} = A_1(x,y)$, $\Phi$ is an undetermined function of a single argument. $D_{xx}$ is completely reducible, there holds $D_{xx} = Lclm(l(\Phi_1), l(\Phi_2))$ for any two choices $\Phi_1$ and $\Phi_2$.*

*b) A right factor $l \equiv \partial_x + \frac{1}{2}A_1\partial_y + r(x,y)$ exists if (9) has the single rational solution $r(x,y)$.*

*c) Two right factors $l_i \equiv \partial_x + \frac{1}{2}A_1\partial_y + r_i(x,y)$ exist if (9) has the rational solutions $r_1$ and $r_2$. $D_{xx}$ is completely reducible, there holds $D_{xx} = Lclm(l_1, l_2)$.*

$$a^2 - A_1 a + A_2 = 0, \tag{7}$$

$$a_x + (A_1 - a)a_y + A_3 a + (A_1 - 2a)b = A_4, \tag{8}$$

$$b_x + (A_1 - a)b_y - b^2 + A_3 b = A_5. \tag{9}$$

## Decomposition of Second Order Operators: 2

**Theorem 4** *Any differential operator*

$$\partial_{xx} + A_1\partial_{xy} + A_2\partial_{yy} + A_3\partial_x + A_4\partial_y + A_5 \tag{10}$$

*decomposes into largest completely reducible factors according to one of the types*

$$\mathcal{L}_{xx}^{(1)} : L_{xx} \text{ or } \mathcal{L}_{xx}^{(2)} : L_x^{(1)} L_x^{(2)}. \tag{11}$$

$L_{xx}$ *and* $L_x$ *are completely reducible operators with leading terms* $\partial_{xx}$ *or* $\partial_x$ *respectively. The upper index distinguishes different copies within a given type.*

**Corollary 3** *Let* $l_x$ *and* $l_{xx}$ *denote irreducible operators with leading term* $\partial_x$ *or* $\partial_{xx}$ *respectively. An additional upper index distinguishes different copies of the respective operator.* $\Phi$ *is an undetermined function of a single argument. The types* $\mathcal{L}_{xx}^{(i,j)}$ *defined below are refinements of the type* $\mathcal{L}_{xx}^{(i)}$ *given in Theorem 4.*

$$\begin{gathered}\mathcal{L}_{xx}^{(1,1)} : l_{xx},\ \ \mathcal{L}_{xx}^{(1,2)} : Lclm(l_x^{(1)}, l_x^{(2)}),\ \ \mathcal{L}_{xx}^{(1,3)} : Lclm(l_x(\Phi)),\\ \mathcal{L}_{xx}^{(2,1)} : l_x^{(1)} l_x^{(2)}.\end{gathered} \tag{12}$$

## $\boxed{\textbf{Decomposition of Second Order Operators: Examples}}$

Example 1

$$L \equiv \partial_{xx} - \frac{2y}{x}\partial_{xy} + \frac{y^2}{x^2}(1 - x^2y^2)\partial_{yy} + \frac{2y}{x^2}\partial_y.$$

Factors $l_{1,2} = \partial_x - (\frac{y}{x} \pm xy^2)\partial_y \pm xy$, type $\mathcal{L}_{xx}^{(1,2)}$.

$< L >= Lclm(l_1, l_2), \quad Gcrd(l_1, l_2) = < \partial_x - \frac{1}{x}, \partial_y - \frac{1}{y} >$

Example 2 *(Miller 1932)*

$$L \equiv \partial_{xx} + \frac{2y}{x}\partial_{xy} + \frac{y^2}{x^2}\partial_{yy} + \frac{1}{x}\partial_x + \frac{y}{x^2}\partial_y - \frac{1}{x^2}$$

$$< L >= Lclm(\partial_x + \frac{y}{x}\partial_y + \frac{1}{x}\frac{1 + x^2\Phi(\phi)}{1 - x^2\Phi(\phi)}, \text{ type } \mathcal{L}_{xx}^{(1,3)}$$

Example 3 *(Forsyth 1906)*

$$L \equiv \partial_{xx} - \partial_{yy} + \frac{4}{x+y}\partial_x.$$

$$L = (\partial_x + \partial_y + \frac{2}{x+y})(\partial_x - \partial_y + \frac{2}{x+y}), \text{ type } \mathcal{L}_{xx}^{(2,1)}$$

## Decomposition of Third Order Operators: 1

**Theorem 5** *Any differential operator*

$$\partial_{xxx}+A_1\partial_{xxy}+A_2\partial_{xyy}+A_3\partial_{yyy}+A_4\partial_{xx}+A_5\partial_{xy}+A_6\partial_{yy}+A_7\partial_x+A_8\partial_y+A_9 \tag{13}$$

*decomposes into largest* <u>*completely reducible*</u> *components corresponding to one of the types* $\mathcal{L}^{(k)}_{xxx}$, $k=1,\ldots,7$, *defined as follows.*

$$\mathcal{L}^{(1)}_{xxx}: L_{xxx},\quad \mathcal{L}^{(2)}_{xxx}: L_{xx}L_x,\quad \mathcal{L}^{(3)}_{xxx}: L^{(1)}_xL^{(2)}_xL^{(3)}_x,\quad \mathcal{L}^{(4)}_{xxx}: L_xL_{xx},$$

$$\mathcal{L}^{(5)}_{xxx}: \begin{cases} L^{(1)}_{xx}L^{(1)}_x \\ L^{(2)}_{xx}L^{(2)}_x \end{cases},\quad \mathcal{L}^{(6)}_{xxx}: \begin{cases} L_{xx}L^{(1)}_x \\ L^{(2)}_xL^{(3)}_xL^{(4)}_x \end{cases},\quad \mathcal{L}^{(7)}_{xxx}: \begin{cases} L^{(1)}_xL^{(2)}_xL^{(3)}_x \\ L^{(4)}_xL^{(5)}_xL^{(6)}_x \end{cases}$$

$L_{xxx}$, $L_{xx}$ *and* $L_x$ *are completely reducible operators with leading terms* $\partial_{xxx}$, $\partial_{xx}$ *or* $\partial_x$ *respectively. Upper indices distinguish different copies within a type definition.*

## Decomposition of Third Order Operators: 2

**Corollary 4** *Let $l_x$, $l_{xx}$ and $l_{xxx}$ denote irreducible operators with leading term $\partial_x$, $\partial_{xx}$ or $\partial_{xxx}$ respectively. The types $\mathcal{L}_{xx}^{(i,j)}$ defined below are refinements of the type $\mathcal{L}_{xx}^{(i)}$, $i = 1, \ldots, 7$ defined in Theorem 5.*

$$\mathcal{L}_{xxx}^{(1,1)} : l_{xxx},\ \mathcal{L}_{xxx}^{(1,2)} : Lclm(l_x^{(1)}, l_x^{(2)}, l_x^{(3)}),\ \mathcal{L}_{xxx}^{(1,3)} : Lclm(l_{xx}, l_x).$$

$$\mathcal{L}_{xxx}^{(2,1)} : l_{xx}l_x,\ \mathcal{L}_{xxx}^{(2,2)} : Lclm(l_x^{(1)}, l_x^{(2)})l_x^{(3)},\ \ \mathcal{L}_{xxx}^{(2,3)} : Lclm(l_x^{(1)}(\Phi))l_x^{(2)}.$$

$$\mathcal{L}_{xxx}^{(3,1)} : l_x^{(1)}l_x^{(2)}l_x^{(3)}.$$

$$\mathcal{L}_{xxx}^{(4,1)} : l_x l_{xx},\ \mathcal{L}_{xxx}^{(4,2} : l_x^{(1)} Lclm(l_x^{(2)}, l_x^{(3)}),\ \mathcal{L}_{xxx}^{(4,3)} : l_x^{(1)} Lclm(l_x^{(2)}(\Phi)).$$

*In addition there are 10 non-unique decompositions.*

$$\mathcal{L}_{xxx}^{(5,1)} : \begin{cases} l_{xx}^{(1)} l_x^{(1)} \\ l_{xx}^{(2)} l_x^{(2)} \end{cases},\ \mathcal{L}_{xxx}^{(5,2)} : \begin{cases} l_{xx} l_x^{(1)} \\ Lclm(l_x^{(2)}, l_x^{(3)}) l_x^{(4)} \end{cases},$$

$$\mathcal{L}_{xxx}^{(5,3)} : \begin{cases} l_{xx} l_x^{(1)} \\ Lclm(l_x^{(2)}(\Phi))) l_x^{(4)} \end{cases},\ \mathcal{L}_{xxx}^{(5,4)} : \begin{cases} Lclm(l_x^{(1)} l_x^{(2)}) l_x^{(3)} \\ Lclm(l_x^{(4)} l_x^{(5)}) l_x^{(6)} \end{cases},$$

$$\mathcal{L}_{xxx}^{(5,5)} : \begin{cases} Lclm(l_x^{(1)} l_x^{(2)}) l_x^{(3)} \\ Lclm(l_x^{(4)}(\Phi)) l_x^{(6)} \end{cases},\ \mathcal{L}_{xxx}^{(5,6)} : \begin{cases} Lclm(l_x^{(1)}(\Phi)) l_x^{(3)} \\ Lclm(l_x^{(4)}(\Psi)) l_x^{(6)} \end{cases}$$

$$\mathcal{L}_{xxx}^{(6,1)} : \begin{cases} l_{xx} l_x^{(1)} \\ l_x^{(2)} l_x^{(3)} l_x^{(4)} \end{cases},\ \mathcal{L}_{xxx}^{(6,2)} : \begin{cases} l_x^{(1)} l_x^{(2)} l_x^{(3)} \\ Lclm(l_x^{(4)} l_x^{(5)}) l_x^{(6)} \end{cases},\ \mathcal{L}_{xxx}^{(6,3)} : \begin{cases} l_x^{(1)} l_x^{(2)} l_x^{(3)} \\ Lclm(l_x^{(4)}(\Phi)) l_x^{(5)} \end{cases}$$

$$\mathcal{L}_{xxx}^{(7,1)} : \begin{cases} l_x^{(1)} l_x^{(2)} l_x^{(3)} \\ l_x^{(4)} l_x^{(5)} l_x^{(6)} \end{cases}.$$

## Decomposition of Second Order Operators: Examples

Example 1 *(Generalized Blumberg Example)*

$$\partial_{xxx} - (c-3)x\partial_{xxy} - (c-2)x^2\partial_{xyy}$$
$$+c\partial_{xx} + (c+2x)\partial_{xy} - (c-2)x^2\partial_{yy}$$
$$+(c-1)\partial_x + (cx + c - x)\partial_y$$

For or $c = 2$ Blumberg's example, decomposition type $\mathcal{L}_{xxx}^{(5,2)}$

For $c = 3$ decomposition type $\mathcal{L}_{xxx}^{(5,1)}$

$$\partial_{xxx} - x^2\partial_{xyy} + 3\partial_{xx} + (2x+3)\partial_{xy} - x^2\partial_{yy} + 2\partial_x + (2x+3)\partial_y$$
$$= \begin{cases} (\partial_{xx} - x^2\partial_{yy} + 2\partial_x + (2x+3)\partial_y)(\partial_x + 1) \\ (\partial_{xx} - x\partial_{xy} + 3\partial_x - (x-1)\partial_y + 2)(\partial_x + x\partial_y). \end{cases}$$

Example 2

$$L \equiv \partial_{xxx} + (x+y-1)\partial_{xxy} - (x+y)\partial_{xyy} - (x-y-1)\partial_{xx} - (x^2 + xy - x$$
$$-(x+y)\partial_{yy} - (xy + x - y + 1)\partial_x - (x^2 + xy + y)\partial_y - xy - 1.$$

Decomposition type $\mathcal{L}_{xxx}^{(2,2)}$

$$L = (\partial_x + x\partial_y + 1)(\partial_x + \partial_y + y)(\partial_x + 1).$$

## What next ?

→ Galois theory for systems of linear pde's of low order with finite-dimensional solution space.

→ Find upper bound for order of linear ode generating overideal (↔ Find bound for number of Laplace iterations)

→ Describe solution spaces for various decomposition types.

→ Galois Theory for linear second-order pde's in two variables.

→ How are Lie symmetries related to the various decomposition types?

→ Find rational solutions of first-order ode's.

A. Eremenko, *Rational Solutions of First-Order Differential Equations*, Ann. Acad. Scient. FennicæMath. **23**, 181-190 (1998); G. Chen, Y. Ma, Algorithmic Reduction and Rational General Solutions of First-Order Algebraic Differential Equations, in: Differential Equations with Symbolic Computation, D. Wang and Z. Zhen, eds, Birkhäuser Verlag, Basel, 2005.

→ Generalization of the above to three or more variables.

# Website: `www.alltypes.de`

→ Demo's and Tutorials.

→ Documentation of Typesystem and Userfunctions

→ Databases of differential equations & literature.

→ Interactive access to ALLTYPES/Reduce System

## ALLTYPES: An ALgebraic Language and TYPE System

→ Mathematical structure, ∼ 100 parametrized algebraic types.

→ Polymorphic functions and methods for implementation.

→ Special language for type manipulations. Example:

`p:=2*x**2*y+5*x*y+3*x-2;` → `p|DPOLY(Q,{x,y},LEX)|;`

## REDUCE Computer Algebra System

→ Efficient arithmetic for polynomials & rational functions over various number domains.

→ Algorithms for polynomial factorization and Gröbner bases.

→ Algorithms for integration.

→ Interactive user interface.