Analysis of Plane Curve Singularities

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Plane Curve Singularities

A plane curve C is the zero set of a polynomial $F(x,y)$ in $\mathbb{C}^2$.

A singularity is a point $p \in C$ where $\operatorname{grad} F = 0$.

Some tools that are used for studying/classifying curve singularities: multiplicity, Milnor-number, Tjurina-number, δ-invariant, Puiseux pairs, resolution tree, Poincaré series, analytic equivalence, equisingularity, topological equivalence, ...

Numerical Difficulties

When the coefficients of the polynomial are slightly perturbed in random direction, then the curve becomes nonsingular.

(Example from Maple help.)

ϵ-points and ϵ-singularities

Let $0 < \epsilon < 1$. By definition, a point $p \in \mathbb{C}^2$ is n ϵ-point of the curve $C : F(x, y) = 0$ iff

$$|F(p)| \leq \epsilon ||F||,$$

where $||F||$ is the 2-norm of the coefficient vertor of F.

And the point is an ϵ-singularity of multiplicity r iff, in addition, it is an ϵ-point of all its derivatives of order up to $r - 1$.

ϵ-points and ϵ-singularities have been introduced by S. Pérez-Díaz, J. R. Sendra, and J. Sendra (papers in TUCS 2004, A3L 2005, CAGD 2005).

Theorems on ϵ-points

Theorem 1: If p is an ϵ-point of C, then there exists a point $q \in C$ close to p. There is a precise bound depending on the degree of F, on the size of $||p||$, and on the ϵ-multiplicity of p on C.

Theorem 2: If C has degree d and p is an ϵ-singularity of multiplicity $d-1$, then there is a curve C' close to C with a rational parametrization.

Singularity Knots

Assume that $o := (0,0)$ is a singularity of $C \subset \mathbb{C}^2$.
Identify $\mathbb{C}^2$ with $\mathbb{R}^4$ and intersect C with a small 3-sphere S with center o.
This gives a 1-dimensional subset of S.
Use stereographic projection from a point $q \in S$ outside C to obtain
a number of closed curves in $\mathbb{R}^3$:
this is the *knot* of the singularity.

(Pictures from an old homepage of Mark van Hoeij.)

Basic Knot Theory

A knot is a closed loop in $\mathbb{R}^3$.
A link is a finite union of disjoint knots.
Two knots/links are equivalent iff there is an orientation-preserving homeomorphism from $\mathbb{R}^3$ to itself taking one link/knot into the other.

A regular projection of a knot/link is a projection onto the plane such that the preimage of a point consists of one or two points, and in the latter case the projection is an ordinary node with transversal self-intersection.

A diagram of a knot/link is the image under a regular projection, together with the information at crossings which branch goes over and which under. This is visualized by "broken arcs".

(Examples from lecture notes by Louis H. Kauffman.)

Knot Invariants

A knot invariant is a function from knot diagrams to some discrete set (e.g. $\mathbb{Z}$ or $\mathbb{Z}[t]$), which is invariant under *Reidemeister moves*
(best explained by picture from the same notes).

Theorem 4. Two knots/links are equivalent iff some diagram of one can be transformed to some diagram of the other by a finite number of Reidemeister moves.

Knot invariants are a way to prove that two given knots/links are not equivalent.

Example: 3-colouring number

A 3-colouring of a diagram is a colouring of its broken arcs by 3 colours, such that the 3 arcs meeting at a crossing have either the same colour or 3 distince colours (see examples).

The number of distinct 3-colourings is called 3-colouring number; it is a knot invariant.

The most celebrated example of a knot invariant is the Jones polynomial. It is not preserved by reflections.

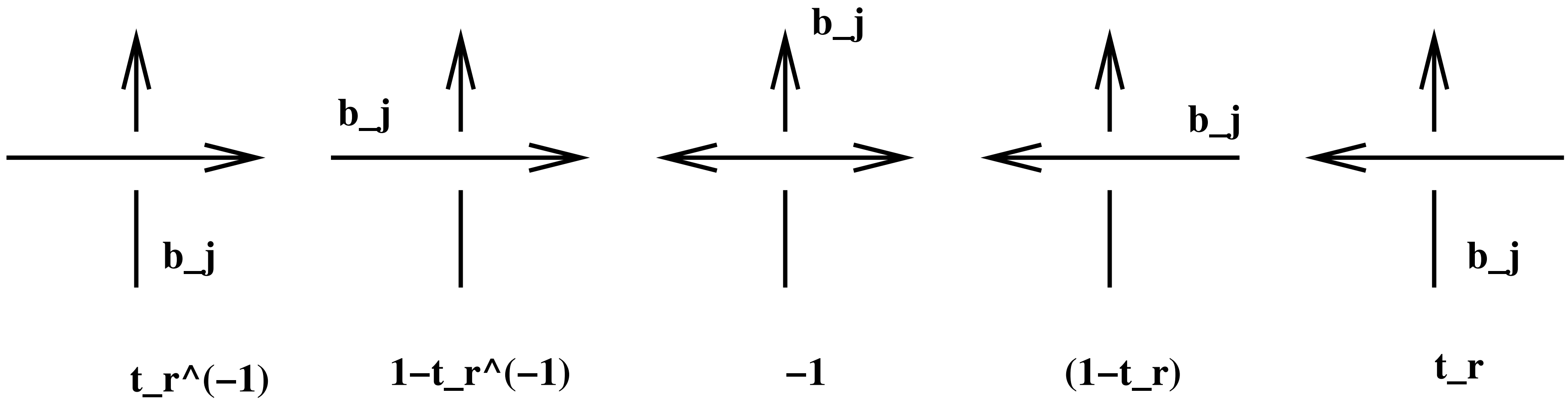
The Alexander Polynomial

The A.P. was introduced by Alexander in 1928. It depends solely on the fundamental group of the complement of the knot in $\mathbb{R}^3$.

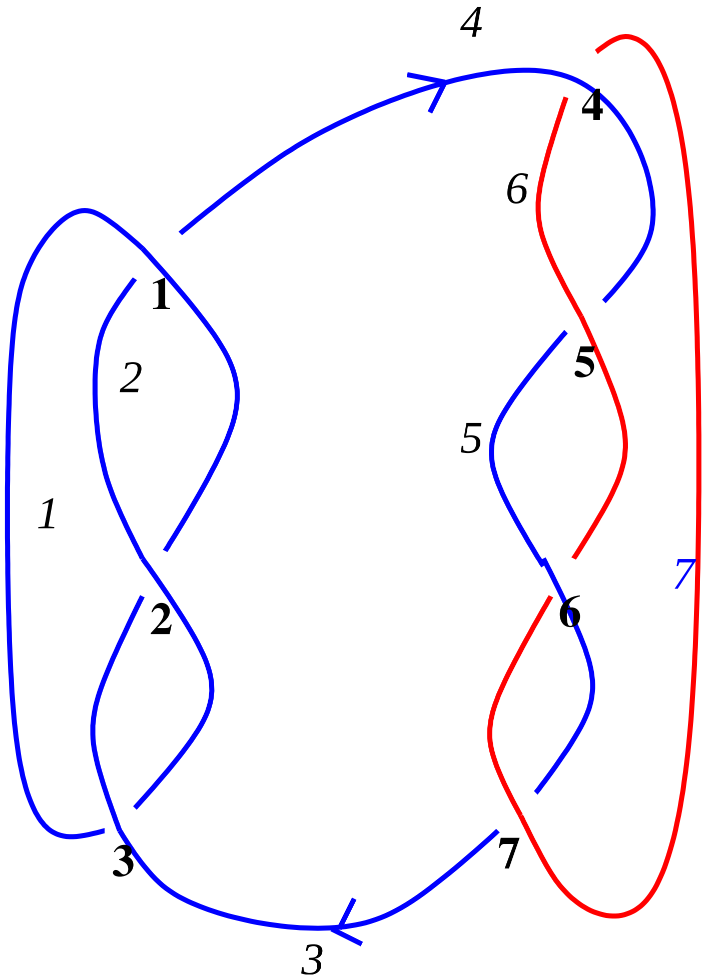
Let L be a link with m knots. The multivariate A.P. is a Laurent polynomial $\Delta_L \in \mathbb{Z}[x_1, \ldots, x_m, x_1^{-1}, \ldots, x_m^{-1}]$, which is defined up to a factor of $\pm x_1^{e_1} \cdots x_m^{e_m}$.
Let $b_1, \ldots, b_n$ be the arcs in a diagram of L.
Let $p_1, \ldots, p_n$ be the crossings of the diagram.
Choose an orientation of all knots in the link.
Define the Alexander matrix $A := (a_{ij})_{i,j=1,\ldots,n}$ as in the Figure.
Then Δ_L is the greatest common divisor of all minors of A.

The Alexander Polynomial

If the crossing b_i meets the arc b_j as indicated, then the entry a_{ij} is set to the value below. Otherwise $a_{ij} = 0$.



We assume that the horizontal arc belongs to the r-th link.



$$\begin{bmatrix} 1-x & x & 0 & -1 & 0 & 0 & 0 \\ -1 & 1-x & x & 0 & 0 & 0 & 0 \\ x & -1 & 1-x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-x & 0 & -1 & x \\ 0 & 0 & 0 & y & -1 & 1-y & 0 \\ 0 & 0 & 0 & 0 & 1-x & x & -1 \\ 0 & 0 & -1 & 0 & y & 0 & 1-y \end{bmatrix}.$$

Example continued

A closer look on the matrix shows that the last column is linear dependent of the first 6, hence we can always cancel the last column.

Then the gcd of all 7 minors is equal to

$$(1 - x + x^2)(1 + xy).$$

Theorems on the Alexander Polynomial

Theorem 5 (Yamamoto). Two singularity knots are equivalent iff their Alexander polynomials coincide.

Theorem 6 (Campillo, Delgado, Gusein-Zade, ...). The Alexander polynomial determines the multiplicity, Milnor number, δ-invariant, Puiseux pairs, resolution tree, Poincaré series, equisingularity type, topological type. Vice versa, the Alexander polynomial is determined by the (generalized) Poincaré series or by the topological type.

The Alexander Polynomial in Singular

In the implementation of Singular, F. H. Carillo and T. Keilen compute first the resolution graph and the multiplicities of the transform.

Then the Alexander polynomial is determined from these data by a formula by Campillo, Delgado, and Gusein-Zade.

(Example.)

Of course, this is an exact algorithm.

The Singularity Knot in Maple

In the implementation of Maple, van Hoeij replaces the curve C by the finite union of its Puiseaux parametrizations

$$x = t^r, y = a_1 t^{e_1} + a_2 t^{e_2} + \ldots,$$

which are computed by Puiseux expansion up to the critical order.

The 3-sphere is replaced by the polydisk

$$D := \{(x, y) \mid |x| \leq \epsilon_x, |y| \leq \epsilon_y\}.$$

The Singularity Knot in Maple

We assume that C intersects ∂D only in the set defined by $|x| = \epsilon_x$.
This is a full torus.

We can now parametrize the intersection curve using $t := (\epsilon_x)^{\frac{1}{r}} e^{\varphi i}$.

(Example.)

Of course, this is also an exact algorithm.

Numerical Computation of the Alexander Polynomial

In order to avoid the computationally costful stereographic projection, we intersect C with the boundary of the hypercube

$$H := \{(x_R, x_I, y_R, y_I) \mid \max(x_R, x_I) \leq \epsilon_x, \max(y_R, y_I) \leq \epsilon_y\}.$$

This boundary consists of 8 cubes in 3-space. ("Corpus Hypercubus" by S. Dali)

The Choice of ϵ_x and ϵ_y

The discriminant $\Delta(x) := \mathrm{disc}_y(F)$ has a root cluster around zero.
We choose ϵ_x such that $\Delta(x)$ has no additional roots with modulus less than ϵ_x.

Similarily, $F(0, y)$ has a root cluster around zero.
We choose ϵ_y so that $F(0, y)$ has no additional roots with modulus less than ϵ_y.

(Example.)

The Choice of ϵ_x and ϵ_y

Assume that

$$F(x,y) = \sum_{i,j} a_{ij} x^i y^j$$

has an ϵ-singularity of multiplicity r at o, and assume that $a_{0r} \neq 0$ (this can be achieved by a generic rotation). Let M be an upper bound of $a_{0r}^{-1} ||F||_\infty$. Then we choose ϵ_x and ϵ_y such that the inequalities

$$\epsilon/3 < \epsilon_y < 1/(2M+1), \epsilon/3 < \epsilon_x < \epsilon_y/(2M+1)$$

are fulfilled.

Subject to all these constraints, we choose ϵ_x, ϵ_y as big as possible (for the sake of numerical stability).

Good Choice

Once we have found such parameters ϵ_x, ϵ_y, the following situation is guaranteed.

- The curve C intersects ∂H only in the 4 cubes $x_R = \pm\epsilon_x$, $x_I = \pm\epsilon_x$.
- For each value $(x_R, \pm\epsilon_x)$ or $(\pm\epsilon_x, x_I)$, there are exactly r points on the knot.
- The r intersection curves are nonsingular with tangents that are not parallel to the x_R, x_I-plane.

(Example.)

Computation of the Curves

(See also D. A. Aruliah and R. M. Corless, ISSAC 04)
We set $x := t + \epsilon_x i$. Then each of the r curves $(t, y(t)) \in [-\epsilon_x, \epsilon_x] \times \mathbb{C}$ is the solution of an initial value problem

$$\partial_x F(t + \epsilon_x i, y(t)) + y'(t)\partial_y F(t + \epsilon_x i, y(t)) = 0,$$

where the initial value $y(-\epsilon_x)$ is one of the r complex roots of $F(-\epsilon_x + \epsilon_x i, y)$ with modulus less than ϵ_y.

Note that $\partial_y F(t + \epsilon_x i, y(t)) \neq 0$ for $t \in [-\epsilon_x, \epsilon_x]$, by the choice of ϵ_x.
Therefore the initial value problem can be computed with classical methods.
See also the Maple file curveps.mw

Computation of the Diagram

Best explained in an example. See the graphics diag[1-4].gif.