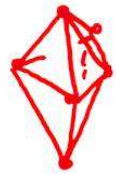
GRÖBNER BASES WITH SPHERICAL OR POLYTOPAL INITIAL IDEAL

$in(I) =$ 

VOLKMAR WELKER (MARBURG, GERMANY)

CONNECTION:

DISCRETE GEOMETRY + GEOMETRIC COMBINATORICS ⟷ GRÖBNER BASES

- GROUNDBREAKING WORK BY BERND STURMFELS

SITUATION:

- $I \trianglelefteq S = k[x_1, \ldots, x_n]$ HOMOGENEOUS IDEAL
- $A = S/I$ GORENSTEIN DOMAIN

PROBLEM:

- FIND TERM ORDER „$\leq$" FOR WHICH $\mathrm{in}_{\leq}(I)$ HAS CERTAIN EXTREMAL PROPERTIES

DO NOT CARE SO MUCH ABOUT ACTUAL GB

BASIC DEFINITIONS: (ALGEBRAIC)

- $A = \bigoplus_{m \geq 0} A_m$ STANDARD GRADED k-ALGEBRA
- $$\mathrm{Hilb}(A, t) = \sum_{m \geq 0} \dim_k A_m \, t^m = \frac{h_0 + \cdots + \overset{\neq 0}{h_r} t^r}{(1-t)^d}$$

dim A

HILBERT-SERIES

- $h(A) = (h_0, \ldots, h_r)$ THE h-VECTOR OF A

EXAMPLE:

- $A = k[x_1 \ldots x_n] = \bigoplus_{m \geq 0} A_m$

GENERATED BY MONOMIAL OF DEGREE m

$$\mathrm{HILB}(A,t) = \sum_{m \geq 0} \binom{n-1+m}{n-1} t^m = \frac{1}{(1-t)^n}$$

$$h(k[x_1 \ldots x_n]) = (1)$$

- $I = \langle x_{11}x_{22} - x_{12}x_{21} \rangle = \left\langle \det \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \right\rangle$

$$A = k[x_{11}, x_{12}, x_{21}, x_{22}] / I$$

$$\mathrm{HILB}(A,t) = \frac{1+t}{(1-t)^3}$$

$$h(A) = (1,1)$$

BASIC DEFINITIONS (COMBINATORICS)

- $\Delta \subseteq 2^{\Omega}$ SIMPLICIAL COMPLEX ON GROUND SET Ω

$$\sigma \subseteq \tau \in \Delta \Rightarrow \sigma \in \Delta$$

$$\leftrightsquigarrow \left\{ \begin{array}{cc} 123, & \\ 12, 13, 23, & 4 \\ 1, 2, 3 & \\ \emptyset & \end{array} \right\}$$

DO NOT DISTINGUISH Δ AND ITS GEOMETRIC REALIZATION

- $I_\Delta \trianglelefteq k[x_\omega \mid \omega \in \Omega]$ STANLEY-REISNER IDEAL

$$I_\Delta = \left\langle \prod_{\omega \in \sigma} x_\omega \;\middle|\; \sigma \notin \Delta \right\rangle$$

- $k[\Delta] = k[x_\omega \mid \omega \in \Omega] / I_\Delta$ SR-RING

$$\mathrm{Hilb}(k[\Delta], t) = \frac{h_0^\Delta + \dots + h_d^\Delta t^d}{(1-t)^d}$$

$$\dim \Delta = \max \{ |\sigma| \mid \sigma \in \Delta \} - 1 = d-1$$

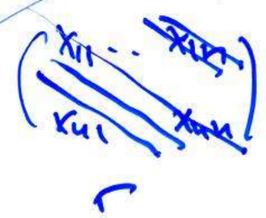
$h(\Delta) = (h_0^{\Delta}, \dots, h_d^{\Delta})$

h-VECTOR OF Δ

☹ $h(\Delta) \neq h(k[\Delta])$ IN GENERAL

☺ $h(\Delta)$ AND $h(k[\Delta])$ DIFFER ONLY BY 0'S ON THE RIGHT

EXAMPLE: $\Delta =$ 



$I_\Delta = \langle x_1x_4, x_2x_4, x_3x_4 \rangle$

$$\mathrm{HILB}(k[\Delta], t) = \frac{1 + t^2 - t^3}{(1-t)^3}$$

HERE $h(k[\Delta]) = h(\Delta) = (1, 0, 1, -1)$

$\Delta = 2^{[n]}$ $[n] = \{1, \dots, n\}$

FULL $(n-1)$-SIMPLEX

$I_\Delta = \langle 0 \rangle = (0)$

$$\mathrm{HILB}(k[\Delta], t) = \frac{1}{(1-t)^n}$$

$h(k[\Delta]) = (1) \neq h(\Delta) = (1, \underbrace{0, \dots, 0}_{n})$

VERY SMALL MOTIVATING OBSERVATION (6)

$$I = \langle x_{11}x_{22} - x_{12}x_{21} \rangle$$

$$A = k[x_{11}, x_{12}, x_{21}, x_{22}]/I$$

CHOOSE TERM ORDER "$\leq$" SUCH THAT $\mathrm{IN}_{\leq}(I) = \langle x_{12}x_{21} \rangle$

THEN $\mathrm{IN}_{\leq}(I) = I_\Delta$ FOR

$\Delta =$ 

$= S^0 * 1\text{-SIMPLEX}$

Γ, Γ' SIMPLICIAL COMPLEXES ON DISJOINT GROUND SETS Ω, Ω'

$$\Gamma * \Gamma' = \{\sigma \cup \sigma' \mid \sigma \in \Gamma, \sigma' \in \Gamma'\}$$

$$\mathrm{HILB}(A, t) = \mathrm{HILB}\left(k[x_{11}, \ldots, x_{22}]/\mathrm{IN}_{\leq}(I), t\right)$$

$$\|$$

$$\mathrm{HILB}(k[\Delta], t)$$

ANOTHER FACT:

$$\mathrm{HILB}\left(k[T * 2^{\Omega}], t\right) = \mathrm{HILB}\left(k[T], t\right) \cdot \frac{1}{(1-t)^{|\Omega|}}$$

So: $$\mathrm{HILB}(A, t) = \frac{1+t}{(1-t)^3} = \underbrace{\frac{1+t}{1-t}}_{\mathrm{HILB}(k[\,\vdots\,],\, t)} \cdot \frac{1}{(1-t)^2}$$

HERE A GORENSTEIN DOMAIN!

QUESTION:

▶ GIVEN $A = S/I$ STANDARD GRADED GORENSTEIN DOMAIN

? IS THERE A TERM ORDER "$\preceq$"

FOR WHICH
- $\mathrm{IN}_{\preceq}(I) = I_{\Delta}$
- Δ = SPHERE $*$ SIMPLEX
- $h(A) = h(\text{SPHERE})$

SO FAR:

NO MOTIVATION NO EXAMPLES

MOTIVATION

CONJECTURE (STANLEY)

A GORENSTEIN DOMAIN

$\Rightarrow h(A)$ UNIMODAL

CONJECTURE (g-CONJECTURE)

Δ SPHERE

$\Rightarrow h(\Delta)$ UNIMODAL

$\Longleftarrow$

IF QUESTION HAS POSITIVE ANSWER

- INTEREST IN $h(A)$ AND $h(\Delta)$ FROM ENUMERATIVE COMBINATORICS

EXAMPLE:

- $S = k[x_{ij} \mid 1 \leq i < j \leq n]$
- generic SKEW-SYMMETRIC MATRIX

$$M = \begin{pmatrix} 0 & x_{12} & x_{13} & \cdots & x_{1n} \\ -x_{12} & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & x_{n-1n} \\ -x_{1n} & \cdots & & -x_{n-1n} & 0 \end{pmatrix}$$

- FOR $1 \le j_1 < \dots < j_r \le n$

SKEW-SYMMETRIC MINOR

$$\begin{pmatrix} 0 & x_{j_1 j_2} & \cdots & x_{j_1 j_r} \\ & \ddots & & \vdots \\ - & & \ddots & x_{j_{r-1} j_r} \\ & & & 0 \end{pmatrix} = M^{j_1 \dots j_r}$$

→ CLASSICAL LINEAR ALGEBRA

$$\det\left(M^{j_1 \dots j_r}\right) = \begin{cases} 0 & r \text{ odd} \\ \underbrace{P^2}_{\text{PFAFFIAN OF } M^{j_1 \dots j_r}} & r \text{ even} \end{cases}$$

- $I_{n,r} = \left\langle P \;\middle|\; \begin{array}{l} \text{PFAFFIAN OF} \\ 2r \times 2r \text{ SKEW-MINOR} \\ \text{OF } M \text{ OF SIZE} \\ 2r \end{array} \right\rangle$

→ CLASSICAL:

$A_{n,r} = S/I_{n,r}$ GORENSTEIN DOMAIN

$$\mathrm{HILB}(A_{n,r}, t) = \frac{h_0 + \dots + h_{(r-1)(n-2r+1)} t^{(r-1)(n-2r+1)}}{(1-t)^{(r-1)(2n-2r+1)}}$$

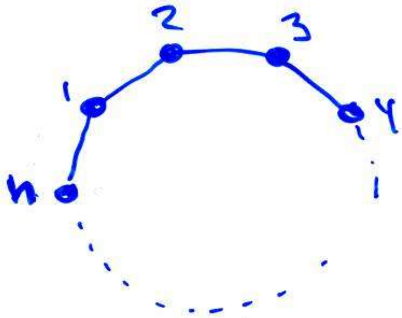
h_i COUNT CERTAIN LATTICE PATHS

THE NO r-CROSSING COMPLEX BY

DRESS, KOOLEN, MOULTON

- MOTIVATED BY MATH. BIOLOGY

- $1,\dots,n$ VERTICES OF CONVEX n-GON



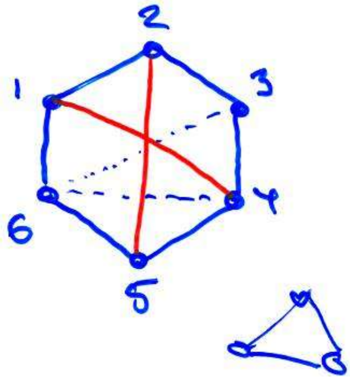
$\Omega_n = \{\{i,j\} \mid 1 \le i < j \le n\}$

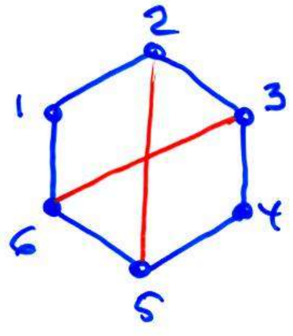
DIAGONALS = VARIABLES

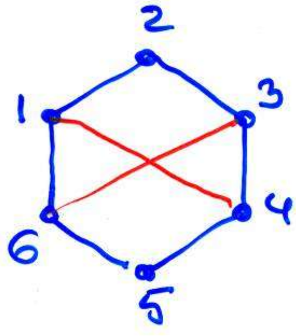
$\Omega_{n,r} = \left\{ \{i,j\} \;\middle|\; \begin{array}{l} \{i,j\} \text{ LEAVES OUT} \\ \ge r-1 \text{ VERTICES ON} \\ \text{EITHER SIDE} \end{array} \right\}$

$\Delta_{n,r} = \left\{ \sigma \subseteq \Omega_{n,r} \;\middle|\; \begin{array}{l} \sigma \text{ CONTAINS NO } r \\ \text{MUTUALLY} \\ \text{CROSSING DIAGONALS} \end{array} \right\}$

$n=6 \quad r=3$







THM: (DRESS, GRÜNEWALD, JONSSON, MOULTON)

$$\Delta_{n,r} \cong S^{(r-1)(n-2r+1)-1}$$

THM: (JONSSON, W.)

THERE IS TERM ORDER $\preceq$ SUCH THAT

$$\mathrm{IN}_{\preceq}(I_{n,r}) = I_{\Delta_{n,r} * \mathrm{SIMPLEX}}$$

→ MORE EXAMPLES

- HIBI-RINGS (REINER-WELKER)
  INCLUDES VIA TORIC DEFORMATION COORDINTE RINGS OF GRASSMANIANS
- GORENSTEIN SEMIGROUP RINGS
  (ATHANASIADIS, HIBI-OHSUGI, BRUNS-RÖMER)
- SYMMETRIC MINORS
  (CONCA, HOSTEN, THOMAS)
- MINORS
  (SOLL-WELKER)