

Coherent Configurations and Association Schemes

Part 3: Galois correspondence between coherent
configurations and permutation groups

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(Finite) Relational Structures

Given:

- a finite set X ,
- a finite index set \mathcal{I} ,
- a family $\{R_i\}_{i \in \mathcal{I}}$ where $R_i \subseteq X^2$ ($i \in \mathcal{I}$).

The pair $(X, \{R_i\}_{i \in \mathcal{I}})$ is called (binary) *relational structure* over X .

We will drop the attribute binary and we say only “relational structure”.

Coherent Configurations

Given:

- a finite set X ,
- a relational structure $(X, \{R_i\}_{i \in \mathcal{I}})$ over X .

$(X, \{R_i\}_{i \in \mathcal{I}})$ is called

coherent configuration of rank $|\mathcal{I}|$ and of degree $|X|$

if:

$$\text{CC1. } \forall i, j : R_i \cap R_j = \emptyset \iff i \neq j,$$

$$\text{CC2. } \bigcup_{i \in \mathcal{I}} R_i = X^2,$$

$$\text{CC3. } \forall i \exists j : R_j = R_i^{-1} = \{(y, x) \mid (x, y) \in R_i\},$$

$$\text{CC4. } \exists \mathcal{J} \subseteq \mathcal{I} : \bigcup_{j \in \mathcal{J}} R_j = \Delta = \{(x, x) \mid x \in X\},$$

CC5. For $(x, y) \in R_k$ the number $|\{z \mid (x, z) \in R_i, (z, y) \in R_j\}|$ does not depend on (x, y) but only on i, j, k . This number is denoted by $p_{i,j}^k$.

Automorphisms of Relational Structures

Given:

- a relational structure $(X, \{R_i\}_{i \in \mathcal{I}})$,
- a permutation $\varphi \in \text{Sym}(X)$,

φ is called *automorphism* of $(X, \{R_i\}_{i \in \mathcal{I}})$ if

$$\forall i \in \mathcal{I} \forall (x, y) \in X^2 : (x, y) \in R_i \iff (x^\varphi, y^\varphi) \in R_i.$$

By $\text{Aut}(X, \{R_i\}_{i \in \mathcal{I}})$ we denote the set of all automorphisms of $(X, \{R_i\}_{i \in \mathcal{I}})$.

Permutation Groups

Given:

- a set X ,
- a subset $G \subseteq \text{Sym}(X)$.

(G, X) is called *permutation group* on X if:

1. $() \in G$,
2. $\varphi \in G \Rightarrow \varphi^{-1} \in G$,
3. $\varphi, \psi \in G \Rightarrow \varphi \cdot \psi \in G$.

For given $(X, \{R_i\}_{i \in \mathcal{I}})$

$$\text{Aut}(X, \{R_i\}_{i \in \mathcal{I}})$$

is always a permutation group.

2-closed Permutation Groups

Given: a permutation group (G, X) .

(G, X) is called *2-closed* if:

there exists a relational structure $(X, \{R_i\}_{i \in \mathcal{I}})$ such that

$$G = \text{Aut}(X, \{R_i\}_{i \in \mathcal{I}}).$$

Let us give a characterization of 2-closed permutation groups using coherent configurations.

Contexts

A context is a triple (G, M, I) where

- a set G of objects (Gegenstände),
- a set M of attributes (Merkmale),
- a relation $I \subseteq G \times M$

For $A \subseteq G$ we define $A' := \{m \in M \mid \forall g \in A : gIm\}$.

For $B \subseteq M$ we define $B' := \{g \in G \mid \forall m \in B : gIm\}$.

The two operators define a Galois-correspondence between $\mathcal{P}(G)$ and $\mathcal{P}(M)$, i.e.

1. $\forall A_1, A_2 \subseteq G : A_1 \subseteq A_2 \Rightarrow A_2' \subseteq A_1'$ (antitonicity 1),
2. $\forall B_1, B_2 \subseteq M : B_1 \subseteq B_2 \Rightarrow B_2' \subseteq B_1'$ (antitonicity 2),
3. $\forall A \subseteq G : A \subseteq A''$ (extensivity 1)
4. $\forall B \subseteq M : B \subseteq B''$ (extensivity 2)

Following we will treat a particular Galois-correspondence between *sets of permutations* and *sets of binary relations*.

The context of permutations and binary relations

Given a finite set X .

We consider the context $\mathbb{K}_X = (\text{Sym}(X), \mathcal{P}(X^2), \triangleright)$.

For $\varphi \in \text{Sym}(X)$, $R \subseteq X^2$ we define

$$\varphi \triangleright R : \iff (x, y) \in R \Rightarrow (x^\varphi, y^\varphi) \in R$$

(φ preserves R / R is invariant for φ .)

Let us study the Galois-closed sets of permutations and of binary relations.

Galois-closed sets

Closed sets of permutations:

$G \subseteq \text{Sym}(X)$ is Galois-closed \iff

$$\exists \{R_i \mid i \in \mathcal{I}\} \subseteq \mathcal{P}(X^2) : G = \text{Aut}(X, \{R_i\}_{i \in \mathcal{I}})$$

i.e. (G, X) is a 2-closed permutation group.

Closed sets of binary relations:

$\{R_i\}_{i \in \mathcal{I}}$ is Galois-closed \iff there exists a 2-closed permutation group (G, X) such that $\{R_i\}_{i \in \mathcal{I}}$ is the set of all binary invariant relations of G .

Let us study these sets in more detail.

Galois closed sets of binary relations

Given:

- a finite set X
- a 2-closed permutation group (G, X)

The binary invariant relations of G are closed under union, intersection and complement (i.e. they form a Boolean lattice).

Since the lattice is finite, it is completely determined by its atoms.

The atoms are called *2-orbits* of (G, X)

If $\{R_i\}_{i \in \mathcal{I}}$ is the complete set of 2-orbits of G , then $(X, \{R_i\}_{i \in \mathcal{I}})$ is a coherent configuration.

Such coherent configurations are called *Schurian coherent configurations*. Schurian coherent configurations inherit the order from the Galois-closed sets of binary relations.

Order on coherent configurations

We extend the natural order on Schurian coherent configurations to all coherent configurations:

Given: $\mathcal{C}_1 = (X, \{R_i\}_{i \in \mathcal{I}})$, $\mathcal{C}_2 = (X, \{S_j\}_{j \in \mathcal{J}})$ coherent configurations.

Let

- $\mathbb{B}(\mathcal{C}_1)$ be all relations that are obtained from $\{R_i\}_{i \in \mathcal{I}}$ by union,
- $\mathbb{B}(\mathcal{C}_2)$ be all relations that are obtained from $\{S_j\}_{j \in \mathcal{J}}$ by union.

We define $\mathcal{C}_1 \leq \mathcal{C}_2$ iff $\mathbb{B}(\mathcal{C}_1) \subseteq \mathbb{B}(\mathcal{C}_2)$.

Question: Is \leq a lattice ordering?

From coherent configurations to coherent algebras

Given: a coherent configuration $(X, \{R_i\}_{i \in \mathcal{I}})$

W.l.o.g. $I = \{1, \dots, r\}$, $X = \{1, \dots, n\}$.

To every R_i we associate $A_i = (a_{x,y}) \in \mathbb{C}^{n \times n}$ by

$$a_{x,y} = \begin{cases} 1 & (x,y) \in R_i \\ 0 & \text{otherwise.} \end{cases}$$

Consider the linear span

$$W = \langle A_1, \dots, A_r \rangle$$

Then W is a self-adjoint unitary matrix-algebra that is closed with respect to the Hadamard-product: $(a_{x,y}) \circ (b_{x,y}) = (a_{x,y}b_{x,y})$.

Selfadjoint unitary matrix algebras that are closed w.r.t.

Hadamard-product are called *coherent algebras*.

From coherent algebras to coherent configurations

Every coherent algebra W has a (canonical) basis of zero-one matrices $\langle A_1, \dots, A_r \rangle$ such that

$$A_i \circ A_j = \begin{cases} 0 & i \neq j \\ A_i & \text{otherwise.} \end{cases}$$

This basis is called *(first) standard basis* of W .

To each matrix $A_i = (a_{x,y})$ we associate $R_i \subseteq X^2$ by

$$R_i = \{(x, y) \mid a_{x,y} = 1\}.$$

With $X = \{1, \dots, n\}$ and $\mathcal{I} = \{1, \dots, r\}$ we have that

$(X, \{R_i\}_{i \in \mathcal{I}})$ is a coherent configuration.

Note: The correspondence between coherent configurations and coherent algebras is one-to-one (up to isomorphism).

Order on coherent algebras

Coherent algebras are naturally ordered by inclusion.

Clearly, this order is a lattice ordering.

The infimum in this lattice is the setwise-intersection.

The supremum is the coherent closure.

If W_1 and W_2 are coherent algebras and \mathcal{C}_1 and \mathcal{C}_2 are the associated coherent configurations then

$$W_1 \subseteq W_2 \iff \mathcal{C}_1 \leq \mathcal{C}_2.$$

In particular the coherent configurations over a set X form a complete lattice.

Infimum of coherent configurations (direct construction)

Given $\mathcal{C}_1 = (X, \{R_i\}_{i \in \mathcal{I}})$, $\mathcal{C}_2 = (X, \{S_j\}_{j \in \mathcal{J}})$ coherent configurations.

Consider the bipartite graph Γ that has as vertex set $\mathcal{I} \dot{\cup} \mathcal{J}$ where $i \in \mathcal{I}$ is connected to $j \in \mathcal{J}$ if $R_i \cap S_j \neq \emptyset$.

Let $\{\mathcal{I}_1 \cup \mathcal{J}_1, \mathcal{I}_2 \cup \mathcal{J}_2, \dots, \mathcal{I}_k \cup \mathcal{J}_k\}$ be the connected components of Γ .

Define

$$T_l = \bigcup_{i \in \mathcal{I}_l} R_i = \bigcup_{j \in \mathcal{J}_l} S_j \quad (l = 1, \dots, k).$$

Then

$$\mathcal{C} = (X, \{T_l\}_{l=1}^k) = \mathcal{C}_1 \wedge \mathcal{C}_2.$$

Coherent closure

Given a set $M \subseteq \mathbb{C}^{n \times n}$.

There is a smallest coherent algebra $\langle\langle M \rangle\rangle$ containing M .

In particular

$$\langle\langle M \rangle\rangle = \bigcap \{W \mid M \subseteq W, W \text{ coherent algebra}\}.$$

This is well-defined since $\mathbb{C}^{n \times n}$ is itself a coherent algebra.

Coherent closure (constructive version)

Given a relational structure $(X, \{R_i\}_{i \in \mathcal{I}})$.

The goal is to find the smallest coherent configuration \mathcal{C} such that $\{R_i\}_{i \in \mathcal{I}} \subseteq \mathbb{B}(\mathcal{C})$.

Step 1) Construct the coarsest partition $\{S_j\}_{j \in \mathcal{J}}$ of X^2 such that each R_i is a union of parts.

Step 2) Construct the coarsest refinement $\{T_k\}_{k \in \mathcal{K}}$ of $\{S_j\}_{j \in \mathcal{J}}$ such that for each T_k we have either $T_k \subseteq \Delta$ or $T_k \cap \Delta = \emptyset$.

Step 3) Construct the coarsest refinement $\{U_l\}_{l \in \mathcal{L}}$ of $\{T_k\}_{k \in \mathcal{K}}$ such that each U_l is either symmetric or asymmetric.

Step 4) Perform the Weisfeiler-Leman algorithm on $\{U_l\}_{l \in \mathcal{L}}$.

Weisfeiler Leman Algorithm (Ideas)

After Step 1 to 4 the relational structure $(X, \{U_l\}_{l \in \mathcal{L}})$ already fulfills almost all axioms of coherent configurations. The following axiom might still not be fulfilled:

CC5. For $(x, y) \in U_k$ the number $|\{z \mid (x, z) \in U_i, (z, y) \in U_j\}|$ does not depend on (x, y) but only on i, j, k .

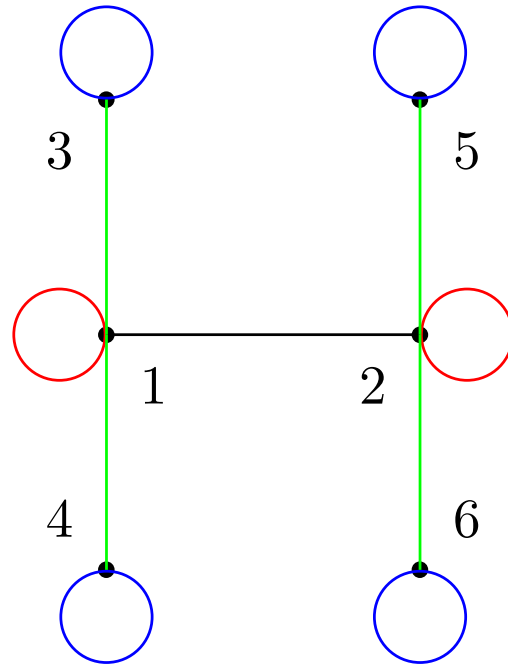
The algorithm proceeds stepwise refining the partition $\{U_l\}_{l \in \mathcal{L}}$ in every step getting closer to the fulfillment of Axiom CC5 without spoiling the other axioms.

The algorithm stops when a stable partition is reached.

WL-stabilization algorithm (Example)

We start with a relational structure that fulfills $CC1, \dots, CC4$.

We consider it as colored graph.



$$\begin{pmatrix} 0 & 3 & 2 & 2 & 4 & 4 \\ 3 & 0 & 4 & 4 & 2 & 2 \\ 2 & 4 & 1 & 4 & 4 & 4 \\ 2 & 4 & 4 & 1 & 4 & 4 \\ 4 & 2 & 4 & 4 & 1 & 4 \\ 4 & 2 & 4 & 4 & 4 & 1 \end{pmatrix}$$

Example continued

We replace the entries of the adjacency matrix by non-commuting variable symbols:

$$\begin{pmatrix} 0 & 3 & 2 & 2 & 4 & 4 \\ 3 & 0 & 4 & 4 & 2 & 2 \\ 2 & 4 & 1 & 4 & 4 & 4 \\ 2 & 4 & 4 & 1 & 4 & 4 \\ 4 & 2 & 4 & 4 & 1 & 4 \\ 4 & 2 & 4 & 4 & 4 & 1 \end{pmatrix} \mapsto \begin{pmatrix} t_0 & t_3 & t_2 & t_2 & t_4 & t_4 \\ t_3 & t_0 & t_4 & t_4 & t_2 & t_2 \\ t_2 & t_4 & t_1 & t_4 & t_4 & t_4 \\ t_2 & t_4 & t_4 & t_1 & t_4 & t_4 \\ t_4 & t_2 & t_4 & t_4 & t_1 & t_4 \\ t_4 & t_2 & t_4 & t_4 & t_4 & t_1 \end{pmatrix}$$

Example continued

We compute the square of this matrix:

$$\begin{pmatrix} t_0 & t_3 & t_2 & t_2 & t_4 & t_4 \\ t_3 & t_0 & t_4 & t_4 & t_2 & t_2 \\ t_2 & t_4 & t_1 & t_4 & t_4 & t_4 \\ t_2 & t_4 & t_4 & t_1 & t_4 & t_4 \\ t_4 & t_2 & t_4 & t_4 & t_1 & t_4 \\ t_4 & t_2 & t_4 & t_4 & t_4 & t_1 \end{pmatrix} \mapsto \begin{pmatrix} x_0 & x_2 & x_3 & x_3 & x_4 & x_4 \\ x_2 & x_0 & x_4 & x_4 & x_3 & x_3 \\ x_5 & x_6 & x_1 & x_7 & x_8 & x_8 \\ x_5 & x_6 & x_7 & x_1 & x_8 & x_8 \\ x_6 & x_5 & x_8 & x_8 & x_1 & x_7 \\ x_6 & x_5 & x_8 & x_8 & x_7 & x_1 \end{pmatrix}$$

where

$$x_0 = t_0^2 + 2t_2^2 + t_3^2 + 2t_4^2$$

$$x_1 = t_1^2 + t_2^2 + 4t_3^2$$

$$x_2 = t_0t_3 + 2t_2t_4 + t_3t_0 + 2t_4t_2, \quad \text{etc.}$$

Example continued

We replace the entries with numbers:

$$\begin{pmatrix} x_0 & x_2 & x_3 & x_3 & x_4 & x_4 \\ x_2 & x_0 & x_4 & x_4 & x_3 & x_3 \\ x_5 & x_6 & x_1 & x_7 & x_8 & x_8 \\ x_5 & x_6 & x_7 & x_1 & x_8 & x_8 \\ x_6 & x_5 & x_8 & x_8 & x_1 & x_7 \\ x_6 & x_5 & x_8 & x_8 & x_7 & x_1 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 2 & 3 & 3 & 4 & 4 \\ 2 & 0 & 4 & 4 & 3 & 3 \\ 5 & 6 & 1 & 7 & 8 & 8 \\ 5 & 6 & 7 & 1 & 8 & 8 \\ 6 & 5 & 8 & 8 & 1 & 7 \\ 6 & 5 & 8 & 8 & 7 & 1 \end{pmatrix}$$

Apply this procedure again until no further colors are introduced.

(In fact the above obtained matrix is already stable)

A remark on the complexity of the WL-stabilization

The above algorithm was implemented directly using relational structures by L. Babel, I. V. Chuvaeva, and M. Klin. It is called STABIL.

The worst case-complexity of this algorithm is roughly $O(n^7)$.

From the point of view of complexity, the best known algorithm is STABCOL (L. Babel, S. Baumann, M. Lüdecke, and G. Tinhofer). It has complexity $O(n^3 \log n)$.

Though STABIL has a much worse theoretical worst-case complexity, it performs in most applications much quicker than STABCOL.

Enumeration of fusions of coherent configurations

Given $\mathcal{C}_1 = (X, \{R_i\}_{i \in \mathcal{I}})$, $\mathcal{C}_2 = (X, \{S_j\}_{j \in \mathcal{J}})$ such that $\mathcal{C}_1 \leq \mathcal{C}_2$.

Then \mathcal{C}_1 is determined by a partition $\{\mathcal{J}_i\}_{i \in \mathcal{I}}$ of \mathcal{J} according to

$$R_i = \bigcup_{j \in \mathcal{J}_i} S_j.$$

\mathcal{C}_1 is a fusion of \mathcal{C}_2 with respect to $\{\mathcal{J}_i\}_{i \in \mathcal{I}}$.

In order to find all fusions of \mathcal{C}_2 , we have to find all partitions of \mathcal{J} whose fusions are coherent configurations.

Strategy for the enumeration of partitions of \mathcal{J}

Step 1 Find feasible candidates for classes \mathcal{J}_i .

Step 2 Construct all partitions of \mathcal{J} that consist only of feasible blocks.

Step 3 Test each such partition if its fusion is a coherent configuration.

A possible (weak) feasibility-condition for blocks is:

$\mathcal{K} \subseteq \mathcal{J}$ is feasible if

$$\forall k_1, k_2 \in \mathcal{K} : \sum_{i \in \mathcal{K}} \sum_{j \in \mathcal{K}} p_{i,j}^{k_1} = \sum_{i \in \mathcal{K}} \sum_{j \in \mathcal{K}} p_{i,j}^{k_2}.$$

How to test a feasible partition?

Given a partition $\{\mathcal{J}_1, \dots, \mathcal{J}_p\}$ of \mathcal{J} into feasible classes.

We have to test if $\forall l, m, o \in \{1, \dots, p\}, \forall k_1, k_2 \in \mathcal{J}_o$:

$$\sum_{i \in \mathcal{J}_l} \sum_{j \in \mathcal{J}_m} p_{i,j}^{k_1} = \sum_{i \in \mathcal{J}_l} \sum_{j \in \mathcal{J}_m} p_{i,j}^{k_2}$$

This number is equal to the structure constant $\hat{p}_{l,m}^o$ of the fusion.

Enumeration of fusions as problem in symbolic computation

Let us consider the following scenario: Given an infinite series $(\mathcal{C}_{s_1, \dots, s_l})$ of coherent configurations — all of the same rank.

Suppose the structure constants of the coherent configurations are given as polynomials in $\mathbb{Z}[s_1, \dots, s_l]$. More precisely, for all i, j, k we have a polynomial $p_{i,j}^k \in \mathbb{Z}[s_1, \dots, s_l]$ such that the corresponding structure constant of $\mathcal{C}_{s_1, \dots, s_l}$ is $p_{i,j}^k(s_1, \dots, s_l)$.

We want to find all fusions of all coherent configurations simultaneously.

Checking the feasibility-conditions involves solving Diophantine systems of polynomial equations.

Let us consider this scenario on hand of an extended example.

Steiner 2-Designs

A *Steiner 2-design* of order (s, t) is an incidence structure $(\mathcal{P}, \mathcal{B}, I)$ where

- \mathcal{P} is a set of points,
- \mathcal{B} is a set of lines,
- each line has $s + 1$ points,
- each point lies on $t + 1$ lines,
- through any two points goes exactly one line.

Flag-algebras

Given a Steiner 2-design $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ of order (s, t) .

Consider the set of flags $\mathcal{F} = \{(P, b) \mid P \in \mathcal{P}, b \in \mathcal{B}, PIb\}$

Define two binary relations on \mathcal{F} :

$$R_L = \{((P_1, b_1), (P_2, b_2)) \mid P_1 \neq P_2, b_1 = b_2\},$$

$$R_N = \{((P_1, b_1), (P_2, b_2)) \mid P_1 = P_2, b_1 \neq b_2\}.$$

Let L and N be the adjacency-matrices of R_L and R_N , respectively.

The *Flag-algebra* of \mathcal{D} is defined as:

$$\mathcal{W}_{\mathcal{F}}(\mathcal{D}) = \langle\langle L, N \rangle\rangle \quad (\text{coherent closure}).$$

This algebra has rank 7. It defines a dihedral scheme in the sense of Zieschang. In particular

$$\mathcal{W}_{\mathcal{F}}(\mathcal{D}) = \langle A_0, \dots, A_6 \rangle \quad \text{where}$$

$$A_0 = I, A_1 = L, A_2 = N, A_3 = LN, A_4 = NL, A_5 = LNL,$$

$$A_6 = NLN - LNL$$

i/j	0	1	2	3	4	5	6
0	1 0 0 0 0 0 0	0 1 0 0 0 0 0	0 0 1 0 0 0 0	0 0 0 1 0 0 0	0 0 0 0 1 0 0	0 0 0 0 0 1 0	0 0 0 0 0 0 1
1	0 1 0 0 0 0 0	s $s - 1$ 0 0 0 0 0	0 0 0 1 0 0 0	0 0 s $s - 1$ 0 0 0	0 0 0 0 0 1 0	0 0 0 0 s $s - 1$ 0	0 0 0 0 0 0 s
2	0 0 1 0 0 0 0	0 0 0 0 1 0 0	t 0 $t - 1$ 0 0 0 0	0 0 0 0 0 1 1	0 t 0 0 $t - 1$ 0 0	0 0 0 s 0 $s - 1$ s	0 0 0 $t - s$ 0 $t - s$ $t - s - 1$
3	0 0 0 1 0 0 0	0 0 0 0 0 1 0	0 t 0 $t - 1$ 0 0 0	0 0 0 0 s $s - 1$ s	ts $t(s - 1)$ 0 0 0 $t - 1$ 0	0 0 s^2 $(s - 1)s$ $(s - 1)s$ $(s - 1)^2$ s^2	0 0 $s(t - s)$ $(s - 1)(t - s)$ $s(t - s)$ $(s - 1)(t - s)$ $s(t - s - 1)$
4	0 0 0 0 1 0 0	0 0 s 0 $s - 1$ 0 0	0 0 0 0 0 1 1	ts 0 $(t - 1)s$ 0 0 $s - 1$ $s - 1$	0 0 0 s 0 $s - 1$ s	0 ts 0 $(s - 1)s$ $(t - 1)s$ $(s - 1)^2$ $(s - 1)s$	0 0 0 $s(t - s)$ 0 $s(t - s)$ $s(t - s - 1)$

Continued on next slide...

i/j	0	1	2	3	4	5	6
5	0	0	0	0	0	ts^2	0
	0	0	0	ts	0	$ts(s-1)$	0
	0	0	0	0	s^2	$(s-1)s^2$	$s^2(t-s)$
	0	s	0	$(t-1)s$	$(s-1)s$	$(s-1)^2s$	$s(s-1)(t-s)$
	0	0	s	$(s-1)s$	$(s-1)s$	$(s-1)^2s$	$s^2(t-s)$
	1	$s-1$	$s-1$	$(s-1)^2$	$(s-1)^2$	$p_{5,5}^5$	$s(s-1)(t-s)$
	0	0	s	$(s-1)s$	s^2	$(s-1)s^2$	$s^2(t-s-1)$
6	0	0	0	0	0	0	$st(t-s)$
	0	0	0	0	0	0	$st(t-s)$
	0	0	0	0	$s(t-s)$	$s^2(t-s)$	$s(s-t+1)(s-t)$
	0	0	0	0	$s(t-s)$	$s^2(t-s)$	$s(s-t+1)(s-t)$
	0	0	$t-s$	$s(t-s)$	$(s-1)(t-s)$	$s(s-1)(t-s)$	$s(s-t+1)(s-t)$
	0	0	$t-s$	$s(t-s)$	$(s-1)(t-s)$	$s(s-1)(t-s)$	$s(s-t+1)(s-t)$
	1	s	$t-s-1$	$s(t-s-1)$	$s(t-s-1)$	$s^2(t-s-1)$	$p_{6,6}^6$

where $p_{6,6}^6 = s^3 - 2ts + 2s - 2ts^2 + 3s^2 - t + t^2s$ and $p_{5,5}^5 = ts + 2s - 1 - 3s^2 + s^3$

Table 4.1: The table of structure constants for Steiner 2-designs

Feasible sets

Let us check if $\{1, 2\}$ can be a class of a partition of a fusion.

We take the vectors:

$$(p_{1,1}^k)_{k=0}^6, \quad (p_{1,2}^k)_{k=0}^6, \quad (p_{2,1}^k)_{k=0}^6, \quad \text{and} \quad (p_{2,2}^k)_{k=0}^6.$$

We sum them up:

$$\begin{bmatrix} s \\ s-1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} t \\ 0 \\ t-1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} s+t \\ \mathbf{s-1} \\ \mathbf{t-1} \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

So we get that $\{1, 2\}$ is feasible only if $s = t$.

Another example of a feasibility test

Let us check if $\{3, 4, 5\}$ can be a class of a partition of a fusion.

We take the vectors:

$$(p_{i,j}^k)_{k=0}^6, \quad \text{for } i, j \in \{3, 4, 5\}.$$

We sum them up and obtain:

$$\begin{bmatrix} st(s+2) \\ t(s^2+2s-1) \\ s(s^2+s+t-1) \\ \mathbf{s(s^2+s+t-2)} \\ \mathbf{s(s^2+s+t-2)} \\ \mathbf{s^3+s^2+st-3s+t-1} \\ s^3+3s^2+s-1 \end{bmatrix}$$

Fusions of the flag-algebras os Steiner 2-Designs

$$\begin{aligned}
 W_0 &= \langle (A_0), (A_1), (A_2), (A_3), (A_4), (A_5), (A_6) \rangle \\
 W_1 &= \langle (A_0), (A_1), (A_2 + A_3 + A_4 + A_5), (A_6) \rangle \\
 W_2 &= \langle (A_0), (A_1), (A_2 + A_3 + A_4 + A_5 + A_6) \rangle \\
 W_3 &= \langle (A_0), (A_1), (A_2 + A_5), (A_3 + A_4), (A_6) \rangle \quad \text{for } (s, t) = (1, t) \\
 W_4 &= \langle (A_0), (A_1 + A_2 + A_3 + A_4 + A_5 + A_6) \rangle \\
 W_5 &= \langle (A_0), (A_1 + A_2 + A_3 + A_4 + A_6), (A_5) \rangle \quad \text{for } (s, t) = (1, t) \\
 W_6 &= \langle (A_0), (A_1 + A_3 + A_4 + A_5 + A_6), (A_2) \rangle \\
 W_7 &= \langle (A_0), (A_1 + A_5), (A_2 + A_3 + A_4 + A_6) \rangle \quad \text{for } (s, t) = (2, t) \\
 W_8 &= \langle (A_0), (A_1 + A_6), (A_2), (A_3 + A_4 + A_5) \rangle \quad \text{for } (s, t) = (s, s + 1) \\
 W_9 &= \langle (A_0), (A_1 + A_6), (A_2 + A_3 + A_4 + A_5) \rangle \quad \text{for } (s, t) = (s, s + 1) \\
 W_{10} &= \langle (A_0), (A_1 + A_6), (A_2 + A_3 + A_4), (A_5) \rangle \quad \text{for } (s, t) = (1, 2) \\
 W_{11} &= \langle (A_0), (A_1 + A_6), (A_2 + A_5), (A_3 + A_4) \rangle \quad \text{for } (s, t) = (3, 4)
 \end{aligned}$$

Isomorphisms between Coherent Configurations

Given coherent configurations $\mathcal{C}_1 = (X, \{R_i\}_{i \in \mathcal{I}})$, $\mathcal{C}_2 = (Y, \{S_j\}_{j \in \mathcal{J}})$.

We say that \mathcal{C}_1 is isomorphic to \mathcal{C}_2 ($\mathcal{C}_1 \cong \mathcal{C}_2$) if

$$\exists \varphi : X \rightarrow Y : \varphi \text{ bijective, } \forall i \in \mathcal{I} \exists j \in \mathcal{J} \forall (x, y) \in R_i : (x^\varphi, y^\varphi) \in S_j.$$

φ is called (combinatorial) isomorphism from \mathcal{C}_1 to \mathcal{C}_2 .

An isomorphism of \mathcal{C}_1 to itself is called *weak automorphism*.

A weak automorphism φ of \mathcal{C}_1 is called (*strong*) *automorphism* if

$$\forall i \in \mathcal{I} \forall (x, y) \in R_i : (x^\varphi, y^\varphi) \in R_i.$$

$$\text{CAut}(\mathcal{C}_1) = \{\text{all weak automorphisms of } \mathcal{C}_1\}$$

$$\text{Aut}(\mathcal{C}_1) = \{\text{all automorphisms of } \mathcal{C}_1\}$$

Normalizers of 2-closed permutation groups

Given a coherent configuration $\mathcal{C} = (X, \{R_i\}_{i \in \mathcal{I}})$.

In general we have that

$$\text{Aut}(\mathcal{C}) \trianglelefteq \text{CAut}(\mathcal{C}).$$

If \mathcal{C} is Schurian, then $\text{CAut}(\mathcal{C})$ is the normalizer of $\text{Aut}(\mathcal{C})$ in $\text{Sym}(X)$.

By definition, $\text{CAut}(\mathcal{C})$ acts naturally on \mathcal{I} . The kernel of this action is $\text{Aut}(\mathcal{C})$.

If $\varphi \in (\text{CAut}(\mathcal{C}), \mathcal{I})$, then

$$\forall i, j, k \in \mathcal{I} : p_{i,j}^k = p_{i^\varphi, j^\varphi}^{k^\varphi}.$$

Algebraic isomorphisms

We generalize the property of weak automorphisms to preserve structure-constants.

Let $\mathcal{C}_1 = (X, \{R_i\}_{i \in \mathcal{I}})$, $\mathcal{C}_2 = (Y, \{S_j\}_{j \in \mathcal{J}})$ be coherent configurations with structure constants $(p_{i,j}^k)$ and $(\hat{p}_{i,j}^k)$, respectively.

A function $\psi : \mathcal{I} \rightarrow \mathcal{J}$ is called *algebraic isomorphism* from \mathcal{C}_1 to \mathcal{C}_2 if it is bijective and

$$\forall i, j, k \in \mathcal{I} : p_{i,j}^k = \hat{p}_{i^\psi, j^\psi}^{k^\psi}.$$

Algebraic automorphisms

An algebraic isomorphism of a coherent configuration $\mathcal{C} = (X, \{R_i\}_{i \in \mathcal{I}})$ to itself is called *algebraic automorphism*.

$$\text{AAut}(\mathcal{C}) = \{\text{all algebraic automorphisms of } \mathcal{C}\}.$$

Generally we have:

$$(\text{CAut}(\mathcal{C}), \mathcal{I}) \leq (\text{AAut}, \mathcal{I}).$$

Elements of $(\text{CAut}(\mathcal{C}), \mathcal{I})$ are called *realizable algebraic automorphisms* of \mathcal{C} .

An idea of a normalizer algorithm for 2-closed permutation groups

Given a 2-closed permutation group (G, X) .

Let $\mathcal{C} = (X, \{R_i\}_{i \in \mathcal{I}})$ be the corresponding Schurian coherent configuration of 2-orbits of (G, X) .

Step 1 Find a set of generators of $\text{Aut}(\mathcal{C})$,

Step 2 Find a set of generators of $\text{AAut}(\mathcal{C})$,

Step 3 Filter the elements of $\text{AAut}(\mathcal{C})$ for realizable algebraic automorphisms.

If we want to test if an algebraic automorphism φ is realizable, we try to construct a color-preserving isomorphism between the colored graphs $(X, \{R_i\}_{i \in \mathcal{I}})$ and $(X, \{R_{i\varphi}\}_{i \in \mathcal{I}})$.

Such an isomorphism is an element of $\text{CAut}(\mathcal{C})$ that realizes φ .

Comparison with the general normalizer algorithm

There is an efficient normalizer algorithm implemented in GAP4.

For some examples this normalizer algorithm performs (somewhat) slowly.

In these cases usually our normalizer-algorithm can solve the problem in a reasonable time.

The main advantage is that most of the time our normalizer algorithm works on the level of structure constants.

This is good because this is a combinatorial object whose size depends on the *rank* and not on the *degree* of the coherent configurations.

Algebraic fusions

Given a coherent configuration $\mathcal{C} = (X, \{R_i\}_{i \in \mathcal{I}})$.

Let $(G, \mathcal{I}) \leq (\text{AAut}(\mathcal{C}), \mathcal{I})$.

Let $\{\mathcal{I}_1, \dots, \mathcal{I}_k\}$ be the orbit-partition of (G, \mathcal{I}) .

Then the fusion of \mathcal{C} with respect to $\{\mathcal{I}_1, \dots, \mathcal{I}_k\}$ is a fusion of \mathcal{C} .

Fusions of \mathcal{C} that are obtained in this way are called

algebraic fusions of \mathcal{C} with respect to (G, \mathcal{I}) .

The algebraic fusion for (G, \mathcal{I}) is an association scheme if and only if (G, \mathcal{I}) acts transitively on the elements of \mathcal{I} that correspond to reflexive relations of \mathcal{C} .

If \mathcal{C} is Schurian and $(G, \mathcal{I}) \leq \text{CAut}(\mathcal{C})$, then the algebraic fusion of \mathcal{C} with respect to (G, \mathcal{I}) is also Schurian.

Algebraic twins

Let $\mathcal{C} = (X, \{R_i\}_{i \in \mathcal{I}})$ be a coherent configuration.

Let $\Pi = \{\mathcal{I}_1, \dots, \mathcal{I}_k\}$ be a partition of \mathcal{I} such that the fusion of \mathcal{C} with respect to Π is a fusion of \mathcal{C} .

If $\varphi \in (\text{AAut}(\mathcal{C}), \mathcal{I})$, then $\Pi^\varphi = \{\mathcal{I}_1^\varphi, \dots, \mathcal{I}_k^\varphi\}$ another partition of \mathcal{I} such that the corresponding fusion is a fusion of \mathcal{C} .

The fusions Π and Π^φ are called *algebraic twins* in the lattice of fusions of \mathcal{C} .

If φ is realizable, then the twins are isomorphic.

Otherwise, it can happen that the twins are *non-isomorphic!*

In general, algebraic twins are algebraically isomorphic.

Demonstration of the concepts (Example)

Consider the group $E_8 \cong \mathbb{Z}_2^3$.

Let $X = \{\text{all 2-element subsets of } E_8\}$.

E_8 acts on X by $\{x, y\} \mapsto \{x + a, y + a\}$.

Let $AP(2) = (X, \{R_i\}_{i \in \mathcal{I}})$ be the Schurian coherent configuration of 2-orbits of (E_8, X) .

It has

- degree 28,
- rank 112,
- 7 fibres.

Groups related to $AP(2)$

$\text{Aut}(AP(2)) = (E_8, X)$	i.e. (E_8, X) is 2-closed
$(\text{CAut}(AP(2)), X) \cong E_{2^{14}} \rtimes GL(3, 2)$	order: $2^{17} \cdot 3 \cdot 7$
$(\text{CAut}(AP(2)), \mathcal{I})$	has order $2^{14} \cdot 3 \cdot 7$
$(\text{AAut}(AP(2)), \mathcal{I}) \cong E_{2^{14}} \rtimes GL(3, 2)$	order: $2^{17} \cdot 3 \cdot 7$

The index of $(\text{CAut}(AP(2)), \mathcal{I})$ in $(\text{AAut}(AP(2)), \mathcal{I})$ is **8**.

Hence there are algebraic automorphisms that are not realizable.

Some interesting twins

#175, #176 These are the two only quasithin schemes on 28 points. #176 is non-Schurian.

#74, #75 #74 is the famous pseudocyclic scheme found by Mathon #75 was discovered by Hollmann. It is non-Schurian.

#109, #110 *This pair of twins is new!* Both schemes arise as algebraic fusions of $AP(2)$.

#6, #7 These are the most famous twins. #6 is the triangular graph $T(8)$ and #7 is one of the Chang-graphs.

COCO

COCO is a computer-algebra system for computing with the Galois-correspondence between coherent configurations and permutation groups.

Core functionality:

CGR: a function that takes as input a permutation group and computes the corresponding Schurian coherent configuration,

INM: a function that takes a coherent configuration and computes its structure constants,

SUB: a function that takes the structure constants and computes all fusions of the coherent configuration

AUT: a function that takes a set of fusions of a coherent configuration and computes their automorphism-groups.

Improvements to COCO

Meanwhile a few experimental improvements were made to COCO.

In particular a function for computing the normalizer of a 2-closed group was added.

Some additional functionality was created outside of COCO using the Computer Algebra System GAP.

This functions were used, e.g. to treat the coherent configuration $AP(2)$ from the example above.

COCO-II

A new version of COCO is now in the process of development. It is planned to be released as a share-package of GAP

In this new version will be contained improved version of the algorithm of COCO.

Main improvements will be the *use of algebraic automorphisms* while computing fusions of a coherent configuration.

This allows to handle configurations of much higher rank than before.

Main developers are *Sven Reichard*, *Misha Klin*, and *Ch. Pech*.

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