

A survey on combinatorial duality approach

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Macaulay [12, 13], introducing the notion of *inverse systems*, proposed duality as a strong combinatorial tool for performing ideal theory investigation. This conference is a survey of the recent combinatorial algorithms stemmed from Macaulay's ideas and the description they offer on the structure of a 0-dimensional ideal.

Duality Let $\mathcal{P} := k[X_1, \dots, X_n]$, $\mathfrak{m} := (X_1, \dots, X_n)$ be the maximal at the origin and

$$\mathcal{T} := \{X_1^{a_1} \cdots X_n^{a_n} : (a_1, \dots, a_n) \in \mathbb{N}^n\}.$$

Denote, for each $\tau \in \mathcal{T}$, $M(\tau) : \mathcal{P} \rightarrow k$ the functional defined by

$$M(\tau) = c(f, \tau), \forall f = \sum_{t \in \mathcal{T}} c(f, t)t \in \mathcal{P}$$

and $\mathbb{M} := \{M(\tau) : \tau \in \mathcal{T}\}$,

We prove Macaulay's duality [12, 13, 8, 18] between \mathfrak{m} -closed ideals $\mathfrak{l} \subset \mathcal{P}$ and \mathcal{P} -modules of functionals $\Lambda \subset \text{Span}_k\{\mathbb{M}\}$.

Macaulay Bases As a consequence we formalize the notion of *Macaulay Basis* [14, 1] as a "canonical" basis of the dual module of an \mathfrak{m} -primary ideal.

Cerlienco–Mureddu Correspondence Cerlienco–Mureddu propose [2, 3, 4, 7] a combinatorial algorithm which, to each finite set of points $Z \subset k^n$, associates an order ideal $\mathbf{N} \subset \mathcal{T}$ and a correspondence $\Phi : Z \leftrightarrow \mathbf{N}$ which satisfies $\mathbf{N} = \mathbf{N}_{<}(\mathfrak{l})$ where $<$ is the lex ordering on \mathcal{T} and \mathfrak{l} the radical ideal consisting of all polynomials vanishing at the points in Z .

Macaulay's Algorithm Macaulay [12, 13] proposed an algorithm which computes a generating set of the dual module.

Gröbner [8] used it in order to compute the decomposition of a primary ideals into irreducible components and realized [20] that it allowed, for a polynomial ideal, to compute its reduced irreducible decomposition introduced by Emmy Noether [19].

Reduced Irreducible Decomposition Emmy Noether [19] proved that, in a noetherian ring R , each ideal $\mathfrak{a} \subset R$ has a *reduced irreducible representation* $\mathfrak{a} = \bigcap_{j=1}^r \mathfrak{i}_j$ as intersection of finitely many irreducible ideals \mathfrak{i}_j satisfying

- $\forall j \in \{1, \dots, r\}, \mathfrak{i}_j \not\supset \bigcap_{\substack{h=1 \\ j \neq h}}^r \mathfrak{i}_h$ and
- there is no irreducible ideal $\mathfrak{i}_j' \supset \mathfrak{i}_j$ such that $\mathfrak{a} = \left(\bigcap_{\substack{h=1 \\ j \neq h}}^r \mathfrak{i}_h \right) \cap \mathfrak{i}_j'$.

Unfortunately such decomposition is not unique, since it strongly depends on the choice of a frame of coordinates. As it is to be expected, also Macaulay's Algorithm used by Gröbner to explicitly compute a reduced irreducible representation, depends on the choice of a frame of coordinates.

Lazard Structural Theorem Lazard [11, 9, 10] described the structure of the lexicographical Gröbner bases of a polynomial ideal $I \subset k[X, Y]$.

Axis-of-Evil Theorem The Axis-of-Evil Theorem [15, 16, 17] describes the combinatorial structure [Gröbner and border basis, linear and Gröbner representation] of a 0-dimensional ideal

$$I = \bigcap \mathfrak{q}_i \subset \mathcal{P}, \sqrt{\mathfrak{q}_i} = (X_1 - a_{i1}, \dots, X_n - a_{in})$$

in terms of a *Macaulay representation*, i.e. of its roots (a_{i1}, \dots, a_{in}) and of the Macaulay basis of each \mathfrak{q}_i .

Such description is "algorithmical" in terms of elementary combinatorial tools and linear interpolation and extends Cerlienco–Mureddu Correspondence and Lazard's Structural Theorem.

The proof is essentially a direct application of Möller's Algorithm [6, 5].

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