

Symbolic Summation, Riemann Zeta Values, and Quantum Field Theory

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Plan

Motivation

- Quantum field theory
 - cutting edge calculations in particle physics

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- Given expression $g(n)$ (depending on n) find expression $f(n)$, such that

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 - polynomial, hypergeometric and harmonic sums and beyond
- Riemann zeta values and multiple polylogarithms

Elementary Particle Physics

- Standard Model
 - successfully description for interactions of elementary particles
 - gauge theory $SU(3) \times SU(2) \times U(1)$ (internal symmetries)

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 - interaction of matter (quarks, leptons, ...) through particle exchange (gluons, photons, W/Z -bosons, ...)
 - cross sections from perturbative calculations in quantum field theory (Feynman diagrams)

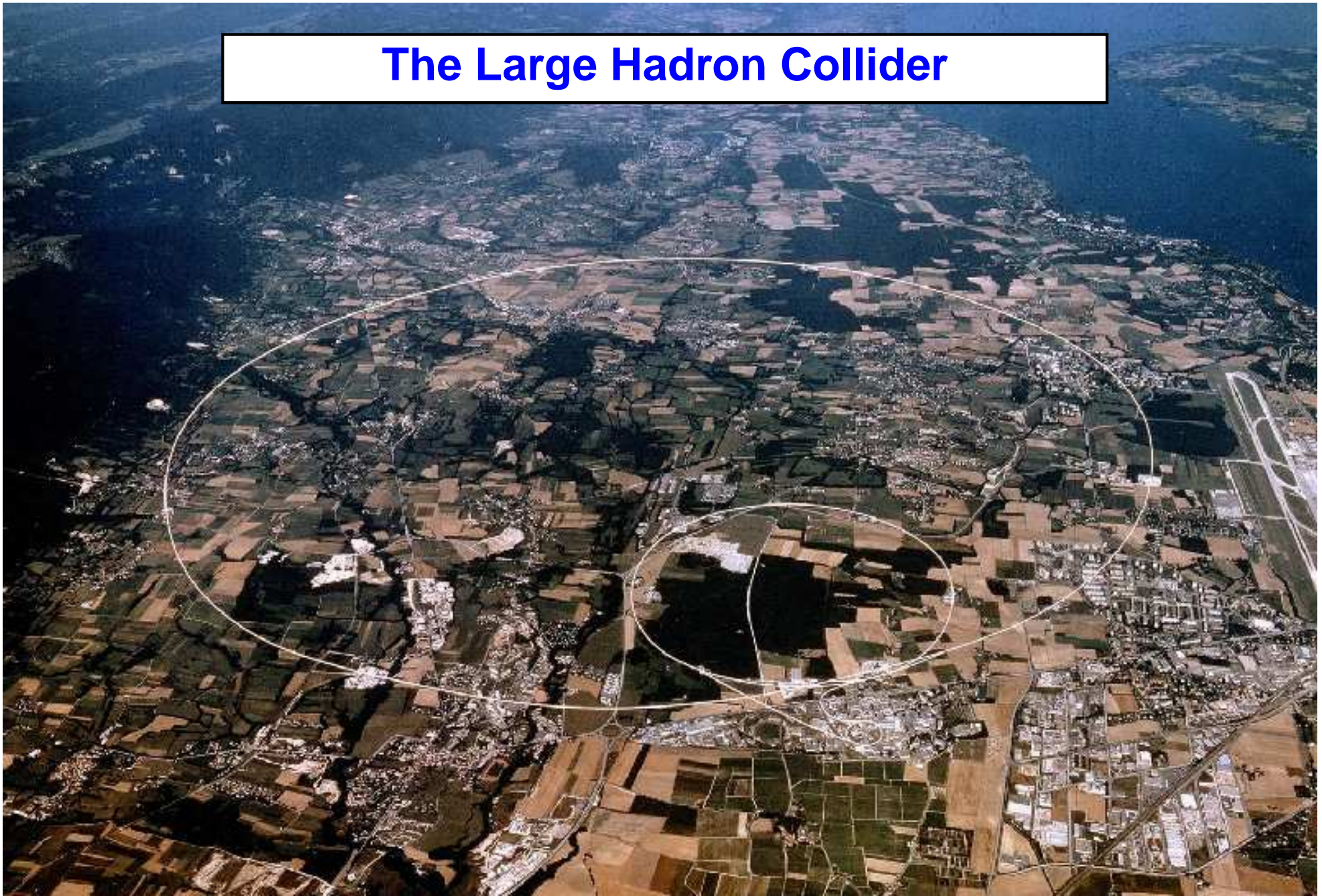
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 - Large Hadron Collider (LHC) at CERN (Genava)
 - collisions of high energy protons (energy frontier)

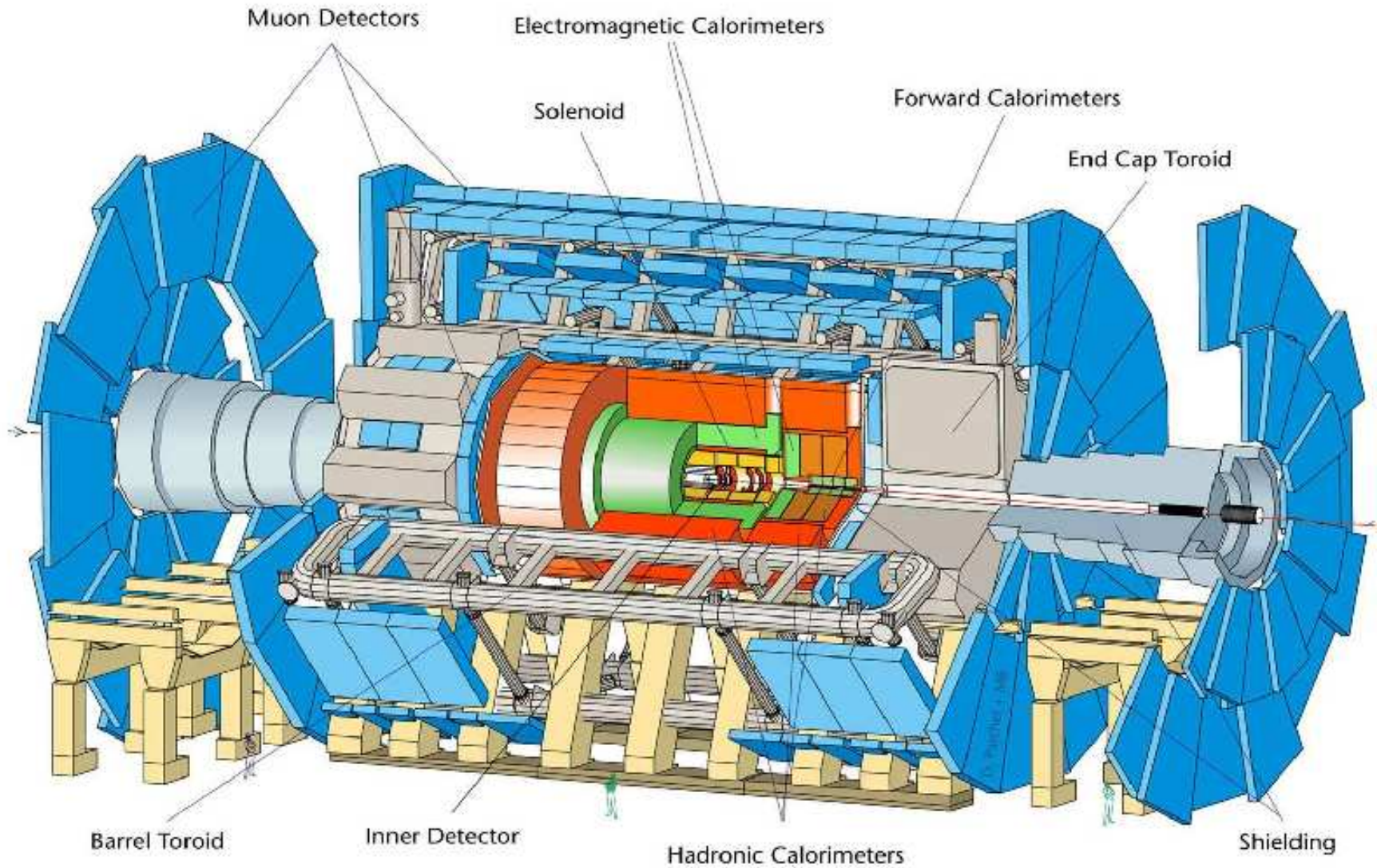
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- Discovery searches for
 - **Higgs** production and **new physics** phenomena

The Large Hadron Collider



The ATLAS Detector

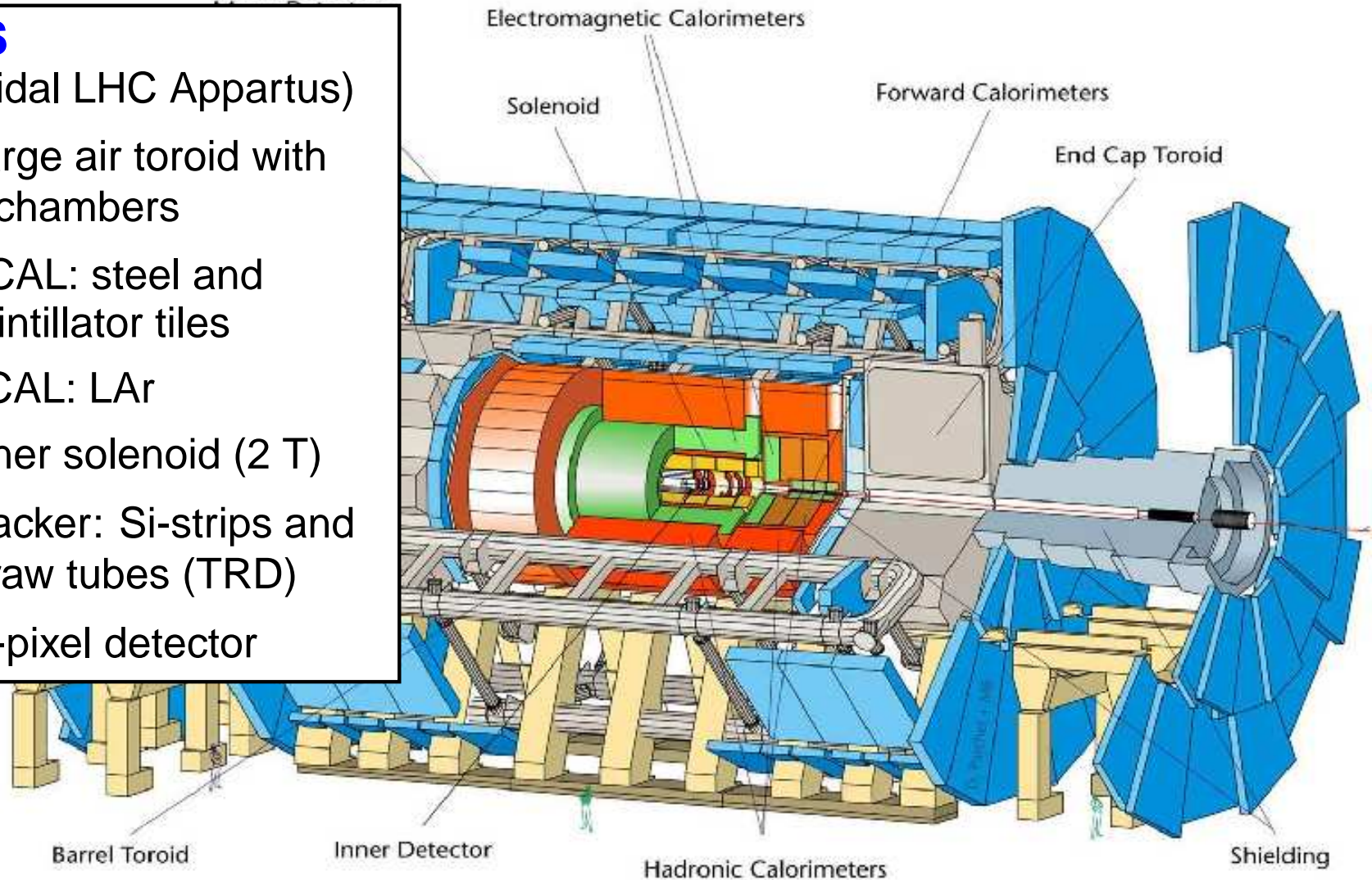


The ATLAS Detector

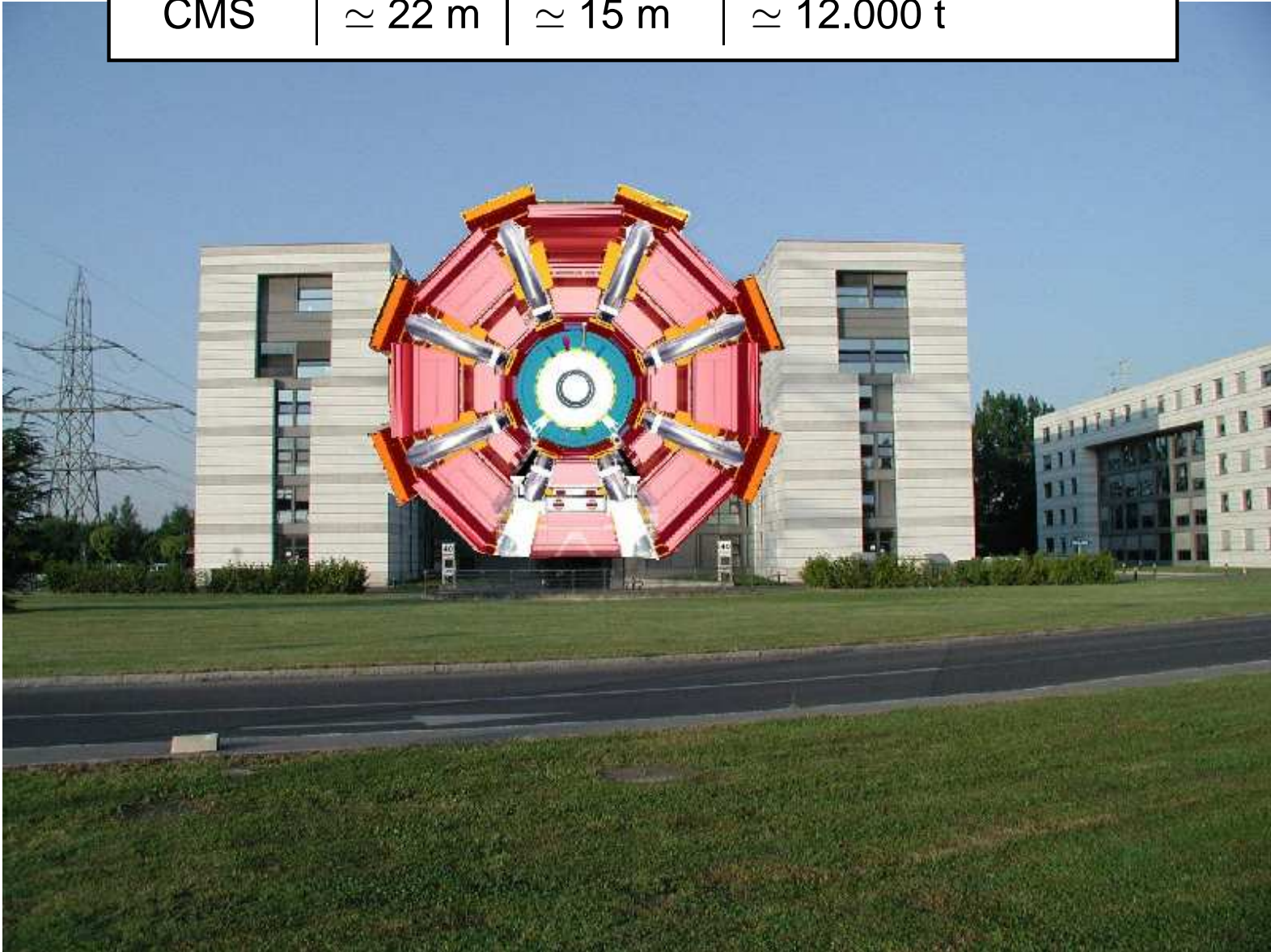
ATLAS

(A Toroidal LHC Apparatus)

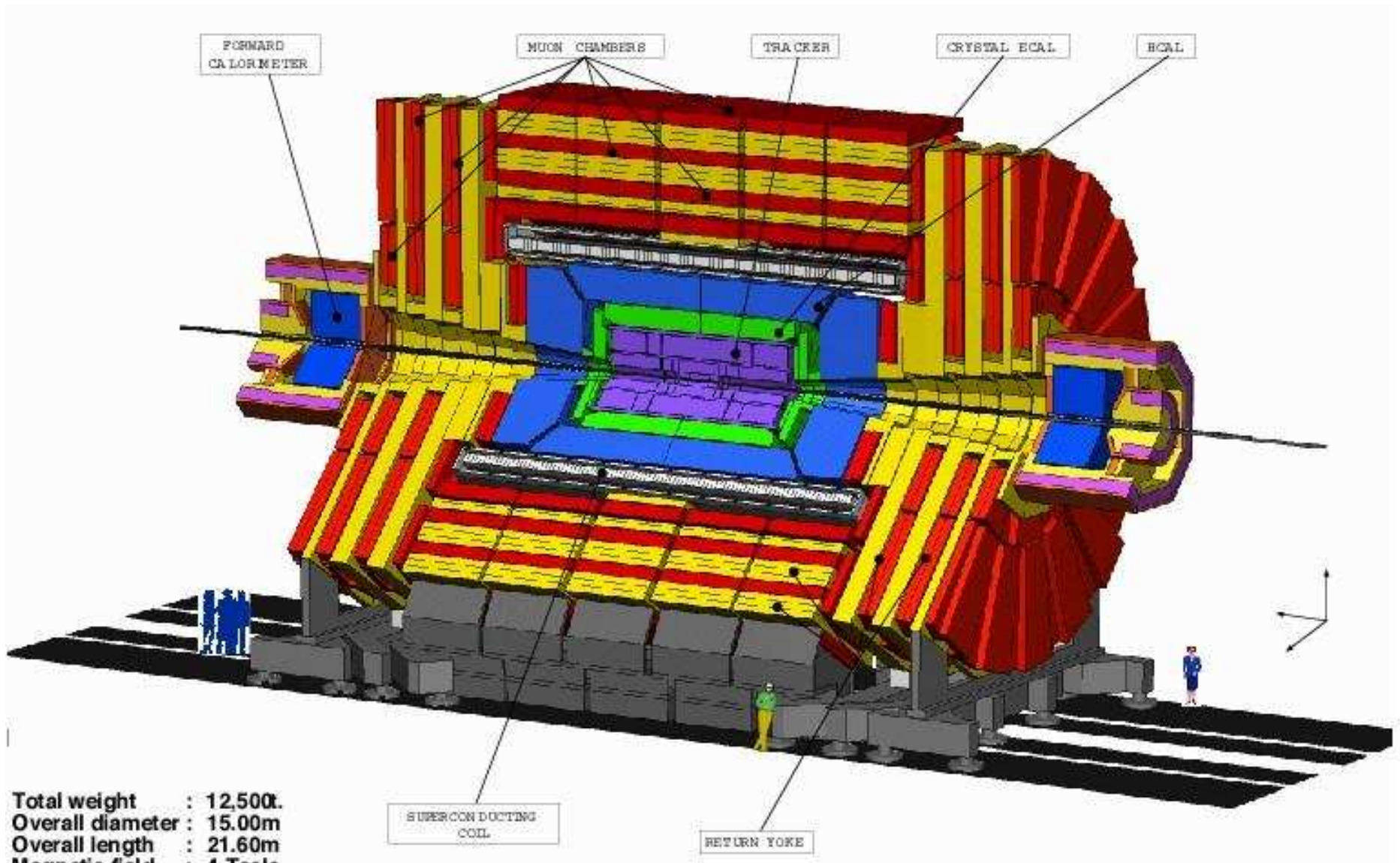
- Large air toroid with μ -chambers
- HCAL: steel and scintillator tiles
- ECAL: LAr
- Inner solenoid (2 T)
- Tracker: Si-strips and straw tubes (TRD)
- Si-pixel detector



	length	diameter	weight
ATLAS	$\simeq 46$ m	$\simeq 25$ m	$\simeq 7.000$ t
CMS	$\simeq 22$ m	$\simeq 15$ m	$\simeq 12.000$ t



The CMS Detector



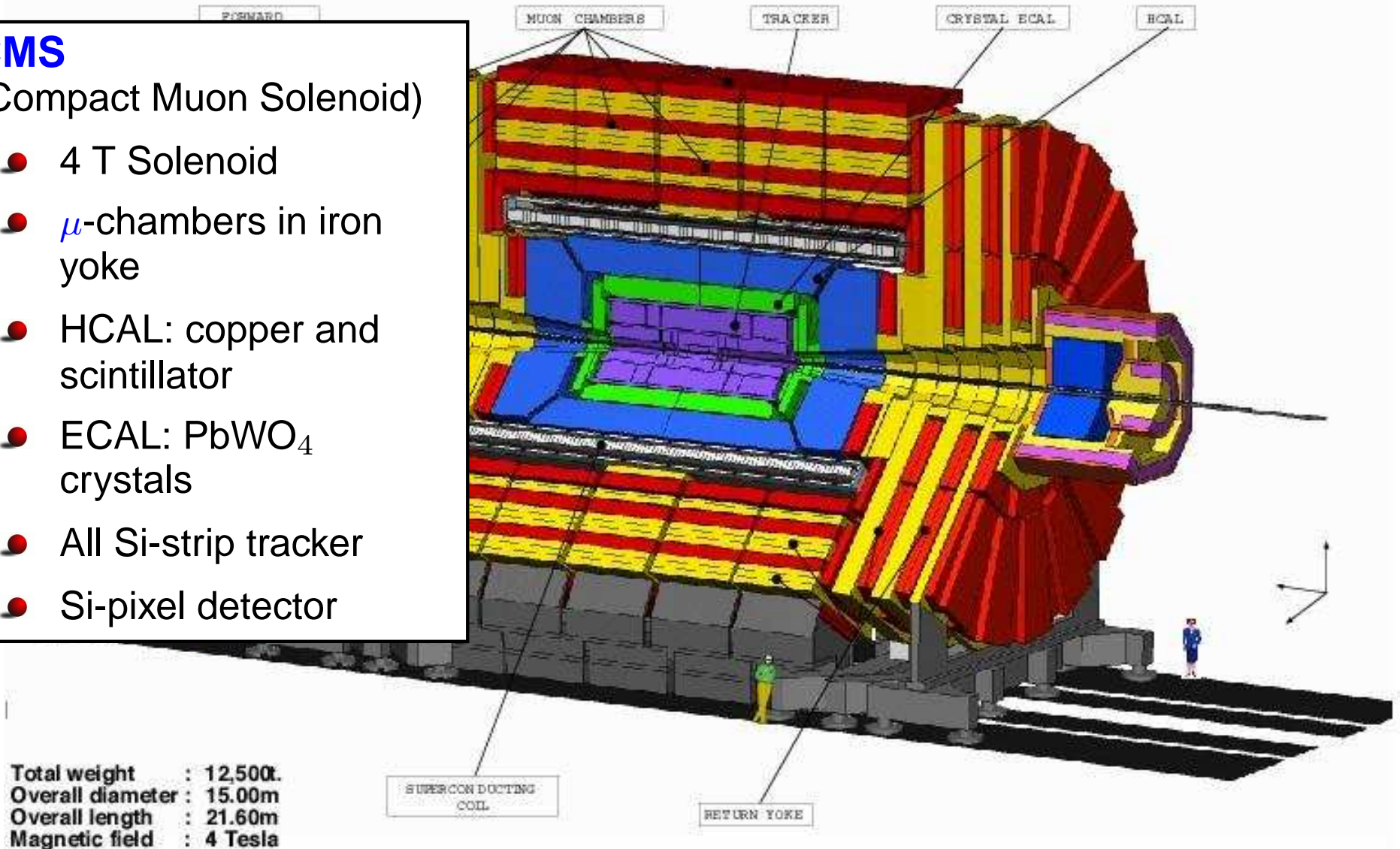
Total weight : 12,500t.
Overall diameter : 15.00m
Overall length : 21.60m
Magnetic field : 4 Tesla

The CMS Detector

CMS

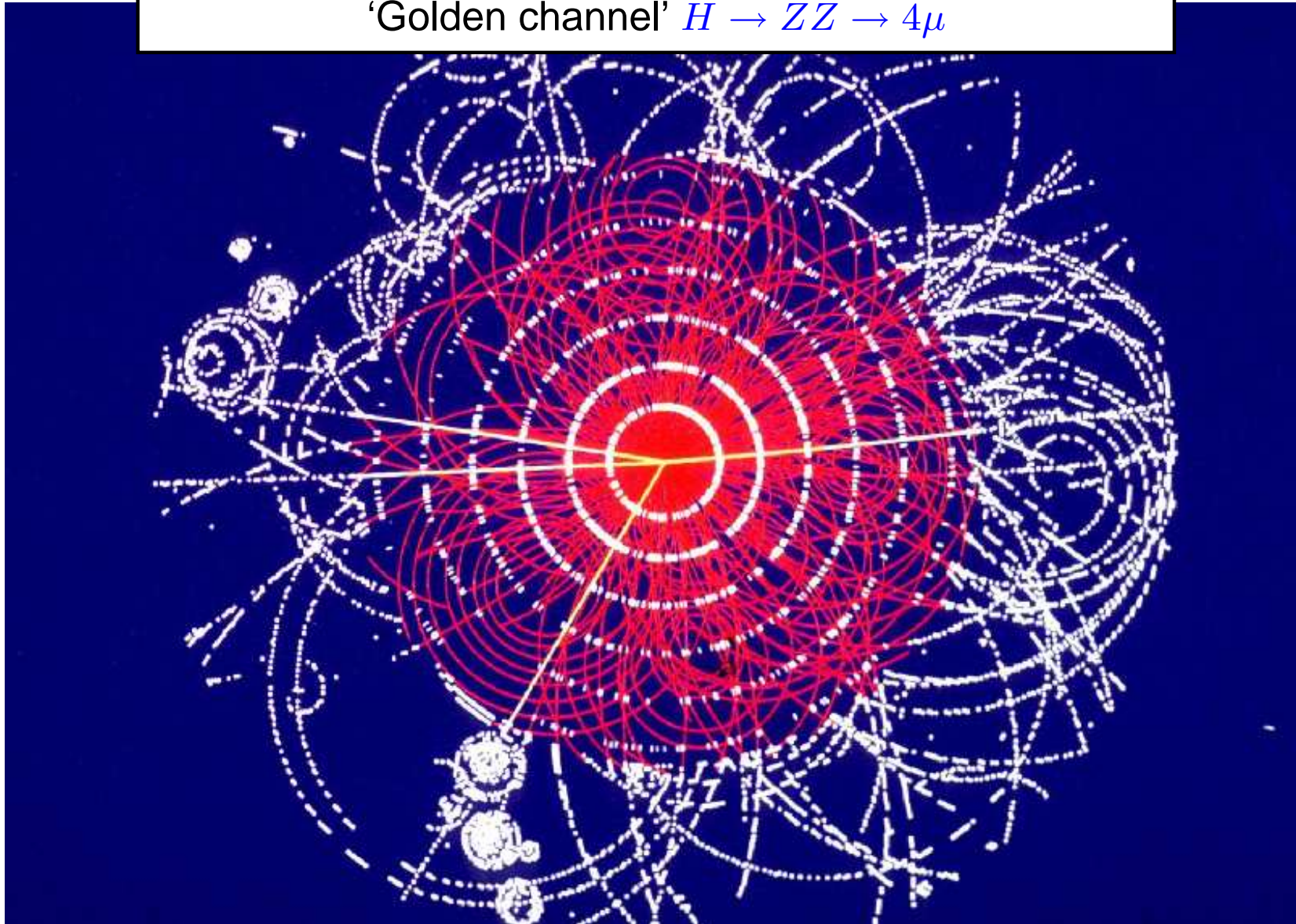
(Compact Muon Solenoid)

- 4 T Solenoid
- μ -chambers in iron yoke
- HCAL: copper and scintillator
- ECAL: PbWO_4 crystals
- All Si-strip tracker
- Si-pixel detector



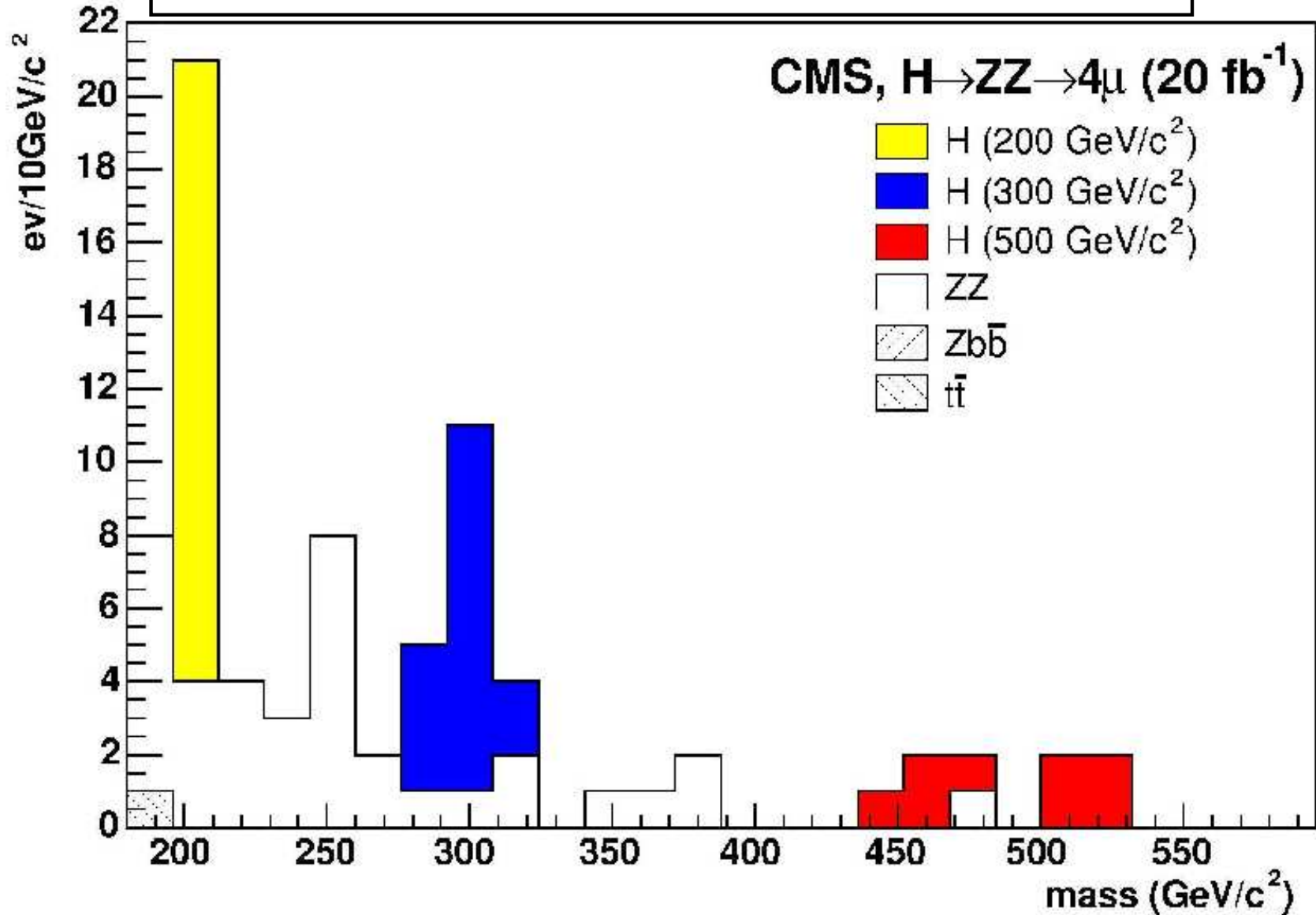
Higgs search

'Golden channel' $H \rightarrow ZZ \rightarrow 4\mu$



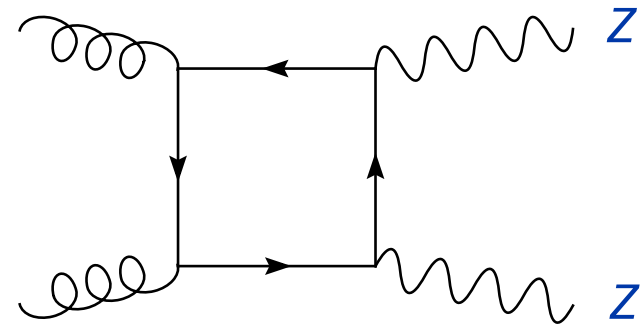
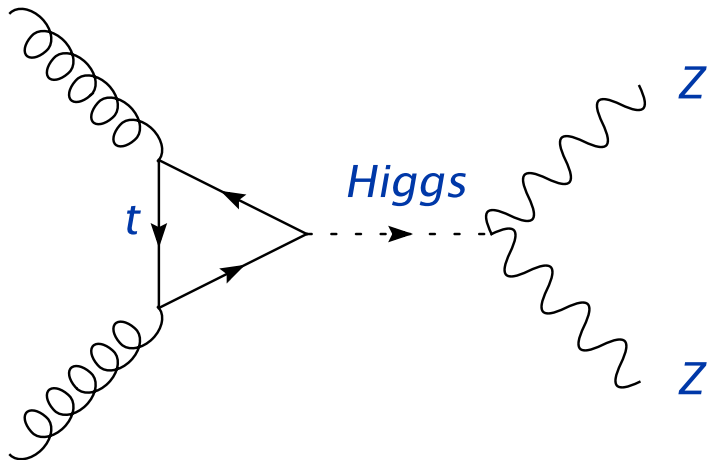
Higgs mass search

Sensitivity study for various Higgs masses



Quantum field theory

- Perturbative predictions for cross sections of scattering processes
 - series expansion in coupling constant



- Example: possible diagrams for $gg \rightarrow ZZ$
 - Higgs signal (left) and QCD background (right)

Perturbation theory in a nut shell

- Feynman diagrams for a given process (tensor integrals, Lorentz structure, ...)
- Feynman integral (l -loops, n -propagators)

$$I(D; \nu_1, \dots, \nu_n) = \int d^D p_1 \dots d^D p_l \frac{1}{(p_1^2)^{\nu_1} \dots (p_n^2)^{\nu_n}}$$

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- Dimensional regularization
 - Bollini, Giambiagi '72; Ashmore '72; Cicuti, Montaldi '72; 't Hooft, Veltman '72
 - all momenta and polarisation vectors in $D = 4 - 2\epsilon$ dimensions
 - manifest gauge invariance

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- all momenta and polarisation vectors in $D = 4 - 2\epsilon$ dimensions
 - manifest gauge invariance
- Classification
 - topology, n -point function (number of loops, legs)
 - scales (number of non-vanishing scalar products of momenta and masses)

- Renormalization

- finite results for physical observables O

- bare quantities O^{bare} as Laurent series in ϵ

- multiplicative renormalization $O^{\text{bare}} = ZO^{\text{ren}}$ ($\overline{\text{MS}}$ -scheme) with constants $Z = \sum_i \alpha^i \frac{1}{\epsilon^i} \gamma^{(i)}$ (schematically)

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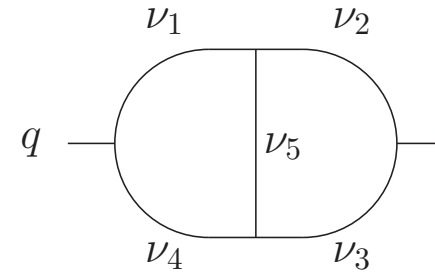
Difficulty

- Calculation of Feynman (loop) integrals

From Feynman integrals to sums

- Example: massless two-loop two-point function

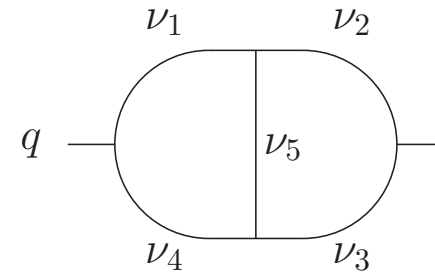
$$T1 = \int d^D p_1 d^D p_2 \frac{1}{(p_1^2)^{\nu_1} (p_2^2)^{\nu_2} (p_3^2)^{\nu_3} (p_4^2)^{\nu_4} (p_5^2)^{\nu_5}}$$



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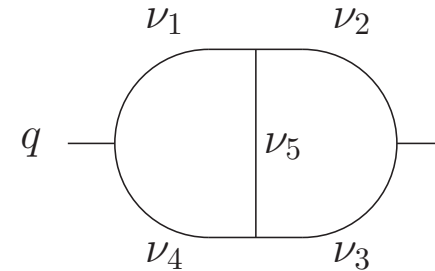
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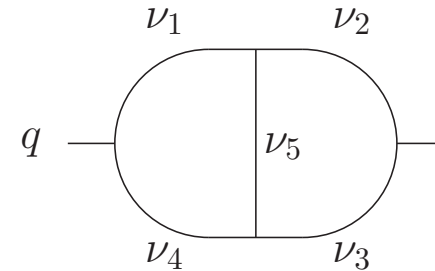
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- History (all powers of propagators $\nu_i = 1 + \epsilon$)
 - ϵ^3, ϵ^4 (ζ_6, ζ_7) Kazakov '84, Kazakov '85
 - ϵ^5 ($\zeta_8, \zeta_{6,2}$) Broadhurst '86
 - ϵ^6 (ζ_9) Barfoot, Broadhurst '88
 - ϵ^7, ϵ^8 ($\zeta_{10}, \zeta_{8,2}, \zeta_{11}, \zeta_{8,2,1}$) Broadhurst, Gracey, Kreimer '96
 - ϵ^9 ($\zeta_{12}, \zeta_{10,2}, \zeta_{8,2,1,1}$) Broadhurst '02

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 - $\epsilon^{10}, \epsilon^{11}, \epsilon^{12}, \epsilon^{13}$ Vermaseren '03 [based on Bierenbaum, Weinzierl '03]

Solution

- All order solution: multiple zeta values are indeed sufficient for massless two-loop two-point function [Bierenbaum, Weinzierl '03](#)
 - Mellin-Barnes representation (sum over residues)

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The two-loop integral can therefore be written as a sum of three terms,

$$\hat{I}^{(2,5)} = T_{(1)} + T_{(2)} + T_{(3)}, \quad (29)$$

where each term is obtained by taking the residues of one Gamma function from the set $\Gamma(-\sigma)$, $\Gamma(-\sigma + m - \varepsilon - \nu_{35})$ and $\Gamma(-\sigma - \tau - m + \varepsilon + \nu_{14})$. Explicitly,

$$T_{(k)} = c \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{2\pi i} \int_{\gamma_2 - i\infty}^{\gamma_2 + i\infty} d\tau \frac{\Gamma(-\tau)\Gamma(-\tau + m - \varepsilon - \nu_{25})\Gamma(\tau + m - \varepsilon - \nu_1)}{\Gamma(-\tau + \nu_1)} H_{(k)}, \quad (30)$$

with

$$H_{(1)} = \frac{\Gamma(-n + m - \varepsilon - \nu_{35})\Gamma(n + m - \varepsilon - \nu_4)}{\Gamma(-n + \nu_4)} \times \frac{\Gamma(-\tau - n - m + \varepsilon + \nu_{14})\Gamma(\tau + n - m + \varepsilon + \nu_{235})\Gamma(\tau + n + \nu_5)}{\Gamma(\tau + n + 2m - 2\varepsilon - \nu_{14})},$$

$$H_{(2)} = \frac{\Gamma(-n - m + \varepsilon + \nu_{35})\Gamma(n + 2m - 2\varepsilon - \nu_{345})}{\Gamma(-n - m + \varepsilon + \nu_{345})}$$

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Symbolic Summation

Polynomial summation

- Examples

$$\sum_{i=0}^{n-1} i = \frac{1}{2}n(n-1)$$

$$\sum_{i=0}^{n-1} i^2 = \frac{1}{6}n(n-1)(2n-1)$$

$$\sum_{i=0}^{n-1} i^3 = \frac{1}{4}n^2(n-1)^2$$

$$\sum_{i=0}^{n-1} i^4 = \frac{1}{30}n(n-1)(2n-1)(3n^2-3n-1)$$

Polynomial summation

- Define rising factorials as $f^{\overline{m}} = f(x)f(x+1)\dots f(x+m-1)$
(also known as Pochhammer symbols $(x)_m$)
- Define falling factorials as $f^{\underline{m}} = f(x)f(x-1)\dots f(x-m+1)$

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- Difference operator acts on falling factorials
 - Differential operator D acts in continuum as $D(x^m) = mx^{m-1}$

$$\Delta(x^m) = mx^{\underline{m-1}}$$

$$\sum_{i=0}^{n-1} i^m = \frac{1}{m+1}n^{\underline{m+1}}$$

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- Conversion of polynomial powers x^m
(decomposition with Stirling numbers of second kind $\left\{ \begin{matrix} m \\ i \end{matrix} \right\}$)

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$$\sum_{i=0}^{n-1} i = \sum_{i=0}^{n-1} i^{\underline{1}} = \frac{1}{2}n^{\underline{2}} = \frac{1}{2}n(n-1)$$

Hypergeometric summation

Definition

- Hypergeometric function ${}_mF_n$

$${}_mF_n \left(\begin{matrix} a_1, \dots, a_m \\ b_1, \dots, b_n \end{matrix} \middle| z \right) = \sum_{i \geq 0} \frac{a_1^{\overline{i}} \dots a_m^{\overline{i}}}{b_1^{\overline{i}} \dots b_n^{\overline{i}}} \frac{z^i}{i!}$$

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Examples

$${}_0F_0 \left(\left| z \right. \right) = \sum_{i \geq 0} \frac{z^i}{i!} = e^z$$

$${}_2F_1 \left(\begin{matrix} a, 1 \\ 1 \end{matrix} \middle| z \right) = \sum_{i \geq 0} a^{\bar{i}} \frac{z^i}{i!} = \frac{1}{(1-z)^a}$$

$${}_2F_1 \left(\begin{matrix} 1, 1 \\ 2 \end{matrix} \middle| z \right) = z \sum_{i \geq 0} \frac{1^{\bar{i}} 1^{\bar{i}}}{2^{\bar{i}}} \frac{z^i}{i!} = -\ln(1-z)$$

Ratios

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- Given a hypergeometric term g , is there hypergeometric term f such that $\Delta f = g$?

$$f_{n+1} - f_n = g_n$$

Gospers algorithm

- Gospers algorithm for indefinite hypergeometric summation determines f_n from a given recursion

$$f_n = f_{n-1} + g_{n-1} = f_{n-2} + g_{n-1} + g_{n-2} = \cdots = f_0 + \sum_{k=0}^{n-1} g_k$$

- Idea: recursive algorithm; telescoping

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Wilf-Zeilberger algorithm

- WZ algorithm
 - definite hypergeometric summation
 - Idea: telescoping

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Definite vs. indefinite summation

- Examples

$$\sum_k \binom{n}{k} = \sum_k \binom{n}{k}$$

$$\sum_{k=0}^n \binom{n}{k} = 2^n$$

Harmonic summation

- Harmonic sums $S_{m_1, \dots, m_k}(n)$

Gonzalez-Arroyo, Lopez, Ynduráin '79; Vermaseren '98; S.M., Uwer, Weinzierl '01

- recursive definition $S_{\pm m_1, \dots, m_k}(n) = \sum_{i=1}^n \frac{(\pm 1)^i}{i^{m_1}} S_{m_2, \dots, m_k}(i)$

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- Particle physics

- dimensional regularization $D = 4 - 2\epsilon$ requires expansion of the Gamma-function around positive integers values ($n \geq 0$)

$$\frac{\Gamma(n + 1 + \epsilon)}{\Gamma(1 + \epsilon)} = \Gamma(n + 1) \exp \left(- \sum_{k=1}^{\infty} \epsilon^k \frac{(-1)^k}{k} S_k(n) \right)$$

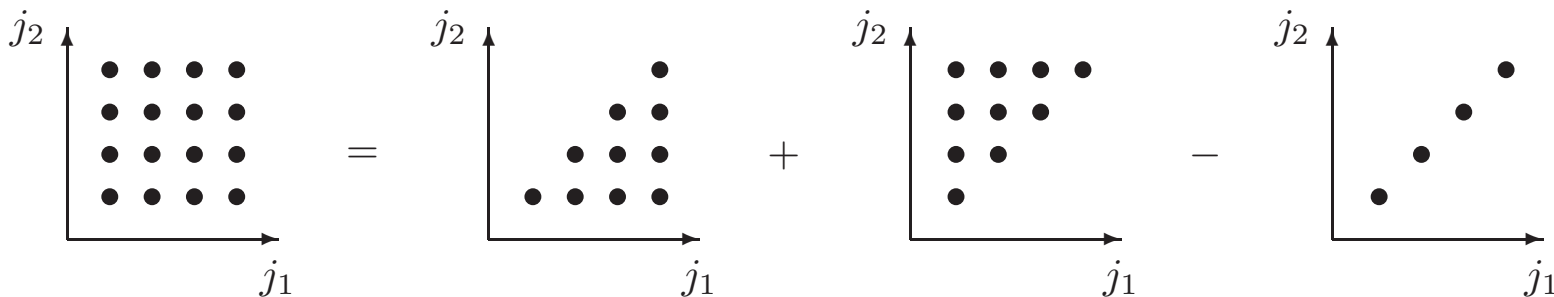
Algorithms for harmonic sums

- Multiplication (Hopf algebra)
 - basic formula (recursion)

$$\begin{aligned}
 S_{m_1, \dots, m_k}(n) \times S_{m'_1, \dots, m'_l}(n) &= \sum_{j_1=1}^n \frac{1}{j_1^{m_1}} S_{m_2, \dots, m_k}(j_1) S_{m'_1, \dots, m'_l}(j_1) \\
 &+ \sum_{j_2=1}^n \frac{1}{j_2^{m'_1}} S_{m_1, \dots, m_k}(j_2) S_{m'_2, \dots, m'_l}(j_2) \\
 &- \sum_{j=1}^n \frac{1}{j^{m_1+m'_1}} S_{m_2, \dots, m_k}(j) S_{m'_2, \dots, m'_l}(j)
 \end{aligned}$$

- Proof uses decomposition

$$\sum_{i=1}^n \sum_{j=1}^n a_{ij} = \sum_{i=1}^n \sum_{j=1}^i a_{ij} + \sum_{j=1}^n \sum_{i=1}^j a_{ij} - \sum_{i=1}^n a_{ii}$$



Algorithms for harmonic sums (cont'd)

- Convolution (sum over $n - j$ and j)

$$\sum_{j=1}^{n-1} \frac{1}{j^{m_1}} S_{m_2, \dots, m_k}(j) \frac{1}{(n-j)^{n_1}} S_{n_2, \dots, n_l}(n-j)$$

- Conjugation

$$- \sum_{j=1}^n \binom{n}{j} (-1)^j \frac{1}{j^{m_1}} S_{m_2, \dots, m_k}(j)$$

- Binomial convolution (sum over **binomial**, $n - j$ and j)

$$- \sum_{j=1}^{n-1} \binom{n}{j} (-1)^j \frac{1}{j^{m_1}} S_{m_2, \dots, m_k}(j) \frac{1}{(n-j)^{n_1}} S_{n_2, \dots, n_l}(n-j)$$

Multiple scales

- Generalized sums $S(n; m_1, \dots, m_k; x_1, \dots, x_k)$

- recursive definition

$$S(n; m_1, \dots, m_k; x_1, \dots, x_k) = \sum_{i=1}^n \frac{x_1^i}{i^{m_1}} S(i; m_2, \dots, m_k; x_2, \dots, x_k)$$

- multiple scales x_1, \dots, x_k
- depth k , weight $w = m_1 + \dots + m_k$

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- Special cases

$$S(\infty; m_1, \dots, m_k; x_1, \dots, x_k) \rightarrow \text{Li}_{m_k, \dots, m_1}(x_k, \dots, x_1)$$

multiple polylogarithms

Goncharov '98; Borwein, Bradley, Broadhurst, Lisonek '99

$$S(\infty; m_1, \dots, m_k; x, 1, \dots, 1) \rightarrow H_{m_1, \dots, m_k}(x)$$

harmonic polylogarithms

Remiddi, Vermaseren '98

$$S(n; m_1, \dots, m_k; 1, \dots, 1) \rightarrow S_{m_1, \dots, m_k}(n)$$

Euler-Zagier sums

Algorithms for nested sums

- Same structures as for harmonic sums, in particular
 - multiplication
$$S(n; m_1, \dots; x_1, \dots) \times S(n; m'_1, \dots; x'_1, \dots)$$
 - convolution
 - conjugation
 - binomial convolution
- Recursive algorithms analogous to harmonic sums solve multiple nested sums

Higher transcendental functions

- Expansion of higher transcendental functions in small parameter
 - expansion parameter ϵ occurs in the rising factorials (Pochhammer symbols)

- Hypergeometric function

$${}_2F_1(a, b; c, x_0) = \sum_{i=0}^{\infty} \frac{a^{\overline{i}} b^{\overline{i}}}{c^{\overline{i}}} \frac{x_0^i}{i!}$$

- First Appell function

$$F_1(a, b_1, b_2; c; x_1, x_2) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{a^{\overline{m_1+m_2}} b_1^{\overline{m_1}} b_2^{\overline{m_2}}}{c^{\overline{m_1+m_2}}} \frac{x_1^{m_1}}{m_1!} \frac{x_2^{m_2}}{m_2!}$$

- Second Appell function

$$F_2(a, b_1, b_2; c_1, c_2; x_1, x_2) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{a^{\overline{m_1+m_2}} b_1^{\overline{m_1}} b_2^{\overline{m_2}}}{c_1^{\overline{m_1}} c_2^{\overline{m_2}}} \frac{x_1^{m_1}}{m_1!} \frac{x_2^{m_2}}{m_2!}$$

Harmonic polylogarithms

- Harmonic polylogarithms $H_{m_1, \dots, m_k}(x)$
Remiddi, Vermaseren '99
 - physical quantities in momentum (x)-space

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- basic functions of lowest weight

$$H_0(x) = \ln x, \quad H_1(x) = -\ln(1-x), \quad H_{-1}(x) = \ln(1+x)$$

- higher functions defined by recursion

$$H_{m_1, \dots, m_w}(x) = \int_0^x dz f_{m_1}(z) H_{m_2, \dots, m_w}(z)$$

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- (Inverse) Mellin transformation

$$\tilde{f}(N) = \int_0^1 dx x^N f(x)$$

- unique mapping $\frac{H_{m_1, \dots, m_w}(x)}{(1 \pm x)} \longleftrightarrow S_{n_1, \dots, n_{w+1}}(N)$

Beyond

Binomial and inverse binomial sums

- Feynman integrals with massive propagators lead to structures

$$\chi = \frac{1 - \sqrt{1 - 4x}}{1 + \sqrt{1 - 4x}}, \quad x = \frac{\chi}{(1 + \chi)^2}, \quad \sqrt{1 - 4x} = \frac{1 - \chi}{1 + \chi}.$$

- Mellin-Barnes representations lead to new classes of sums
 - two-loop two-point functions [Davydychev, Kalmykov '02](#)

$$\sum_{k=1}^{\infty} \binom{2k}{k} \frac{x^k}{k^a}$$

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- Roots of unity $\exp\left(\frac{2\pi ip}{q}\right)$ from expansions of transcendental functions around rational values [Weinzierl '04](#)

Applications in particle physics

Feynman integrals from deep-inelastic scattering

- Two-scale problem
 - virtuality of the exchanged gauge boson Q^2
 - scalar product of boson's and nucleon's momenta, $2P \cdot Q$
 - Bjorken's variable $x = Q^2 / (2P \cdot Q)$ with $0 \leq x \leq 1$ (dimensionless)

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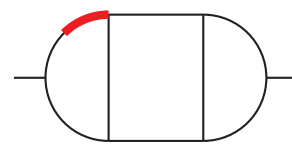
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- Scalar diagram with external momenta P and Q (four-point function with underlying ladder topology)

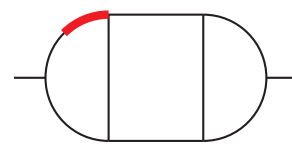

$$= \int \prod_n^3 d^D p_n \frac{1}{(P - p_1)^2} \frac{1}{p_1^2 \cdots p_8^2}$$

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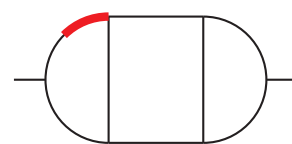
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$$= \int \prod_n^3 d^D p_n \frac{1}{(P - p_1)^2} \frac{1}{p_1^2 \cdots p_8^2}$$

- N -th moment: coefficient of $(2P \cdot Q)^N$



$$= \frac{(2P \cdot Q)^N}{(Q^2)^{N+\alpha}} C_N$$

Difference equations

- Single-step difference equation in N
 - extremely simple example

$$\begin{array}{c} \text{Diagram 1} \end{array} = -\frac{N+3+3\epsilon}{N+2} \frac{2p \cdot q}{q^2} \begin{array}{c} \text{Diagram 2} \end{array} + \frac{2}{N+2} \begin{array}{c} \text{Diagram 3} \end{array}$$

The diagrams are Feynman diagrams for a two-loop process. Each diagram consists of a circle with two vertical lines inside. The top and bottom arcs are labeled with '1'. Diagram 1 has a red arc connecting the two vertical lines at the top. Diagram 2 is identical to Diagram 1. Diagram 3 has a red arc connecting the two vertical lines at the top, and a '2' is placed above the right vertical line.

Difference equations

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$$\text{Diagram} = -\frac{N+3+3\epsilon}{N+2} \frac{2p \cdot q}{q^2} \text{Diagram} + \frac{2}{N+2} \text{Diagram}$$

- Formal equation

$$\mathbf{I}(\mathbf{N}) = -\frac{N+3+3\epsilon}{N+2} \mathbf{I}(\mathbf{N} - \mathbf{1}) + \frac{2}{N+2} \mathbf{G}(\mathbf{N})$$

Difference equations

- Single-step difference equation in N
 - extremely simple example

$$\text{Diagram} = -\frac{N+3+3\epsilon}{N+2} \frac{2p \cdot q}{q^2} \text{Diagram} + \frac{2}{N+2} \text{Diagram}$$

- Formal equation, formal solution

$$\mathbf{I}(N) = (-1)^N \frac{\prod_{j=1}^N (j+3+3\epsilon)}{\prod_{j=1}^N (j+2)} \mathbf{I}(\mathbf{0}) + (-1)^N \sum_{i=1}^N (-1)^j \frac{\prod_{j=i+1}^N (j+3+3\epsilon)}{\prod_{j=i}^N (j+2)} \mathbf{G}(\mathbf{i})$$

Difference equations

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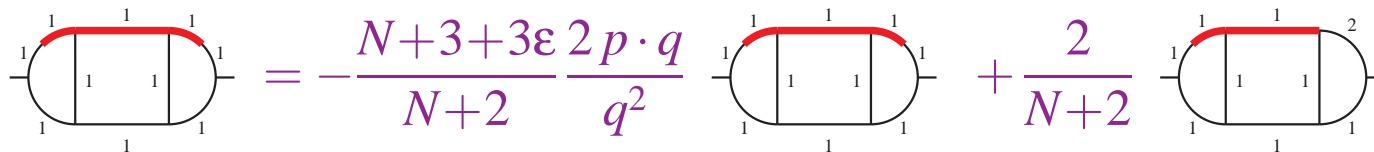
$$\mathbf{I(N)} = (-1)^N \frac{\prod_{j=1}^N (j+3+3\epsilon)}{\prod_{j=1}^N (j+2)} \mathbf{I(0)} + (-1)^N \sum_{i=1}^N (-1)^i \frac{\prod_{j=i+1}^N (j+3+3\epsilon)}{\prod_{j=i}^N (j+2)} \mathbf{G(i)}$$

$$\mathbf{I(0)} = -\frac{2}{3} \frac{1}{\epsilon^2} + \frac{23}{3} \frac{1}{\epsilon} - 42$$

$$\mathbf{G(i)} = \frac{(-1)^i}{\epsilon^2} \frac{2}{3} \left(\frac{S_1(i+2)}{i+2} - \frac{S_{1,2}(i)}{2} - \frac{S_2(i+1)}{2(i+1)} - S_2(i) - \frac{1}{(i+1)^2} - \frac{1}{(i+2)^2} \right) + \dots$$

Difference equations

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 - extremely simple example



$$\text{Diagram} = -\frac{N+3+3\epsilon}{N+2} \frac{2p \cdot q}{q^2} \text{Diagram} + \frac{2}{N+2} \text{Diagram}$$

- Formal equation, formal solution, input to solution

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- **Upshot**

- automatic build-up of **nested sums**
- efficient implementation in FORM

I(N)=

$$\begin{aligned} & \text{sign}(N) * \text{ep}^{-2} * (4/3 * S(R(1), 1 + N) * \text{den}(1 + N) + 8/3 * S(R(1), 1 + N) * \text{den}(1 + N)^2 + 4/3 * S(R(1), 2 + N) * \text{den}(2 + N) + 4/3 * S(R(1), 2 + N) * \text{den}(2 + N)^2 + 4/3 * S(R(1), N) + 2/3 * S(R(1, 2), N) + 2/3 * S(R(2), 1 + N) * \text{den}(1 + N) + 2/3 * S(R(2), 2 + N) * \text{den}(2 + N) - 2 * S(R(2), N) - 4/3 * S(R(2), N) * N + 4 * S(R(2, 1), N) + 4/3 * S(R(2, 1), N) * N - 6 * S(R(3), N) - 2 * S(R(3), N) * N - 8/3 * \text{den}(1 + N)^2 - 4 * \text{den}(1 + N)^3 - 4/3 * \text{den}(2 + N)^2 - 2 * \text{den}(2 + N)^3) \\ & + \text{sign}(N) * \text{ep}^{-1} * (- 16 * S(R(1), 1 + N) * \text{den}(1 + N) - 88/3 * S(R(1), 1 + N) * \text{den}(1 + N)^2 - 20/3 * S(R(1), 1 + N) * \text{den}(1 + N)^3 - 16 * S(R(1), 2 + N) * \text{den}(2 + N) - 44/3 * S(R(1), 2 + N) * \text{den}(2 + N)^2 - 10/3 * S(R(1), 2 + N) * \text{den}(2 + N)^3 - 20 * S(R(1), N) + 8/3 * S(R(1, 1), 1 + N) * \text{den}(1 + N) + 8/3 * S(R(1, 1), 1 + N) * \text{den}(1 + N)^2 + 8/3 * S(R(1, 1), 2 + N) * \text{den}(2 + N) + 8/3 * S(R(1, 1), N) + 10/3 * S(R(1, 1, 2), N) + 10/3 * S(R(1, 2), 1 + N) * \text{den}(1 + N) + 10/3 * S(R(1, 2), 2 + N) * \text{den}(2 + N) - 16 * S(R(1, 2), N) - 4 * S(R(1, 2), N) * N + 14 * S(R(1, 2, 1), N) + 4 * S(R(1, 2, 1), N) * N - 24 * S(R(1, 3), N) - 6 * S(R(1, 3), N) * N - 58/3 * S(R(2), 1 + N) * \text{den}(1 + N) - 40/3 * S(R(2), 1 + N) * \text{den}(1 + N)^2 - 46/3 * S(R(2), 2 + N) * \text{den}(2 + N) - 6 * S(R(2), 2 + N) * \text{den}(2 + N)^2 + 56/3 * S(R(2), N) + 20 * S(R(2), N) * N + 10 * S(R(2, 1), 1 + N) * \text{den}(1 + N) + 6 * S(R(2, 1), 2 + N) * \text{den}(2 + N) - 134/3 * S(R(2, 1), N) - 56/3 * S(R(2, 1), N) * N + 16/3 * S(R(2, 1, 1), N) + 8/3 * S(R(2, 1, 1), N) * N - 62/3 * S(R(2, 2), N) - 22/3 * S(R(2, 2), N) * N - 18 * S(R(3), 1 + N) * \text{den}(1 + N) - 12 * S(R(3), 2 + N) * \text{den}(2 + N) + 76 * S(R(3), N) + 100/3 * S(R(3), N) * N - 10 * S(R(3, 1), N) - 10/3 * S(R(3, 1), N) * N + 36 * S(R(4), N) + 12 * S(R(4), N) * N + 32 * \text{den}(1 + N)^2 + 164/3 * \text{den}(1 + N)^3 + 24 * \text{den}(1 + N)^4 + 16 * \text{den}(2 + N)^2 + 82/3 * \text{den}(2 + N)^3 + 12 * \text{den}(2 + N)^4) \\ & + \text{sign}(N) * (100 * S(R(1), 1 + N) * \text{den}(1 + N) + 168 * S(R(1), 1 + N) * \text{den}(1 + N)^2 + 268/3 * S(R(1), 1 + N) * \text{den}(1 + N)^3 - 16/3 * S(R(1), 1 + N) * \text{den}(1 + N)^4 + 100 * S(R(1), 2 + N) * \text{den}(2 + N) + 84 * S(R(1), 2 + N) * \text{den}(2 + N)^2 + 134/3 * S(R(1), 2 + N) * \text{den}(2 + N)^3 - 8/3 * S(R(1), 2 + N) * \text{den}(2 + N)^4 + 160 * S(R(1), N) - 32 * S(R(1, 1), 1 + N) * \text{den}(1 + N) - 80/3 * S(R(1, 1), 1 + N) * \text{den}(1 + N)^2 - 20/3 * S(R(1, 1), 1 + N) * \text{den}(1 + N)^3 - 32 * S(R(1, 1), 2 + N) * \text{den}(2 + N) - 4/3 * S(R(1, 1), 2 + N) * \text{den}(2 + N)^2 - 10/3 * S(R(1, 1), 2 + N) * \text{den}(2 + N)^3 - 40 * S(R(1, 1), N) + 4/3 * S(R(1, 1, 1), 1 + N) * \text{den}(1 + N) - 40/3 * S(R(1, 1, 1), 1 + N) * \text{den}(1 + N)^2 + 4/3 * S(R(1, 1, 1), 2 + N) * \text{den}(2 + N) - 44/3 * S(R(1, 1, 1), 2 + N) * \text{den}(2 + N)^2 + 4/3 * S(R(1, 1, 1), N) + 38/3 * S(R(1, 1, 1, 2), N) + 38/3 * S(R(1, 1, 2), 1 + N) * \text{den}(1 + N) + 38/3 * S(R(1, 1, 2), 2 + N) * \text{den}(2 + N) - 68 * S(R(1, 1, 2), N) - 12 * S(R(1, 1, 2), N) * N + 42 * S(R(1, 1, 2, 1), N) + 12 * S(R(1, 1, 2, 1), N) * N - 76 * S(R(1, 1, 3), N) - 18 * S(R(1, 1, 3), N) * N - 170/3 * S(R(1, 2), 1 + N) * \text{den}(1 + N) + 40/3 * S(R(1, 2), 1 + N) * \text{den}(1 + N)^2 - 134/3 * S(R(1, 2), 2 + N) * \text{den}(2 + N) + 14 * S(R(1, 2), 2 + N) * \text{den}(2 + N)^2 + 430/3 * S(R(1, 2), N) + 60 * S(R(1, 2), N) * N + 30 * S(R(1, 2, 1), 1 + N) * \text{den}(1 + N) + 18 * S(R(1, 2, 1), 2 + N) * \text{den}(2 + N) - 452/3 * S(R(1, 2, 1), N) - 56 * S(R(1, 2, 1), N) * N + 74/3 * S(R(1, 2, 1, 1), N) + 8 * S(R(1, 2, 1, 1), N) * N - 248/3 * S(R(1, 2, 2), N) - 22 * S(R(1, 2, 2), N) * N - 58 * S(R(1, 3), 1 + N) * \text{den}(1 + N) - 40 * S(R(1, 3), 2 + N) * \text{den}(2 + N) + 886/ \end{aligned}$$

$$\begin{aligned} & 3 * S(R(1, 3), N) + 100 * S(R(1, 3), N) * N - 116/3 * S(R(1, 3, 1), N) - 10 * S(R(1, 3, 1), N) * N + 410/3 * S(R(1, 4), N) + 36 * S(R(1, 4), N) * N + 186 * S(R(2), 1 + N) * \text{den}(1 + N) + 448/3 * S(R(2), 1 + N) * \text{den}(1 + N)^2 + 160/3 * S(R(2), 1 + N) * \text{den}(1 + N)^3 + 138 * S(R(2), 2 + N) * \text{den}(2 + N) + 206/3 * S(R(2), 2 + N) * \text{den}(2 + N)^2 + 80/3 * S(R(2), 2 + N) * \text{den}(2 + N)^3 - 70 * S(R(2), N) - 160 * S(R(2), N) * N - 338/3 * S(R(2, 1), 1 + N) * \text{den}(1 + N) - 64/3 * S(R(2, 1), 1 + N) * \text{den}(1 + N)^2 - 206/3 * S(R(2, 1), 2 + N) * \text{den}(2 + N) - 10/3 * S(R(2, 1), 2 + N) * \text{den}(2 + N)^2 + 760/3 * S(R(2, 1), N) + 140 * S(R(2, 1), N) * N + 50/3 * S(R(2, 1, 1), 1 + N) * \text{den}(1 + N) + 26/3 * S(R(2, 1, 1), 2 + N) * \text{den}(2 + N) - 170/3 * S(R(2, 1, 1), N) - 100/3 * S(R(2, 1, 1), N) * N - 12 * S(R(2, 1, 1, 1), N) + 4/3 * S(R(2, 1, 1, 1), N) * N + 38/3 * S(R(2, 1, 2), N) - 2/3 * S(R(2, 1, 2), N) * N - 182/3 * S(R(2, 2), 1 + N) * \text{den}(1 + N) - 116/3 * S(R(2, 2), 2 + N) * \text{den}(2 + N) + 676/3 * S(R(2, 2), N) + 308/3 * S(R(2, 2), N) * N - 118/3 * S(R(2, 2, 1), N) - 18 * S(R(2, 2, 1), N) * N + 296/3 * S(R(2, 3), N) + 36 * S(R(2, 3), N) * N + 694/3 * S(R(3), 1 + N) * \text{den}(1 + N) + 188/3 * S(R(3), 1 + N) * \text{den}(1 + N)^2 + 448/3 * S(R(3), 2 + N) * \text{den}(2 + N) + 80/3 * S(R(3), 2 + N) * \text{den}(2 + N)^2 - 1454/3 * S(R(3), N) - 290 * S(R(3), N) * N - 86/3 * S(R(3, 1), 1 + N) * \text{den}(1 + N) - 56/3 * S(R(3, 1), 2 + N) * \text{den}(2 + N) + 440/3 * S(R(3, 1), N) + 164/3 * S(R(3, 1), N) * N - 10 * S(R(3, 1, 1), N) - 10/3 * S(R(3, 1, 1), N) * N + 80 * S(R(3, 2), N) + 80/3 * S(R(3, 2), N) * N + 302/3 * S(R(4), 1 + N) * \text{den}(1 + N) + 194/3 * S(R(4), 2 + N) * \text{den}(2 + N) - 434 * S(R(4), N) - 556/3 * S(R(4), N) * N - 8 * S(R(4, 1), N) - 8/3 * S(R(4, 1), N) * N - 150 * S(R(5), N) - 50 * S(R(5), N) * N - 200 * \text{den}(1 + N)^2 - 380 * \text{den}(1 + N)^3 - 896/3 * \text{den}(1 + N)^4 - 100 * \text{den}(1 + N)^5 - 100 * \text{den}(2 + N)^2 - 190 * \text{den}(2 + N)^3 - 448/3 * \text{den}(2 + N)^4 - 50 * \text{den}(2 + N)^5); \end{aligned}$$

$I(N) =$

$$\begin{aligned} & \text{sign}(N) * \text{ep}^{-2} * (4/3 * S(R(1), 1 + N) * \text{den}(1 + N) + 8/3 * S(R(1), 1 + N) * \text{den}(1 + N)^2 + 4/3 * S(R(1), 2 + N) * \text{den}(2 + N) + 4/3 * S(R(1), 2 + N) * \text{den}(2 + N)^2 + 4/3 * S(R(1), N) + 2/3 * S(R(1, 2), N) + 2/3 * S(R(2), 1 + N) * \text{den}(1 + N) + 2/3 * S(R(2), 2 + N) * \text{den}(2 + N) - 2 * S(R(2), N) - 4/3 * S(R(2), N) * N + 4 * S(R(2, 1), N) + 4/3 * S(R(2, 1), N) * N - 6 * S(R(3), N) - 2 * S(R(3), N) * N - 8/3 * \text{den}(1 + N)^2 - 4 * \text{den}(1 + N)^3 - 4/3 * \text{den}(2 + N)^2 - 2 * \text{den}(2 + N)^3) \\ & + \text{sign}(N) * \text{ep}^{-1} * (- 16 * S(R(1), 1 + N) * \text{den}(1 + N) - 88/3 * S(R(1), 1 + N) * \text{den}(1 + N)^2 - 20/3 * S(R(1), 1 + N) * \text{den}(1 + N)^3 - 16 * S(R(1), 2 + N) * \text{den}(2 + N) - 44/3 * S(R(1), 2 + N) * \text{den}(2 + N)^2 - 10/3 * S(R(1), 2 + N) * \text{den}(2 + N)^3 - 20 * S(R(1), N) + 8/3 * S(R(1, 1), 1 + N) * \text{den}(1 + N) + 8/3 * S(R(1, 1), 1 + N) * \text{den}(1 + N)^2 + 8/3 * S(R(1, 1), 2 + N) * \text{den}(2 + N) + 8/3 * S(R(1, 1), N) + 10/3 * S(R(1, 1, 2), N) + 10/3 * S(R(1, 2), 1 + N) * \text{den}(1 + N) + 10/3 * S(R(1, 2), 2 + N) * \text{den}(2 + N) - 16 * S(R(1, 2), N) - 4 * S(R(1, 2), N) * N + 14 * S(R(1, 2, 1), N) + 4 * S(R(1, 2, 1), N) * N - 24 * S(R(1, 3), N) - 6 * S(R(1, 3), N) * N - 58/3 * S(R(2), 1 + N) * \text{den}(1 + N) - 40/3 * S(R(2), 1 + N) * \text{den}(1 + N)^2 - 46/3 * S(R(2), 2 + N) * \text{den}(2 + N) - 6 * S(R(2), 2 + N) * \text{den}(2 + N)^2 + 56/3 * S(R(2), N) + 20 * S(R(2), N) * N + 10 * S(R(2, 1), 1 + N) * \text{den}(1 + N) + 6 * S(R(2, 1), 2 + N) * \text{den}(2 + N) - 134/3 * S(R(2, 1), N) - 56/3 * S(R(2, 1), N) * N + 16/3 * S(R(2, 1, 1), N) + 8/3 * S(R(2, 1, 1), N) * N - 62/3 * S(R(2, 2), N) - 22/3 * S(R(2, 2), N) * N - 18 * S(R(3), 1 + N) * \text{den}(1 + N) - 12 * S(R(3), 2 + N) * \text{den}(2 + N) + 76 * S(R(3), N) + 100/3 * S(R(3), N) * N - 10 * S(R(3, 1), N) - 10/3 * S(R(3, 1), N) * N + 36 * S(R(4), N) + 12 * S(R(4), N) * N + 32 * \text{den}(1 + N)^2 + 164/3 * \text{den}(1 + N)^3 + 24 * \text{den}(1 + N)^4 + 16 * \text{den}(2 + N)^2 + 82/3 * \text{den}(2 + N)^3 + 12 * \text{den}(2 + N)^4) \\ & + \text{sign}(N) * (100 * S(R(1), 1 + N) * \text{den}(1 + N) + 168 * S(R(1), 1 + N) * \text{den}(1 + N)^2 + 268/3 * S(R(1), 1 + N) * \text{den}(1 + N)^3 - 16/3 * S(R(1), 1 + N) * \text{den}(1 + N)^4 + 100 * S(R(1), 2 + N) * \text{den}(2 + N) + 84 * S(R(1), 2 + N) * \text{den}(2 + N)^2 + 134/3 * S(R(1), 2 + N) * \text{den}(2 + N)^3 - 8/3 * S(R(1), 2 + N) * \text{den}(2 + N)^4 + 160 * S(R(1), N) - 32 * S(R(1, 1), 1 + N) * \text{den}(1 + N) - 80/3 * S(R(1, 1), 1 + N) * \text{den}(1 + N)^2 - 20/3 * S(R(1, 1), 1 + N) * \text{den}(1 + N)^3 - 32 * S(R(1, 1), 2 + N) * \text{den}(2 + N) - 4/3 * S(R(1, 1), 2 + N) * \text{den}(2 + N)^2 - 10/3 * S(R(1, 1), 2 + N) * \text{den}(2 + N)^3 - 40 * S(R(1, 1), N) + 4/3 * S(R(1, 1, 1), 1 + N) * \text{den}(1 + N) - 40/3 * S(R(1, 1, 1), 1 + N) * \text{den}(1 + N)^2 + 4/3 * S(R(1, 1, 1), 2 + N) * \text{den}(2 + N) - 44/3 * S(R(1, 1, 1), 2 + N) * \text{den}(2 + N)^2 + 4/3 * S(R(1, 1, 1), N) + 38/3 * S(R(1, 1, 1, 2), N) + 38/3 * S(R(1, 1, 2), 1 + N) * \text{den}(1 + N) + 38/3 * S(R(1, 1, 2), 2 + N) * \text{den}(2 + N) - 68 * S(R(1, 1, 2), N) - 12 * S(R(1, 2), N) * N + 42 * S(R(1, 1, 2, 1), N) + 12 * S(R(1, 1, 2, 1), N) * N - 76 * S(R(1, 1, 3), N) - 18 * S(R(1, 1, 3), N) * N - 170/3 * S(R(1, 2), 1 + N) * \text{den}(1 + N) + 40/3 * S(R(1, 2), 1 + N) * \text{den}(1 + N)^2 - 134/3 * S(R(1, 2), 2 + N) * \text{den}(2 + N) + 14 * S(R(1, 2), 2 + N) * \text{den}(2 + N)^2 + 430/3 * S(R(1, 2), N) + 60 * S(R(1, 2), N) * N + 30 * S(R(1, 2, 1), 1 + N) * \text{den}(1 + N) + 18 * S(R(1, 2, 1), 2 + N) * \text{den}(2 + N) - 452/3 * S(R(1, 2, 1), N) - 56 * S(R(1, 2, 1), N) * N + 74/3 * S(R(1, 2, 1, 1), N) + 8 * S(R(1, 2, 1, 1), N) * N - 248/3 * S(R(1, 2, 2), N) - 22 * S(R(1, 2, 2), N) * N - 58 * S(R(1, 3), 1 + N) * \text{den}(1 + N) - 40 * S(R(1, 3), 2 + N) * \text{den}(2 + N) + 886/ \end{aligned}$$

$$\begin{aligned} & 3 * S(R(1, 3), N) + 100 * S(R(1, 3), N) * N - 116/3 * S(R(1, 3, 1), N) - 10 * S(R(1, 3, 1), N) * N + 410/3 * S(R(1, 4), N) + 36 * S(R(1, 4), N) * N + 186 * S(R(2), 1 + N) * \text{den}(1 + N) + 448/3 * S(R(2), 1 + N) * \text{den}(1 + N)^2 + 160/3 * S(R(2), 1 + N) * \text{den}(1 + N)^3 + 138 * S(R(2), 2 + N) * \text{den}(2 + N) + 206/3 * S(R(2), 2 + N) * \text{den}(2 + N)^2 + 80/3 * S(R(2), 2 + N) * \text{den}(2 + N)^3 - 70 * S(R(2), N) - 160 * S(R(2), N) * N - 338/3 * S(R(2, 1), 1 + N) * \text{den}(1 + N) - 64/3 * S(R(2, 1), 1 + N) * \text{den}(1 + N)^2 - 206/3 * S(R(2, 1), 2 + N) * \text{den}(2 + N) - 10/3 * S(R(2, 1), 2 + N) * \text{den}(2 + N)^2 + 760/3 * S(R(2, 1), N) + 140 * S(R(2, 1), N) * N + 50/3 * S(R(2, 1, 1), 1 + N) * \text{den}(1 + N) + 26/3 * S(R(2, 1, 1), 2 + N) * \text{den}(2 + N) - 170/3 * S(R(2, 1, 1), N) - 100/3 * S(R(2, 1, 1), N) * N - 12 * S(R(2, 1, 1, 1), N) + 4/3 * S(R(2, 1, 1, 1), N) * N + 38/3 * S(R(2, 1, 2), N) - 2/3 * S(R(2, 1, 2), N) * N - 182/3 * S(R(2, 2), 1 + N) * \text{den}(1 + N) - 116/3 * S(R(2, 2), 2 + N) * \text{den}(2 + N) + 676/3 * S(R(2, 2), N) + 308/3 * S(R(2, 2), N) * N - 118/3 * S(R(2, 2, 1), N) - 18 * S(R(2, 2, 1), N) * N + 296/3 * S(R(2, 3), N) + 36 * S(R(2, 3), N) * N + 694/3 * S(R(3), 1 + N) * \text{den}(1 + N) + 188/3 * S(R(3), 1 + N) * \text{den}(1 + N)^2 + 448/3 * S(R(3), 2 + N) * \text{den}(2 + N) + 80/3 * S(R(3), 2 + N) * \text{den}(2 + N)^2 - 1454/3 * S(R(3), N) - 290 * S(R(3), N) * N - 86/3 * S(R(3, 1), 1 + N) * \text{den}(1 + N) - 56/3 * S(R(3, 1), 2 + N) * \text{den}(2 + N) + 440/3 * S(R(3, 1), N) + 164/3 * S(R(3, 1), N) * N - 10 * S(R(3, 1, 1), N) - 10/3 * S(R(3, 1, 1), N) * N + 80 * S(R(3, 2), N) + 80/3 * S(R(3, 2), N) * N + 302/3 * S(R(4), 1 + N) * \text{den}(1 + N) + 194/3 * S(R(4), 2 + N) * \text{den}(2 + N) - 434 * S(R(4), N) - 556/3 * S(R(4), N) * N - 8 * S(R(4, 1), N) - 8/3 * S(R(4, 1), N) * N - 150 * S(R(5), N) - 50 * S(R(5), N) * N - 200 * \text{den}(1 + N)^2 - 380 * \text{den}(1 + N)^3 - 896/3 * \text{den}(1 + N)^4 - 100 * \text{den}(1 + N)^5 - 100 * \text{den}(2 + N)^2 - 190 * \text{den}(2 + N)^3 - 448/3 * \text{den}(2 + N)^4 - 50 * \text{den}(2 + N)^5); \end{aligned}$$

Result for $I(N)$ in the G-scheme

Splitting functions in a nut shell

- Calculate anomalous dimensions (Mellin moments of splitting functions) \rightarrow divergence of Feynman diagrams in dimensional regularization $D = 4 - 2\epsilon$

$$\gamma_{ij}^{(n)}(N) = - \int_0^1 dx x^{N-1} P_{ij}^{(n)}(x)$$

Splitting functions in a nut shell

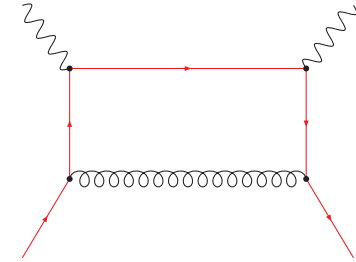
- Calculate anomalous dimensions (Mellin moments of splitting functions) \rightarrow divergence of Feynman diagrams in dimensional regularization $D = 4 - 2\epsilon$

$$\gamma_{ij}^{(n)}(N) = - \int_0^1 dx x^{N-1} P_{ij}^{(n)}(x)$$

- **One-loop** Feynman diagrams

\rightarrow in total 18 for $\gamma_{ij}^{(0)} / P_{ij}^{(0)}$

(pencil + paper)



Splitting functions in a nut shell

- Calculate anomalous dimensions (Mellin moments of splitting functions) \rightarrow divergence of Feynman diagrams in dimensional regularization $D = 4 - 2\epsilon$

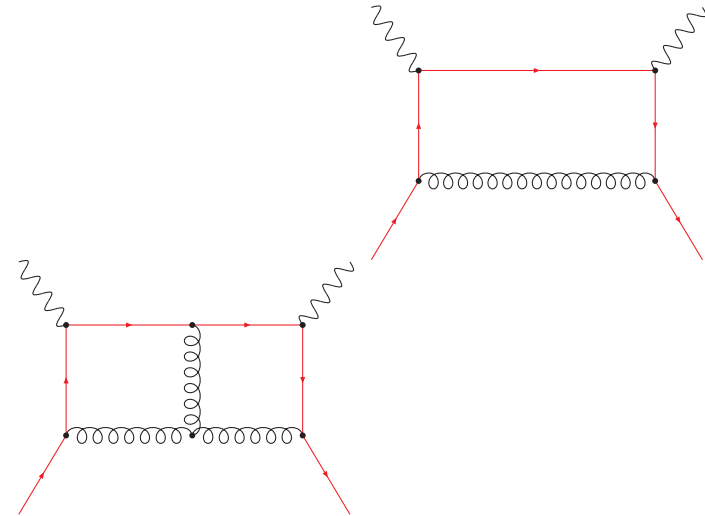
$$\gamma_{ij}^{(n)}(N) = - \int_0^1 dx x^{N-1} P_{ij}^{(n)}(x)$$

- **One-loop** Feynman diagrams

\rightarrow in total 18 for $\gamma_{ij}^{(0)} / P_{ij}^{(0)}$
(pencil + paper)

- **Two-loop** Feynman diagrams

\rightarrow in total 350 for $\gamma_{ij}^{(1)} / P_{ij}^{(1)}$
(simple computer algebra)



Splitting functions in a nut shell

- Calculate anomalous dimensions (Mellin moments of splitting functions) \rightarrow divergence of Feynman diagrams in dimensional regularization $D = 4 - 2\epsilon$

$$\gamma_{ij}^{(n)}(N) = - \int_0^1 dx x^{N-1} P_{ij}^{(n)}(x)$$

- **One-loop** Feynman diagrams

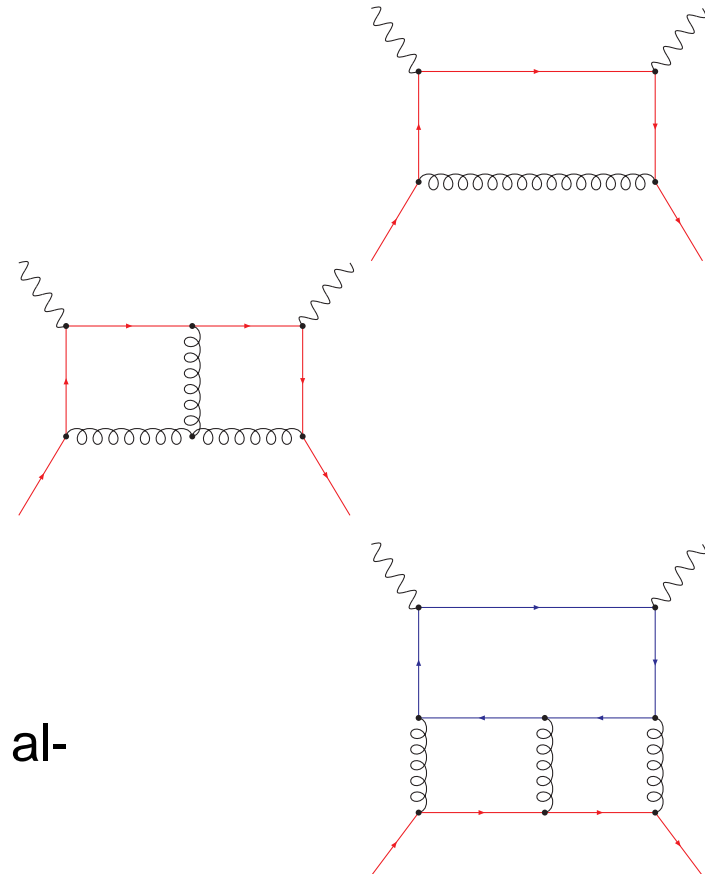
\rightarrow in total 18 for $\gamma_{ij}^{(0)} / P_{ij}^{(0)}$
(pencil + paper)

- **Two-loop** Feynman diagrams

\rightarrow in total 350 for $\gamma_{ij}^{(1)} / P_{ij}^{(1)}$
(simple computer algebra)

- **Three-loop** Feynman diagrams

\rightarrow in total 9607 for $\gamma_{ij}^{(2)} / P_{ij}^{(2)}$
(cutting edge technology \rightarrow computer algebra system FORM [Vermaseren '89-'05](#))



LO and NLO splitting functions

$$P_{\text{ns}}^{(0)}(x) = C_F(2p_{\text{qq}}(x) + 3\delta(1-x))$$

$$P_{\text{ps}}^{(0)}(x) = 0$$

$$P_{\text{qg}}^{(0)}(x) = 2n_f p_{\text{qg}}(x)$$

$$P_{\text{gq}}^{(0)}(x) = 2C_F p_{\text{gq}}(x)$$

$$P_{\text{gg}}^{(0)}(x) = C_A(4p_{\text{gg}}(x) + \frac{11}{3}\delta(1-x)) - \frac{2}{3}n_f\delta(1-x)$$

$$\begin{aligned} P_{\text{ns}}^{(1)+}(x) &= 4C_A C_F \left(p_{\text{qq}}(x) \left[\frac{67}{18} - \zeta_2 + \frac{11}{6}H_0 + H_{0,0} \right] + p_{\text{qq}}(-x) \left[\zeta_2 + 2H_{-1,0} - H_{0,0} \right] \right. \\ &\quad \left. + \frac{14}{3}(1-x) + \delta(1-x) \left[\frac{17}{24} + \frac{11}{3}\zeta_2 - 3\zeta_3 \right] \right) - 4C_F n_f \left(p_{\text{qq}}(x) \left[\frac{5}{9} + \frac{1}{3}H_0 \right] + \frac{2}{3}(1-x) \right. \\ &\quad \left. + \delta(1-x) \left[\frac{1}{12} + \frac{2}{3}\zeta_2 \right] \right) + 4C_F^2 \left(2p_{\text{qq}}(x) \left[H_{1,0} - \frac{3}{4}H_0 + H_2 \right] - 2p_{\text{qq}}(-x) \left[\zeta_2 + 2H_{-1,0} \right. \right. \\ &\quad \left. \left. - H_{0,0} \right] - (1-x) \left[1 - \frac{3}{2}H_0 \right] - H_0 - (1+x)H_{0,0} + \delta(1-x) \left[\frac{3}{8} - 3\zeta_2 + 6\zeta_3 \right] \right) \end{aligned}$$

$$\begin{aligned} P_{\text{ns}}^{(1)-}(x) &= P_{\text{ns}}^{(1)+}(x) + 16C_F \left(C_F - \frac{C_A}{2} \right) \left(p_{\text{qq}}(-x) \left[\zeta_2 + 2H_{-1,0} - H_{0,0} \right] - 2(1-x) \right. \\ &\quad \left. - (1+x)H_0 \right) \end{aligned}$$

$$P_{\text{ps}}^{(1)}(x) = 4C_F n_f \left(\frac{20}{9} \frac{1}{x} - 2 + 6x - 4H_0 + x^2 \left[\frac{8}{3}H_0 - \frac{56}{9} \right] + (1+x) \left[5H_0 - 2H_{0,0} \right] \right)$$

$$\begin{aligned} P_{\text{qg}}^{(1)}(x) &= 4C_A n_f \left(\frac{20}{9} \frac{1}{x} - 2 + 25x - 2p_{\text{qg}}(-x)H_{-1,0} - 2p_{\text{qg}}(x)H_{1,1} + x^2 \left[\frac{44}{3}H_0 - \frac{218}{9} \right] \right. \\ &\quad \left. + 4(1-x) \left[H_{0,0} - 2H_0 + xH_1 \right] - 4\zeta_2 x - 6H_{0,0} + 9H_0 \right) + 4C_F n_f \left(2p_{\text{qg}}(x) \left[H_{1,0} + H_{1,1} + H_2 \right. \right. \\ &\quad \left. \left. - \zeta_2 \right] + 4x^2 \left[H_0 + H_{0,0} + \frac{5}{2} \right] + 2(1-x) \left[H_0 + H_{0,0} - 2xH_1 + \frac{29}{4} \right] - \frac{15}{2} - H_{0,0} - \frac{1}{2}H_0 \right) \end{aligned}$$

$$\begin{aligned} P_{\text{gq}}^{(1)}(x) &= 4C_A C_F \left(\frac{1}{x} + 2p_{\text{gq}}(x) \left[H_{1,0} + H_{1,1} + H_2 - \frac{11}{6}H_1 \right] - x^2 \left[\frac{8}{3}H_0 - \frac{44}{9} \right] + 4\zeta_2 - 2 \right. \\ &\quad \left. - 7H_0 + 2H_{0,0} - 2H_1 x + (1+x) \left[2H_{0,0} - 5H_0 + \frac{37}{9} \right] - 2p_{\text{gq}}(-x)H_{-1,0} \right) - 4C_F n_f \left(\frac{2}{3}x \right. \\ &\quad \left. - p_{\text{gq}}(x) \left[\frac{2}{3}H_1 - \frac{10}{9} \right] \right) + 4C_F^2 \left(p_{\text{gq}}(x) \left[3H_1 - 2H_{1,1} \right] + (1+x) \left[H_{0,0} - \frac{7}{2} + \frac{7}{2}H_0 \right] - 3H_{0,0} \right. \\ &\quad \left. + 1 - \frac{3}{2}H_0 + 2H_1 x \right) \end{aligned}$$

$$\begin{aligned} P_{\text{gg}}^{(1)}(x) &= 4C_A n_f \left(1-x - \frac{10}{9}p_{\text{gg}}(x) - \frac{13}{9} \left(\frac{1}{x} - x^2 \right) - \frac{2}{3}(1+x)H_0 - \frac{2}{3}\delta(1-x) \right) + 4C_A^2 \left(27 \right. \\ &\quad \left. + (1+x) \left[\frac{11}{3}H_0 + 8H_{0,0} - \frac{27}{2} \right] + 2p_{\text{gg}}(-x) \left[H_{0,0} - 2H_{-1,0} - \zeta_2 \right] - \frac{67}{9} \left(\frac{1}{x} - x^2 \right) - 12H_0 \right. \\ &\quad \left. - \frac{44}{3}x^2 H_0 + 2p_{\text{gg}}(x) \left[\frac{67}{18} - \zeta_2 + H_{0,0} + 2H_{1,0} + 2H_2 \right] + \delta(1-x) \left[\frac{8}{3} + 3\zeta_3 \right] \right) + 4C_F n_f \left(2H_0 \right. \\ &\quad \left. + \frac{2}{3} \frac{1}{x} + \frac{10}{3}x^2 - 12 + (1+x) \left[4 - 5H_0 - 2H_{0,0} \right] - \frac{1}{2}\delta(1-x) \right). \end{aligned}$$

NNLO non-singlet splitting functions

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$$\begin{aligned}
 P_{qq}^{(2)}(x) = & 16C_F C_F \gamma \left(\frac{1}{3} P_{qq}^{(1)}(x) \left[\frac{10}{3} \zeta_3 - \frac{209}{36} 9\zeta_5 - \frac{167}{18} H_0 + 2H_0\zeta_2 - 7H_{0,0} - 2H_{0,0} \right. \right. \\
 & + 3H_{0,0} - H_0 \left. \right] + \frac{1}{3} P_{qq}^{(1)}(-x) \left[\frac{2}{3} \zeta_3 - \frac{5}{3} \zeta_2 - H_{-2,0} - 2H_{-1,1} - \frac{10}{3} H_{-1,0} - H_{-1,0} \right. \\
 & + 2H_{-1,2} + \frac{1}{2} H_0\zeta_2 + \frac{5}{6} H_{0,0} + H_{0,0,0} - H_1 \left. \right] + (1-x) \left[\frac{1}{6} \zeta_3 - \frac{237}{24} H_0 + \frac{1}{6} H_{0,0} - H_1 \right. \\
 & - (1+x) \left[\frac{2}{3} H_{-1,0} + \frac{1}{2} H_1 \right] + \frac{1}{3} \zeta_3 + H_0 + \frac{1}{6} H_{0,0} + \delta(1-x) \left[\frac{5}{4} \zeta_4 - \frac{167}{54} \zeta_5 + \frac{1}{20} \zeta_2^2 + \frac{25}{18} \zeta_3 \right] \\
 & + 16C_F C_F^2 \left(P_{qq}^{(1)}(x) \left[\frac{5}{6} \zeta_3 - \frac{69}{20} \zeta_2^2 - H_{-3,0} - 3H_{-2,1} - 14H_{-2,-1,0} + 3H_{-2,0} + 5H_{-2,0} \right. \right. \\
 & - 4H_{-2,2} - \frac{151}{48} H_0 + \frac{11}{12} H_0\zeta_2 + \frac{1}{2} H_0\zeta_3 - \frac{13}{24} H_{0,0} - 4H_0\zeta_5 + \frac{23}{12} H_{0,0,0} + 5H_{0,0,0} + \frac{3}{2} H_1 \\
 & - 24H_{1,1} - 16H_{1,-2,0} + \frac{67}{9} H_{1,0} - 2H_{1,0}\zeta_2 + \frac{31}{3} H_{1,0,0} + 11H_{1,0,0,0} + 8H_{1,1,0,0} - 8H_{1,3} + 2H_1 \\
 & + \frac{67}{9} H_2 - 2H_2\zeta_2 + \frac{11}{3} H_{2,0} + 5H_{2,0,0} + H_{3,0} \left. \right] + P_{qq}^{(1)}(-x) \left[\frac{1}{2} \zeta_3^2 - \frac{67}{9} \zeta_2 + \frac{31}{3} \zeta_3 + 5H_{-3,0} \right. \\
 & - 32H_{-2,1} - 4H_{-2,-1,0} - \frac{31}{6} H_{-2,0} + 21H_{-2,0,0} + 30H_{-2,2} - \frac{31}{3} H_{-1,1} - 42H_{-1,1} + \frac{9}{4} H_0 \\
 & - 4H_{-1,-2,0} + 56H_{-1,-1,1} - 36H_{-1,-1,0,0} - 56H_{-1,-1,2} - \frac{134}{9} H_{-1,0} - 42H_{-1,0}\zeta_2 - H_{3,0} \\
 & + 32H_{-1,3} - \frac{21}{8} H_{-1,0,0} + 17H_{-1,0,0,0} + \frac{31}{6} H_{-1,1,2} + 2H_{-1,2,0} + \frac{13}{12} H_0\zeta_2 + \frac{29}{3} H_0\zeta_3 + \frac{67}{9} H_{0,0} \\
 & + 13H_{0,0}\zeta_2 + \frac{89}{12} H_{0,0,0} - 5H_{0,0,0,0} - 7H_2\zeta_2 - \frac{31}{6} H_3 + 10H_1 \left. \right] + (1-x) \left[\frac{133}{36} 4H_{0,0,0,0} \right. \\
 & - \frac{167}{4} \zeta_5 - 2H_0\zeta_3 - 2H_{-3,0} + H_{-2,1} + 2H_{-2,-1,0} - 3H_{-2,0,0} - \frac{77}{4} H_{0,0,0} - \frac{209}{6} H_1 - 7H_1\zeta_2 \\
 & + 4H_{1,0,0} + \frac{13}{3} H_1 \left. \right] + (1+x) \left[\frac{43}{2} \zeta_2 - 3\zeta_2^2 + \frac{22}{3} H_{-2,0} - 31H_{-1,1} - 14H_{-1,-1,0} - \frac{13}{3} H_{-1,0} \right. \\
 & + 24H_{-1,2} + 23H_{-1,0,0} + \frac{5}{2} H_0\zeta_2 + 5H_0\zeta_3 + \frac{1457}{48} H_0 - \frac{1025}{36} H_{0,0} - \frac{155}{6} H_2 + H_2\zeta_2 - 15H_3 \\
 & + 2H_{2,0,0} - 3H_1 - 5\zeta_3 - \frac{1}{2} \zeta_2^2 + 50\zeta_5 - 2H_{-3,0} - 7H_{-2,0} - H_0\zeta_3 - \frac{37}{2} H_0\zeta_5 - \frac{243}{9} H_0 \\
 & - 2H_0\zeta_2 + \frac{185}{6} H_{0,0} - 22H_{0,0,0} - 4H_{0,0,0,0} + \frac{3}{2} H_2 + 6H_1 + \delta(1-x) \left[\frac{151}{6} \zeta_3 + \frac{205}{24} \zeta_2^2 \right. \\
 & - \frac{247}{60} \zeta_2^2 + \frac{211}{12} \zeta_3 + \frac{15}{2} \zeta_3 \left. \right] + 16C_F^2 C_F \left(P_{qq}^{(1)}(x) \left[\frac{245}{28} \zeta_3 - \frac{67}{18} \zeta_2 + \frac{12}{5} \zeta_2^2 + \frac{1}{2} \zeta_3 + \frac{1043}{216} H_0 \right. \right. \\
 & + H_{-3,0} + 4H_{-2,-1,0} - \frac{3}{2} H_{-2,0} - H_{-2,0,0} + 2H_{-2,2} - \frac{31}{12} H_0\zeta_2 + 4H_0\zeta_3 + \frac{389}{72} H_{0,0} - 2H_{0,0} \\
 & - H_{0,0,0} + 9H_1\zeta_2 + 6H_1 - 2H_1\zeta_2 - \frac{11}{4} H_{1,0,0} - 3H_{1,0,0,0} - 4H_{1,1,0,0} + 4H_{1,3} + \frac{31}{12} H_{0,0,0} \\
 & + \frac{11}{12} H_3 + H_1 \left. \right] + P_{qq}^{(1)}(-x) \left[\frac{67}{18} \zeta_3 - \zeta_2^2 - \frac{11}{4} \zeta_3 - H_{-3,0} + 8H_{-2,1} + \frac{11}{6} H_{-2,0} - 4H_{-2,0,0} \right. \\
 & - 3H_{-1,0,0,0} + \frac{11}{3} H_{-1,1} - 12H_{-1,1} - 16H_{-1,-1,1} - 58H_{-1,-1,0,0} + 16H_{-1,-1,2} + \frac{67}{9} H_{-1,0}
 \end{aligned}$$

$$\begin{aligned}
 & - 8H_{-2,2} + 11H_{-1,0}\zeta_2 + \frac{11}{6} H_{-1,0,0} - \frac{11}{3} H_{-1,2} - 8H_{-1,3} - \frac{3}{2} H_0 - \frac{1}{6} H_0\zeta_2 - 4H_0\zeta_3 - \frac{67}{18} H_{0,0} \\
 & - 3H_0\zeta_5 + \frac{31}{12} H_{0,0,0} + H_{0,0,0,0} + 2H_2\zeta_2 + \frac{11}{6} H_2 + 2H_1 \left. \right] + (1-x) \left[\frac{1883}{108} \zeta_3 - \frac{1}{2} H_{0,0,0,0} + 11H_1 \right. \\
 & - H_{-2,-1,0} + \frac{1}{2} H_{-3,0} - \frac{1}{2} H_{-2,1} + \frac{1}{2} H_{-2,0,0} + \frac{323}{36} H_0 + H_0\zeta_2 - \frac{13}{3} H_{0,0} - \frac{5}{2} H_{0,0,0} + 2H_1\zeta_2 \\
 & - 2H_{1,0,0} \left. \right] + (1+x) \left[8H_{-1,1}\zeta_2 + 4H_{-1,-1,0} - \frac{8}{3} H_{-1,0} - 5H_{-1,0,0} - 6H_{-1,2} - \frac{13}{3} \zeta_3 + \frac{5}{8} \zeta_2^2 \right. \\
 & - \frac{43}{4} \zeta_3 - \frac{5}{2} H_{-2,0} - \frac{11}{2} H_0\zeta_2 - \frac{1}{2} H_0\zeta_3 - \frac{5}{4} H_{0,0}\zeta_2 + 7H_2 - \frac{1}{4} H_{2,0,0} + 3H_1 + \frac{3}{4} H_1 \left. \right] + \frac{1}{2} H_{0,0}\zeta_2 \\
 & + \frac{1}{4} \zeta_2^2 - \frac{8}{9} \zeta_2 + \frac{17}{9} \zeta_3 + H_{-2,0} - \frac{19}{2} H_0 + \frac{5}{2} H_{0,0}\zeta_2 - H_0\zeta_3 + \frac{13}{3} H_{0,0} + \frac{5}{2} H_{0,0,0} + \frac{1}{2} H_{0,0,0,0} \\
 & - \delta(1-x) \left[\frac{1657}{276} \zeta_3 - \frac{281}{27} \zeta_2^2 + \frac{8}{9} \zeta_2^2 + \frac{97}{9} \zeta_3 - \frac{5}{2} \zeta_3 \right] + 16C_F^2 \gamma \left(\frac{1}{3} P_{qq}^{(1)}(x) \left[H_0 - \frac{1}{3} \zeta_3 + \frac{1}{3} H_1 \right. \right. \\
 & + (1-x) \left[\frac{15}{144} \zeta_3 + \frac{1}{4} H_1 \right] - \delta(1-x) \left[\frac{17}{144} \zeta_3 - \frac{5}{27} \zeta_2 + \frac{5}{9} \zeta_3 \right] + 16C_F^2 \gamma \left(\frac{1}{3} P_{qq}^{(1)}(-x) \left[\zeta_3 - 4H_{1,0,0} \right. \right. \\
 & - \frac{55}{16} \zeta_3 + H_0 + H_0\zeta_2 + \frac{3}{2} H_{0,0} - H_{0,0,0} - \frac{10}{3} H_{1,0} - \frac{10}{3} H_2 - 2H_{2,0} - 2H_1 \left. \right] + \frac{2}{3} P_{qq}^{(1)}(-x) \left[\frac{5}{3} \zeta_2 \right. \\
 & - \frac{2}{3} \zeta_3 + H_{-2,0} + 2H_{-1,1} + \frac{10}{3} H_{-1,0} + H_{-1,0,0} - 2H_{-1,2} - \frac{1}{2} H_0\zeta_2 - \frac{5}{2} H_{0,0} - H_{0,0,0} + H_1 \left. \right] \\
 & + (1-x) \left[\frac{10}{9} + \frac{10}{18} H_{0,0} - \frac{4}{3} H_1 + \frac{2}{3} H_1 + \frac{4}{3} H_2 \right] + (1+x) \left[\frac{2}{3} H_{-1,0} - \frac{25}{24} H_0 + \frac{1}{2} H_{0,0} \right] + \frac{2}{3} H_0 \\
 & - \frac{7}{9} H_{0,0} + \frac{4}{3} H_2 - \delta(1-x) \left[\frac{23}{16} \zeta_3 - \frac{29}{12} \zeta_2 + \frac{17}{6} \zeta_3 \right] + 16C_F^3 \left(P_{qq}^{(1)}(x) \left[\frac{9}{10} \zeta_2^2 - 2H_{-3,0} \right. \right. \\
 & + 6H_{-2,1} + 12H_{-2,-1,0} - 6H_{-2,0,0} - \frac{3}{10} H_0 - \frac{3}{2} H_0\zeta_2 + H_0\zeta_3 + \frac{18}{5} H_{0,0} - 2H_{0,0,0} + 8H_{1,3} \\
 & + 12H_1\zeta_2 + 8H_{1,0} - 6H_{1,0,0} - 4H_{1,0,0,0} + 4H_{2,0} - 3H_{2,0} + 2H_{1,0,0} + 4H_{1,2} \\
 & + 4H_{0,0} + 4H_{1,1} + 2H_1 \left. \right] + P_{qq}^{(1)}(-x) \left[\frac{43}{12} \zeta_2^2 - \frac{9}{2} \zeta_2 + 32H_{-2,1} + 8H_{-2,-1,0} + 3H_{-2,0,0} \right. \\
 & - 26H_{-2,0,0} - 28H_{-2,2} + 6H_{-1,1}\zeta_2 + 36H_{-1,1}\zeta_3 + 8H_{-1,-2,0} - 48H_{-1,-1,1} + 40H_{-1,-1,0,0} \\
 & + 48H_{-1,-1,2} + 40H_{-1,0}\zeta_2 + 3H_{-1,0,0} - 22H_{-1,0,0,0} - 6H_{-1,2} - 4H_{-1,2,0} - 32H_{-1,3} - \frac{3}{2} H_0 \\
 & - \frac{3}{2} H_0\zeta_2 - 13H_0\zeta_3 - 14H_0\zeta_5 - \frac{9}{2} H_{0,0,0,0} + 6H_{0,0,0,0} + 6H_2\zeta_2 + 3H_1 + 2H_{1,0} + 12H_1 \left. \right] \\
 & + (1-x) \left[2H_{-3,0} - \frac{31}{4} H_{-2,0,0} + 4H_{-2,0,0,0} + H_0\zeta_2 - 3H_{0,0,0,0} + 35H_1 + 6H_1\zeta_2 - H_{1,0} + \delta(1-x) \left[\frac{67}{24} \zeta_2 \right. \right. \\
 & + (1+x) \left[\frac{37}{4} \zeta_2^2 - \frac{93}{4} \zeta_2 - \frac{81}{2} \zeta_3 - 15H_{-2,0} + 30H_1 + \zeta_3 + 12H_{-1,0} - 2H_{-1,0} - 26H_{-1,0,0} \right. \\
 & - 24H_{-1,2} - \frac{539}{16} H_0 - 28H_0\zeta_2 + \frac{191}{8} H_{0,0} + 20H_{0,0,0} + \frac{85}{4} H_2 - 3H_{2,0,0} - 2H_{0,0} + 13H_3 \\
 & - H_1 \left. \right] + \zeta_2 + 3\zeta_3 + 4H_{-3,0} + 10H_{-2,0} + \frac{67}{2} H_0 + 6H_0\zeta_2 + 19H_0\zeta_3 - 25H_{0,0} - 17H_{0,0,0} \\
 & - 2H_1 - H_{2,0} - 4H_1 + \delta(1-x) \left[\frac{29}{132} - 2\zeta_5 + \frac{9}{2} \zeta_5^2 + \frac{18}{5} \zeta_2^2 + \frac{17}{4} \zeta_3 - 15\zeta_3 \right] \left. \right] .
 \end{aligned}$$

$$\begin{aligned}
 P_{qg}^{(2)}(x) = & P_{qg}^{(2)}(x) + 16C_F C_F \left(C_F - \frac{C_A}{3} \right) \left(P_{qg}^{(1)}(-x) \left[\frac{134}{9} \zeta_3 - 4\zeta_2^2 - 11\zeta_3 - 4H_{-3,0} \right. \right. \\
 & + 32H_{-2,1} - \frac{22}{3} H_{-2,0} - 16H_{-2,0,0} - 32H_{-2,2} + \frac{268}{9} H_{-1,0} + 44H_{-1,0}\zeta_2 + 48H_{-1,1}\zeta_3 - 64H_{-1,-1}\zeta_2 \\
 & + 32H_{-1,-1,0,0} + 64H_{-1,-1,2} + \frac{268}{9} H_{-1,0} + 44H_{-1,0}\zeta_2 + \frac{22}{3} H_{-1,0,0} - 12H_{-1,0,0,0} - \frac{44}{3} H_{-1,2} \\
 & - 32H_{-1,3} - 3H_0 - \frac{2}{3} H_0\zeta_2 - 16H_0\zeta_3 - \frac{134}{9} H_{0,0} - 12H_{0,0}\zeta_2 - \frac{31}{3} H_{0,0,0} + 4H_{0,0,0,0} + 8H_2\zeta_2 \\
 & + \frac{22}{3} H_3 + 8H_1 \left. \right] + (1-x) \left[\frac{367}{18} \zeta_2^2 + 2H_{-3,0} - 2H_{-2,1} - 4H_{-2,-1,0} - 10H_{-2,0} - 2H_{0,0} \right. \\
 & + 2H_{-2,0,0} + 2H_0\zeta_2 + H_0\zeta_3 - H_{0,0,0,0} + 8H_1\zeta_2 + \frac{140}{3} H_1 \left. \right] + (1+x) \left[32H_{-1,1}\zeta_2 - 18\zeta_2 \right. \\
 & - 23\zeta_3 + \frac{26}{3} H_{-1,0} - 16H_{-1,0,0} - 32H_{-1,2} - \frac{481}{18} H_0 - 29H_0\zeta_2 + 5H_{0,0,0} + 24H_1 + \frac{26}{3} H_2 \left. \right] \\
 & - \zeta_2 - 2\zeta_3 + 32H_0 + 14H_0\zeta_2 + 2H_{0,0,0} - 16H_3 \left. \right] + 16C_F^2 \gamma \left(C_F - \frac{C_A}{3} \right) \left(P_{qg}^{(1)}(-x) \left[2\zeta_3 \right. \right. \\
 & - \frac{20}{9} \zeta_3 + \frac{4}{3} H_{-2,0} - \frac{8}{3} H_{-1,1} - \frac{40}{9} H_{-1,0} - \frac{4}{3} H_{-1,0,0} + \frac{8}{3} H_{-1,2} + \frac{2}{3} H_0\zeta_2 + \frac{20}{9} H_{0,0} + \frac{4}{3} H_{0,0,0} \\
 & - \frac{4}{3} H_1 \left. \right] + (1-x) \left[\frac{61}{9} H_1 \right] + (1+x) \left[2H_{0,0} - \frac{8}{9} H_{-1,0} + \frac{41}{9} H_0 - \frac{4}{3} H_1 \right] \\
 & + 16C_F^2 \left(C_F - \frac{C_A}{3} \right) \left(P_{qg}^{(1)}(x) \left[9\zeta_3 - 7\zeta_2^2 + 12H_{-3,0} - 64H_{-2,1} - 16H_{-2,-1,0} - 6H_{-2,0} \right. \right. \\
 & + 52H_{-2,0,0} + 56H_{-2,2} - 12H_{-1,1}\zeta_2 - 72H_{-1,1}\zeta_3 - 16H_{-1,-2,0} + 96H_{-1,-1,1} - 80H_{-1,-1,0,0} \\
 & - 96H_{-1,-1,2} - 80H_{-1,0}\zeta_2 - 6H_{-1,0,0} + 44H_{-1,0,0,0} + 12H_{-1,2} + 8H_{-1,2,0} + 64H_{-1,3} + 3H_0 \\
 & + 3H_0\zeta_2 + 26H_0\zeta_3 + 28H_0\zeta_5 + 9H_{0,0,0} - 12H_{0,0,0,0} - 12H_2\zeta_2 - 6H_1 - 4H_{1,0} - 24H_1 \left. \right] \\
 & - (1-x) \left[15 + 8H_{-3,0} + 8H_{-2,0,0} + 61H_0 + 6H_0\zeta_2 + 2H_0\zeta_3 - 6H_{0,0,0,0} + 12H_2\zeta_2 + 60H_1 \right. \\
 & + 8H_1 \left. \right] + (1+x) \left[24\zeta_3 + 57\zeta_3 + 10H_{-2,0} - 48H_{-1,1}\zeta_2 - 4H_{-1,0} + 40H_{-1,0,0} + 48H_{-1,2} \right. \\
 & + 59H_0\zeta_2 - 22H_{0,0} - 35H_{0,0,0} - 22H_2 - 4H_{2,0} - 44H_3 \left. \right] + 8\zeta_3 - 4\zeta_3^2 - 4H_{-2,0} + 42H_0 \\
 & - 38H_0\zeta_2 + 14H_{0,0} - 16H_2 + 26H_{0,0} + 24H_1 \left. \right) .
 \end{aligned}$$

$$+ \frac{3}{8} H_{0,0} - \frac{1}{2} H_{0,0,0} + \frac{1}{2} H_{0,0,0,0} + H_{-2,0} - H_1 \left. \right) .$$

$$\begin{aligned}
 P_{gg}^{(2)}(x) = & 16\gamma_1 \frac{d^{4n-4d_g}}{d^n} \left(\frac{1}{3} (1-x) \left[\frac{50}{3} + \frac{41}{12} \zeta_3 - \frac{5}{2} \zeta_2^2 - H_{-3,0} + H_{-2,1} - H_{-2,0,0} + \frac{9}{2} H_1 \right. \right. \\
 & + 2H_{-2,-1,0} + \frac{3}{2} H_{0,0}\zeta_2 - \frac{1}{2} H_1\zeta_2 - \frac{3}{4} H_{1,0,0} + \frac{91}{12} H_1 \left. \right] + \frac{1}{3} (1+x) \left[H_{-1,-1,0} - \frac{3}{2} H_{-1,1} + \frac{3}{2} H_1 \right. \\
 & - \frac{13}{6} H_{-1,0} + \frac{1}{2} H_{-1,0,0} + 2H_{-1,2} - \frac{3}{2} H_{-2,0} + \frac{9}{4} H_0\zeta_2 + \frac{29}{12} H_{0,0} + \frac{41}{12} H_0\zeta_2 - H_2\zeta_2 - \frac{1}{2} H_{2,0,0} \\
 & + \frac{2}{3} H_1 \left. \right] - \frac{1}{3} \left(\frac{1}{3} x^2 + x^2 \right) \left[3H_{-1,1}\zeta_2 + 2H_{-1,-1,0} - 2H_{-1,0,0} - 2H_{-1,2} + H_1\zeta_2 \right] + \frac{1}{3} \left[5\zeta_3 - 2H_1 \right. \\
 & + 2H_{-2,0} + 4H_0\zeta_2 - 2H_{0,0,0} + 2H_1\zeta_2 \left. \right] + \frac{1}{24} H_0 + \zeta_3 - \frac{9}{2} \zeta_2^2 + \zeta_3 - H_0\zeta_2 - H_0\zeta_3 - 2H_0\zeta_5
 \end{aligned}$$

NNLO singlet splitting functions

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$$\begin{aligned}
 P_{qq}^{(2)}(x) = & 16C_F C_F \eta \left(\frac{4}{3} x^2 + x^2 \right) \left[\frac{11}{3} H_{1,0} - \frac{14}{9} H_{2,0} + \frac{1}{2} H_{1,1} \zeta_2 - H_{1,1,0} - 2H_{1,1,0,0} \right. \\
 & - H_{1,1,1} \left. \right] + \frac{2}{3} \zeta_2^2 \left[\frac{16}{5} \zeta_2 + H_{2,1} + 9 \zeta_3 + \frac{9}{2} H_{3,0} - \frac{676}{216} + \frac{571}{72} H_4 + \frac{10}{3} H_5 + H_{1,1} \zeta_5 - \frac{1}{2} H_{1,1,1} \right. \\
 & - 3H_{1,1,0} + 2H_{1,1,0,0} + 2H_{1,1,1} \left. \right] + (1-x) \left[\frac{182}{9} \zeta_2 + \frac{158}{3} \zeta_3 + \frac{297}{108} H_4 - \frac{13}{2} H_{2,0} + 3H_{3,0,0} \right. \\
 & - \frac{31}{6} H_{2,0} + 3H_{3,0} + H_{3,0,0} + H_{3,0,1} + 2H_{3,0,2} + 2H_{3,0,3} - 2H_{3,0,4} + \frac{1}{2} H_{3,0,5} + \frac{9}{2} H_{3,0,6} \\
 & + \frac{1}{2} H_{3,0,7} + \frac{1}{2} H_{3,0,8} + H_{3,0,9} + H_{3,0,10} + H_{3,0,11} \left. \right] + \frac{1}{2} \left[\frac{146}{3} \zeta_2 + \frac{31}{6} H_{2,1} + \frac{71}{18} H_{3,1} + \frac{113}{18} \zeta_2^2 - \frac{829}{27} H_4 \right. \\
 & + \frac{2}{3} H_{4,1} + H_{4,1,0} + H_{4,1,1} \left. \right] + (1+x) \left[\frac{112}{3} \zeta_2 + \frac{5}{6} H_{2,1} + \frac{18}{12} H_{3,1} + \frac{18}{18} \zeta_2^2 + \frac{27}{27} H_4 \right. \\
 & + \frac{5}{2} H_{2,0} + \frac{16}{3} H_{3,0} + 6H_{3,1,0} + \frac{31}{6} H_{3,0,0} + \frac{17}{24} H_{3,0,1} + \frac{117}{24} \zeta_2^2 + 9H_{3,0,2} + \frac{5}{2} H_{3,1,1} \zeta_2 + 2H_{3,1,0} \left. \right] \\
 & + \frac{1}{2} H_{2,0,0} - 2H_{1,2} + H_2 \zeta_2 - \frac{7}{2} H_{2,0,0} + H_{1,1,0} - 2H_{2,1,1} + H_{3,1} + \frac{1}{2} H_{3,1,1} \left. \right] + 5H_{2,0} + H_{2,1} \\
 & + H_{2,0,0} - \frac{5}{2} \zeta_2^2 + 4H_{3,0} + 4H_{3,1} \zeta_2 - \frac{32}{9} H_{3,0,0} + \frac{29}{12} H_{3,0,1} - \frac{235}{172} \zeta_2^2 - \frac{511}{12} H_{3,1,1} + \frac{19}{3} H_{3,0,2,0} + \frac{33}{4} H_{3,1,2} - H_3 \\
 & - \frac{11}{2} H_{3,0} \zeta_2 - \frac{11}{2} \zeta_2^3 - \frac{3}{2} H_{2,0} - 10H_{3,0,0} + \frac{2}{3} \zeta_2^2 H_{3,0} + 10 \zeta_2^3 + \frac{1}{8} H_4 + 10 \zeta_2^4 + \frac{8}{3} H_5 + \frac{4}{3} H_6 - \frac{4}{3} H_7 \\
 & - 4 \zeta_2 - H_6 \zeta_2 + H_4 + H_{2,0} - 6H_{2,0,0} \left. \right] + 16C_F \eta^2 \left(\frac{2}{3} H_2 - 2H_3 + \zeta_2^2 \right) \left[H_2 \zeta_2 + 3 + \right. \\
 & \left. - \frac{4}{3} \zeta_2 \left. \right] + \frac{2}{9} (1-x^2) \left[H_{1,1} + \frac{5}{3} H_{1,1} + \frac{7}{3} \right] + (1-x) \left[\frac{1}{6} H_{1,1} - \frac{7}{6} H_{1,1} + \frac{35}{18} H_{1,1} + \frac{185}{54} \right. \\
 & \left. + \frac{1}{6} (1+x) \left[\frac{2}{3} H_2 - \frac{4}{3} \zeta_2 + \zeta_2^2 + H_{2,1} - 2H_{2,1,0} + 2H_{2,1,0,0} + H_{2,1,0,1} \right] + 16C_F \zeta_2 \left(\frac{85}{18} H_{1,1} \right. \right. \\
 & \left. \left. + \frac{25}{4} H_{2,0} + \frac{883}{54} H_{2,1} + \frac{101}{72} \zeta_2 - \frac{73}{72} H_{3,0} + H_3 - 5H_{2,0,0} - H_{2,1,0} - H_{2,1,0,1} + \zeta_2^2 \right) \right. \\
 & \left. + \frac{85}{18} H_{2,0,0} - \frac{109}{18} H_{2,1,0} - \frac{109}{18} H_{2,1,0,0} + \frac{23}{9} H_{2,1,0,1} + \frac{29}{9} H_{2,1,0,2} + \frac{16}{9} H_{2,1,0,3} + 4H_{2,1,0,4} + \frac{1}{2} H_{2,1,0,5} - \frac{26}{3} H_{2,1,0,6} \right. \\
 & \left. + \frac{23}{3} H_{3,0,0} - \frac{4}{3} \zeta_2^2 \left. \right] + \frac{23}{3} H_{3,0,1} - \frac{17}{12} H_{3,0,2} + \frac{523}{144} \zeta_2^2 - \frac{55}{16} H_{3,1,1} + H_{3,1,0} + H_{3,1,1,0} - H_{3,1,1,1} \left. \right] \\
 & + (1-x) \left[\frac{1}{2} H_{3,0,0} + \frac{4}{12} H_{3,1,0} - \frac{17}{12} H_{3,1,1} - \frac{2743}{144} H_{3,1,1,0} - \frac{25}{16} H_{3,1,1,1} + \frac{5}{8} H_{3,1,1,2} - 8H_{3,1,1,3} + 3H_{3,1,1,4} \right. \\
 & + 3H_{3,1,1,5} - 3H_{3,1,1,6} - H_{3,1,1,7} \left. \right] + (1+x) \left[\frac{1669}{276} H_{3,0,0} + 4H_{3,1,0} + 2H_{3,1,0,0} + \frac{27}{10} \zeta_2^2 \right. \\
 & \left. - 7H_{3,0,1} + 6H_{3,0,2} - 4H_{3,0,3} + H_{2,0,0} - 2H_{2,1,0} - 2H_{2,1,1} - 4H_{3,0} - H_{3,1} - 6H_{3,1} \right. \left. \right].
 \end{aligned}$$

$$\begin{aligned}
 P_{qg}^{(2)}(x) = & 16C_F C_F \eta \left(P_{qg}^{(0)}(x) \left[\frac{39}{2} H_2 \zeta_2 - 4H_{1,1,1} + 3H_{2,0,0} - \frac{15}{2} H_2 + 9H_{1,1,0} + 3H_{2,0,0} \right. \right. \\
 & + H_3 \zeta_2 - 2H_{2,1,1} + 4H_2 \zeta_2 - \frac{173}{2} H_3 \zeta_2 - \frac{551}{54} \zeta_2^2 + \frac{49}{2} H_4 + \frac{3}{2} H_{3,0,0} - \frac{1}{2} H_{3,0,1} - \frac{1}{2} H_{3,0,2} \\
 & - \frac{385}{72} H_{3,1,0} - \frac{31}{72} H_{3,1,1} + \frac{113}{72} H_{3,1,2} + \frac{49}{72} H_{3,1,3} + \frac{6}{72} H_{3,0,0,0} + \frac{173}{72} H_{3,0,1,0} + \frac{1259}{72} H_{3,0,2,0} + \frac{2833}{72} H_{3,0,3,0} \\
 & + 6H_{2,1,0} + 3H_{1,2,0} + 9H_{1,0,2} + 6H_{1,1,2} + H_{1,0,0,0} + 3H_{1,1,0,0} - 4H_{1,1,1,0} - 3H_{1,2,0,0} - 6H_{1,2,1,0} \left. \right. \\
 & \left. \left. + 727 H_{1,0} - H_{1,1} \zeta_2 - 2H_{1,2} - \frac{5}{2} H_{2,1} \zeta_2 - H_{3,1,0} - 2H_{3,1,0,0} + 2H_{3,1,1,0} - \frac{3}{2} H_{3,1,0,0,0} \right. \right. \\
 & \left. \left. + 6H_{1,1,0} - 1.0 - 2H_{1,1,3} + 2H_{1,1,2} \right) \right] + \left(\frac{1-x}{2} \right)^2 \left[\frac{1}{2} H_2 \zeta_2 - 2H_{1,0,1} + \frac{10}{3} H_{1,1} \right. \\
 & \left. - 8H_{1,0,0} + \frac{3}{2} H_{1,1} + H_1 \zeta_2 + \frac{19}{2} H_{2,0,0} - \frac{239}{16} H_{2,0,1} + 8(1+x) \left[H_{1,0} - \frac{28}{9} H_{2,0} - 2H_{1,1,0} \right. \right. \\
 & \left. \left. - 8H_{1,0,0} + \frac{3}{2} H_{1,1} + 6 \zeta_2 + \frac{161}{36} H_{2,1} - \frac{2351}{108} \left(\frac{2}{3} x \right)^2 \right) \right] \left[\frac{26}{3} H_{1,0} - \frac{28}{9} H_{2,0} - 2H_{1,1,0} \right. \\
 & \left. - 8H_{1,0,0} + \frac{3}{2} H_{1,1} + 6 \zeta_2 + \frac{161}{36} H_{2,1} - \frac{2351}{108} \left(\frac{2}{3} x \right)^2 \right) \right].
 \end{aligned}$$

$$\begin{aligned}
 & -2H_{1,0} - \frac{11}{2} H_{2,0} - 13H_{1,1} - \frac{13}{2} H_{2,1} + \frac{15}{2} H_{3,1} + \frac{2005}{64} + \frac{157}{64} \zeta_2 + 8 \zeta_3 + \frac{1291}{252} H_4 + \frac{55}{12} H_{4,1} \\
 & + \frac{3}{2} H_5 + \frac{1}{2} H_{6,1} + \frac{27}{4} H_{7,0} - \frac{11}{2} H_{2,0,0} - 8H_{2,0,1} - 4 \zeta_2^2 + \frac{3}{2} H_{2,1,0} - H_{2,2} + \frac{5}{2} H_2 \zeta_2 + 8H_{1,1,0} - \frac{1}{2} H_{1,1,1} \\
 & + 4H_{2,0,1} - H_{1,1,0} - H_{1,1,0} + 7H_2 \zeta_2 + 6H_2 \zeta_3 + 12H_{2,1,0} - 6H_{2,2,0,0} + 3H_{3,1,0} - 3H_{3,0,0} \\
 & + \frac{9}{2} H_{3,0,1} - \frac{35}{8} H_{3,0,2} + 2H_3 + 3H_{3,1,0} + H_{3,1,1} \left. \right] + 16C_F^2 C_F \left(\zeta_2^2 \left[\frac{2}{3} H_2 \zeta_2 - \frac{2105}{18} H_{2,0} + \frac{77}{8} H_{3,0,0} \right. \right. \\
 & \left. \left. - 6H_3 + \frac{16}{3} \zeta_2^2 - 10H_{1,0} - \frac{14}{3} H_{2,0} - \frac{2}{3} H_{1,1} \zeta_2 + \frac{14}{3} H_{2,1} + \frac{10}{3} H_{3,1,0} + \frac{4}{3} H_{3,1,1} + \frac{81}{37} H_{3,0,0} \right. \right. \\
 & \left. \left. + \frac{4}{3} H_{3,1,0} - \frac{104}{9} \zeta_2 - \frac{2}{3} H_{3,1,1} + \frac{145}{18} H_{3,1,2} + \frac{3}{2} H_{3,1,3} + \frac{109}{18} H_{3,1,4} + \frac{8}{3} H_{3,1,5} + 6H_3 \zeta_2 \right. \right. \\
 & \left. \left. + 4H_{2,0} + \frac{27}{2} H_6 \right) + P_{qg}^{(0)}(x) \left[\frac{7}{2} H_2 \zeta_2 + \frac{2592}{2592} H_{2,0} + \frac{1}{2} H_{2,0,0} + \frac{1}{2} H_{2,0,1} + \frac{1}{2} H_{2,0,2} + \frac{1}{2} H_{2,0,3} \right. \right. \\
 & \left. \left. + 4H_{1,1} + \frac{67}{6} H_{1,1,1} + \frac{109}{6} \zeta_2^2 + \frac{17}{3} H_{2,1} - \frac{21}{6} H_{3,1,0} - \frac{11}{2} H_{3,1,1} - \frac{21}{6} \zeta_2^2 + 2H_{3,1,0,0} + H_{3,1,0,1} - 2H_{3,1,0,2} \right. \right. \\
 & \left. \left. + \frac{395}{24} H_{3,1,1,0} - 2H_{3,1,1,1} - H_{1,1} \zeta_2 + \frac{55}{12} H_{3,1,1,0} + 2H_{3,1,1,1} + 4H_{3,1,1,2} - \frac{55}{12} H_{2,1,2} \right. \right. \\
 & \left. \left. + \frac{4}{3} H_{2,1,3} - \frac{5880}{18} \zeta_2 \right) \right. \\
 & + 6H_{1,2,0} + 4H_{1,2,1} + 4H_3 + 3H_{2,1,0} + 3H_{2,1,1} + \frac{211}{18} H_{3,1,1} + 49 \zeta_2^2 \left. \right] + (1+x) \left[11 \zeta_2 + H_{1,1} + \frac{1}{2} H_{2,1,0} + 2H_{1,2,0} - 2H_{1,2,1,0} \right. \\
 & - \frac{109}{12} H_{2,0,0} + H_6 \zeta_2 + \frac{1}{3} \zeta_2^3 + \frac{1}{3} H_2 \zeta_2 + 2H_3 \zeta_2 - \frac{65}{24} H_{2,1,1} - \frac{19}{24} H_{3,1,1} - 4H_{3,0,0} - \frac{4}{3} H_{3,0,1} - 3H_{3,0,2} \\
 & - 7H_{2,0,0} - \frac{3}{2} H_{2,1,2} + \frac{3279}{216} H_{3,1,0} - 4H_{2,2,2} + \frac{49}{6} H_{3,0,0} - 13H_{1,1,1} \zeta_2 - 8H_{1,1,3} \\
 & - 6H_{1,1,0} - 1.0 + 12H_{1,1,0} + 10H_{1,1,1,0} + 12H_{1,1,0,2} - 5H_{1,1,2,0} - 2H_{1,1,2,1,0} - 2H_{1,1,2,2,0} - 2H_{1,1,2,3,0} \\
 & - \frac{11}{12} H_{2,0,0} + (1-x) \left[\frac{41099}{2292} - 3H_{1,2,0} - \frac{3}{2} H_{2,2,0} - \frac{1225}{9} \zeta_2 - 4H_{3,0,0} + \frac{6}{5} H_{3,0,1} - 2H_{3,0,2} \right. \\
 & - 7H_2 \zeta_2 + \frac{97}{12} H_{3,0,0} - \frac{10}{3} H_{3,0,1} - \frac{245}{12} H_{3,0,2} - 8H_{3,0,3,0} \left. \right] + (1+x) \left[\frac{4113}{18} H_{1,1} + \frac{29}{6} H_{2,1,0} - \frac{6}{12} H_{2,1,1} \right. \\
 & \left. + \frac{1}{6} H_{2,1,2} - 12H_{2,1,3} + \frac{31}{12} H_{2,1,4} + \frac{1}{2} H_{2,1,5} + \frac{61}{12} H_{2,1,6} - 4H_{2,1,7} + \frac{1}{2} H_{2,1,8} - \frac{49}{12} H_{2,1,9} - \frac{1}{2} H_{2,1,10} \right. \\
 & \left. + \frac{1}{2} H_{2,1,11} + \frac{55}{9} H_{2,1,12} + \frac{17}{18} H_{2,1,13} + \frac{49}{18} H_{2,1,14} - \frac{13}{2} H_{2,1,15} + \frac{61}{27} H_{2,1,16} + \frac{1}{2} H_{2,1,17} + \frac{92}{27} H_{2,1,18} \right. \\
 & \left. + \frac{1}{2} H_{2,1,19} + \frac{1}{2} H_{2,1,20} + \frac{1}{2} H_{2,1,21} + \frac{1}{2} H_{2,1,22} + \frac{1}{2} H_{2,1,23} + \frac{1}{2} H_{2,1,24} + \frac{1}{2} H_{2,1,25} + \frac{1}{2} H_{2,1,26} + \frac{1}{2} H_{2,1,27} \right. \\
 & \left. + \frac{1}{2} H_{2,1,28} + \frac{1}{2} H_{2,1,29} + \frac{1}{2} H_{2,1,30} + \frac{1}{2} H_{2,1,31} + \frac{1}{2} H_{2,1,32} + \frac{1}{2} H_{2,1,33} + \frac{1}{2} H_{2,1,34} + \frac{1}{2} H_{2,1,35} + \frac{1}{2} H_{2,1,36} + \frac{1}{2} H_{2,1,37} \right. \\
 & \left. + \frac{1}{2} H_{2,1,38} + \frac{1}{2} H_{2,1,39} + \frac{1}{2} H_{2,1,40} + \frac{1}{2} H_{2,1,41} + \frac{1}{2} H_{2,1,42} + \frac{1}{2} H_{2,1,43} + \frac{1}{2} H_{2,1,44} + \frac{1}{2} H_{2,1,45} + \frac{1}{2} H_{2,1,46} + \frac{1}{2} H_{2,1,47} \right. \\
 & \left. + \frac{1}{2} H_{2,1,48} + \frac{1}{2} H_{2,1,49} + \frac{1}{2} H_{2,1,50} + \frac{1}{2} H_{2,1,51} + \frac{1}{2} H_{2,1,52} + \frac{1}{2} H_{2,1,53} + \frac{1}{2} H_{2,1,54} + \frac{1}{2} H_{2,1,55} + \frac{1}{2} H_{2,1,56} + \frac{1}{2} H_{2,1,57} \right. \\
 & \left. + \frac{1}{2} H_{2,1,58} + \frac{1}{2} H_{2,1,59} + \frac{1}{2} H_{2,1,60} + \frac{1}{2} H_{2,1,61} + \frac{1}{2} H_{2,1,62} + \frac{1}{2} H_{2,1,63} + \frac{1}{2} H_{2,1,64} + \frac{1}{2} H_{2,1,65} + \frac{1}{2} H_{2,1,66} + \frac{1}{2} H_{2,1,67} \right. \\
 & \left. + \frac{1}{2} H_{2,1,68} + \frac{1}{2} H_{2,1,69} + \frac{1}{2} H_{2,1,70} + \frac{1}{2} H_{2,1,71} + \frac{1}{2} H_{2,1,72} + \frac{1}{2} H_{2,1,73} + \frac{1}{2} H_{2,1,74} + \frac{1}{2} H_{2,1,75} + \frac{1}{2} H_{2,1,76} + \frac{1}{2} H_{2,1,77} \right. \\
 & \left. + \frac{1}{2} H_{2,1,78} + \frac{1}{2} H_{2,1,79} + \frac{1}{2} H_{2,1,80} + \frac{1}{2} H_{2,1,81} + \frac{1}{2} H_{2,1,82} + \frac{1}{2} H_{2,1,83} + \frac{1}{2} H_{2,1,84} + \frac{1}{2} H_{2,1,85} + \frac{1}{2} H_{2,1,86} + \frac{1}{2} H_{2,1,87} \right. \\
 & \left. + \frac{1}{2} H_{2,1,88} + \frac{1}{2} H_{2,1,89} + \frac{1}{2} H_{2,1,90} + \frac{1}{2} H_{2,1,91} + \frac{1}{2} H_{2,1,92} + \frac{1}{2} H_{2,1,93} + \frac{1}{2} H_{2,1,94} + \frac{1}{2} H_{2,1,95} + \frac{1}{2} H_{2,1,96} + \frac{1}{2} H_{2,1,97} \right. \\
 & \left. + \frac{1}{2} H_{2,1,98} + \frac{1}{2} H_{2,1,99} + \frac{1}{2} H_{2,1,100} + \frac{1}{2} H_{2,1,101} + \frac{1}{2} H_{2,1,102} + \frac{1}{2} H_{2,1,103} + \frac{1}{2} H_{2,1,104} + \frac{1}{2} H_{2,1,105} + \frac{1}{2} H_{2,1,106} + \frac{1}{2} H_{2,1,107} \right. \\
 & \left. + \frac{1}{2} H_{2,1,108} + \frac{1}{2} H_{2,1,109} + \frac{1}{2} H_{2,1,110} + \frac{1}{2} H_{2,1,111} + \frac{1}{2} H_{2,1,112} + \frac{1}{2} H_{2,1,113} + \frac{1}{2} H_{2,1,114} + \frac{1}{2} H_{2,1,115} + \frac{1}{2} H_{2,1,116} + \frac{1}{2} H_{2,1,117} \right. \\
 & \left. + \frac{1}{2} H_{2,1,118} + \frac{1}{2} H_{2,1,119} + \frac{1}{2} H_{2,1,120} + \frac{1}{2} H_{2,1,121} + \frac{1}{2} H_{2,1,122} + \frac{1}{2} H_{2,1,123} + \frac{1}{2} H_{2,1,124} + \frac{1}{2} H_{2,1,125} + \frac{1}{2} H_{2,1,126} + \frac{1}{2} H_{2,1,127} \right. \\
 & \left. + \frac{1}{2} H_{2,1,128} + \frac{1}{2} H_{2,1,129} + \frac{1}{2} H_{2,1,130} + \frac{1}{2} H_{2,1,131} + \frac{1}{2} H_{2,1,132} + \frac{1}{2} H_{2,1,133} + \frac{1}{2} H_{2,1,134} + \frac{1}{2} H_{2,1,135} + \frac{1}{2} H_{2,1,136} + \frac{1}{2} H_{2,1,137} \right. \\
 & \left. + \frac{1}{2} H_{2,1,138} + \frac{1}{2} H_{2,1,139} + \frac{1}{2} H_{2,1,140} + \frac{1}{2} H_{2,1,141} + \frac{1}{2} H_{2,1,142} + \frac{1}{2} H_{2,1,143} + \frac{1}{2} H_{2,1,144} + \frac{1}{2} H_{2,1,145} + \frac{1}{2} H_{2,1,146} + \frac{1}{2} H_{2,1,147} \right. \\
 & \left. + \frac{1}{2} H_{2,1,148} + \frac{1}{2} H_{2,1,149} + \frac{1}{2} H_{2,1,150} + \frac{1}{2} H_{2,1,151} + \frac{1}{2} H_{2,1,152} + \frac{1}{2} H_{2,1,153} + \frac{1}{2} H_{2,1,154} + \frac{1}{2} H_{2,1,155} + \frac{1}{2} H_{2,1,156} + \frac{1}{2} H_{2,1,157} \right. \\
 & \left. + \frac{1}{2} H_{2,1,158} + \frac{1}{2} H_{2,1,159} + \frac{1}{2} H_{2,1,160} + \frac{1}{2} H_{2,1,161} + \frac{1}{2} H_{2,1,162} + \frac{1}{2} H_{2,1,163} + \frac{1}{2} H_{2,1,164} + \frac{1}{2} H_{2,1,165} + \frac{1}{2} H_{2,1,166} + \frac{1}{2} H_{2,1,167} \right. \\
 & \left. + \frac{1}{2} H_{2,1,168} + \frac{1}{2} H_{2,1,169} + \frac{1}{2} H_{2,1,170} + \frac{1}{2} H_{2,1,171} + \frac{1}{2} H_{2,1,172} + \frac{1}{2} H_{2,1,173} + \frac{1}{2} H_{2,1,174} + \frac{1}{2} H_{2,1,175} + \frac{1}{2} H_{2,1,176} + \frac{1}{2} H_{2,1,177} \right. \\
 & \left. + \frac{1}{2} H_{2,1,178} + \frac{1}{2} H_{2,1,179} + \frac{1}{2} H_{2,1,180} + \frac{1}{2} H_{2,1,181} + \frac{1}{2} H_{2,1,182} + \frac{1}{2} H_{2,1,183} + \frac{1}{2} H_{2,1,184} + \frac{1}{2} H_{2,1,185} + \frac{1}{2} H_{2,1,186} + \frac{1}{2} H_{2,1,187} \right. \\
 & \left. + \frac{1}{2} H_{2,1,188} + \frac{1}{2} H_{2,1,189} + \frac{1}{2} H_{2,1,190} + \frac{1}{2} H_{2,1,191} + \frac{1}{2} H_{2,1,192} + \frac{1}{2} H_{2,1,193} + \frac{1}{2} H_{2,1,194} + \frac{1}{2} H_{2,1,195} + \frac{1}{2} H_{2,1,196} + \frac{1}{2} H_{2,1,197} \right. \\
 & \left. + \frac{1}{2} H_{2,1,198} + \frac{1}{2} H_{2,1,199} + \frac{1}{2} H_{2,1,200} + \frac{1}{2} H_{2,1,201} + \frac{1}{2} H_{2,1,202} + \frac{1}{2} H_{2,1,203} + \frac{1}{2} H_{2,1,204} + \frac{1}{2} H_{2,1,205} + \frac{1}{2} H_{2,1,206} + \frac{1}{2} H_{2,1,207} \right. \\
 & \left. + \frac{1}{2} H_{2,1,208} + \frac{1}{2} H_{2,1,209} + \frac{1}{2} H_{2,1,210} + \frac{1}{2} H_{2,1,211} + \frac{1}{2} H_{2,1,212} + \frac{1}{2} H_{2,1,213} + \frac{1}{2} H_{2,1,214} + \frac{1}{2} H_{2,1,215} + \frac{1}{2} H_{2,1,216} + \frac{1}{2} H_{2,1,217} \right. \\
 & \left. + \frac{1}{2} H_{2,1,218} + \frac{1}{2} H_{2,1,219} + \frac{1}{2} H_{2,1,220} + \frac{1}{2} H_{2,1,221} + \frac{1}{2} H_{2,1,222} + \frac{1}{2} H_{2,1,223} + \frac{1}{2} H_{2,1,224} + \frac{1}{2} H_{2,1,225} + \frac{1}{2} H_{2,1,226} + \frac{1}{2} H_{2,1,227} \right. \\
 & \left. + \frac{1}{2} H_{2,1,228} + \frac{1}{2} H_{2,1,229} + \frac{1}{2} H_{2,1,230} + \frac{1}{2} H_{2,1,231} + \frac{1}{2} H_{2,1,232} + \frac{1}{2} H_{2,1,233} + \frac{1}{2} H_{2,1,234} + \frac{1}{2} H_{2,1,235} + \frac{1}{2} H_{2,1,236} + \frac{1}{2} H_{2,1,237} \right. \\
 & \left. + \frac{1}{2} H_{2,1,238} + \frac{1}{2} H_{2,1,239} + \frac{1}{2} H_{2,1,240} + \frac{1}{2} H_{2,1,241} + \frac{1}{2} H_{2,1,242} + \frac{1}{2} H_{2,1,243} + \frac{1}{2} H_{2,1,244} + \frac{1}{2} H_{2,1,245} + \frac{1}{2} H_{2,1,246} + \frac{1}{2} H_{2,1,247} \right. \\
 & \left. + \frac{1}{2} H_{2,1,248} + \frac{1}{2} H_{2,1,249} + \frac{1}{2} H_{2,1,250} + \frac{1}{2} H_{2,1,251} + \frac{1}{2} H_{2,1,252} + \frac{1}{2} H_{2,1,253} + \frac{1}{2} H_{2,1,254} + \frac{1}{2} H_{2,1,255} + \frac{1}{2} H_{2,1,256} + \frac{1}{2} H_{2,1,257} \right. \\
 & \left. + \frac{1}{2} H_{2,1,258} + \frac{1}{2} H_{2,1,259} + \frac{1}{2} H_{2,1,260} + \frac{1}{2} H_{2,1,261} + \frac{1}{2} H_{2,1,262} + \frac{1}{2} H_{2,1,263} + \frac{1}{2} H_{2,1,264} + \frac{1}{2} H_{2,1,265} + \frac{1}{2} H_{2,1,266} + \frac{1}{2} H_{2,1,267} \right. \\
 & \left. + \frac{1}{2} H_{2,1,268} + \frac{1}{2} H_{2,1,269} + \frac{1}{2} H_{2,1,270} + \frac{1}{2} H_{2,1,271} + \frac{1}{2} H_{2,1,272} + \frac{1}{2} H_{2,1,273} + \frac{1}{2} H_{2,1,274} + \frac{1}{2} H_{2,1,275} + \frac{1}{2} H_{2,1,276} + \frac{1}{2} H_{2,1,277} \right. \\
 & \left. + \frac{1}{2} H_{2,1,278} + \frac{1}{2} H_{2,1,279} + \frac{1}{2} H_{2,1,280} + \frac{1}{2} H_{2,1,281} + \frac{1}{2} H_{2,1,282} + \frac{1}{2} H_{2,1,283} + \frac{1}{2} H_{2,1,284} + \frac{1}{2} H_{2,1,285} + \frac{1}{2} H_{2,1,286} + \frac{1}{2} H_{2,1,287} \right. \\
 & \left. + \frac{1}{2} H_{2,1,288} + \frac{1}{2} H_{2,1,289} + \frac{1}{2} H_{2,1,290} + \frac{1}{2} H_{2,1,291} + \frac{1}{2} H_{2,1,292} + \frac{1}{2} H_{2,1,293} + \frac{1}{2} H_{2,1,294} + \frac{1}{2} H_{2,1,295} + \frac{1}{2} H_{2,1,296} + \frac{1}{2} H_{2,1,297} \right. \\
 & \left. + \frac{1}{2} H_{2,1,298} + \frac{1}{2} H_{2,1,299} + \frac{1}{2} H_{2,1,300} + \frac{1}{2} H_{2,1,301} + \frac{1}{2} H_{2,1,302} + \frac{1}{2} H_{2,1,303} + \frac{1}{2} H_{2,1,304} + \frac{1}{2} H_{2,1,305} + \frac{1}{2} H_{2,1,306} + \frac{1}{2} H_{2,1,307} \right. \\
 & \left. + \frac{1}{2} H_{2,1,308} + \frac{1}{2} H_{2,1,309} + \frac{1}{2} H_{2,1,310} + \frac{1}{2} H_{2,1,311} + \frac{1}{2} H_{2,1,312} + \frac{1}{2} H_{2,1,313} + \frac{1}{2} H_{2,1,314} + \frac{1}{2} H_{2,1,315} + \frac{1}{2} H_{2,1,316} + \frac{1}{2} H_{2,1,317} \right. \\
 & \left. + \frac{1}{2} H_{2,1,318} + \frac{1}{2} H_{2,1,319} + \frac{1}{2} H_{2,1,320} + \frac{1}{2} H_{2,1,321} + \frac{1}{2} H_{2,1,322} + \frac{1}{2} H_{2,1,323} + \frac{1}{2} H_{2,1,324} + \frac{1}{2} H_{2,1,325} + \frac{1}{2} H_{2,1,326} + \frac{1}{2} H_{2,1,327} \right. \\
 & \left. + \frac{1}{2} H_{2,1,328} + \frac{1}{2} H_{2,1,329} + \frac{1}{2} H_{2,1,330} + \frac{1}{2} H_{2,1,331} + \frac{1}{2} H_{2,1,332} + \frac{1}{2} H_{2,1,333} + \frac{1}{2} H_{2$$

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- Precision predictions for particle physics at colliders
- Efficient method for scattering cross sections
- Improvements in understanding of quantum field theory

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Riemann zeta values

- Basis of (transcendental) numbers for Feynman diagram calculations
- Generalization to cases of many variables \longrightarrow multiple polylogarithms

Computer algebra implementations

- Symbolic summation for harmonic sums $S_{m_1, \dots, m_k}(n)$
 - SUMMER package (using FORM) Vermaseren '98
- Symbolic summation for generalized sums $S(n; m_1, \dots, m_k; x_1, \dots, x_k)$
 - XSUMMER package (using FORM) S.M., Uwer '05
 - *nestedsums* (using GiNaC) Weinzierl '02