Symbolic Summation, Riemann Zeta Values, and Quantum Field Theory

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Plan

Motivation

- Quantum field theory
  - cutting edge calculations in particle physics

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Task

- Given expression $g(n)$ (depending on n) find expression $f(n)$, such that

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  - polynomial, hypergeometric and harmonic sums and beyond
- Riemann zeta values and multiple polylogarithms

Elementary Particle Physics

- Standard Model
  - successfully description for interactions of elementary particles
  - gauge theory $SU(3) \times SU(2) \times U(1)$ (internal symmetries)

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  - cross sections from perturbative calculations in quantum field theory (Feynman diagrams)

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  - Large Hadron Collider (LHC) at CERN (Genava)
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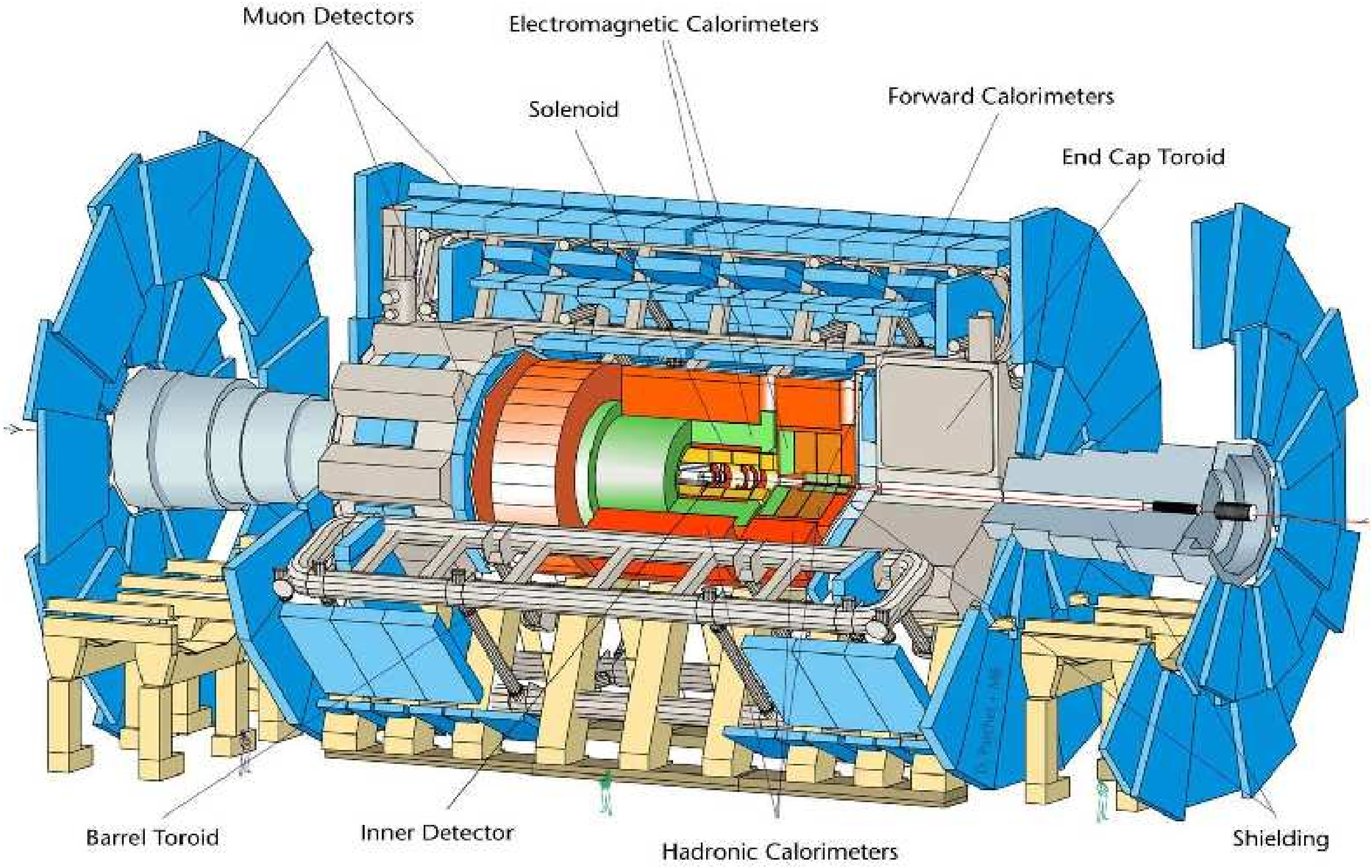
- Discovery searches for
  - **Higgs** production and **new physics** phenomena

The Large Hadron Collider

The ATLAS Detector



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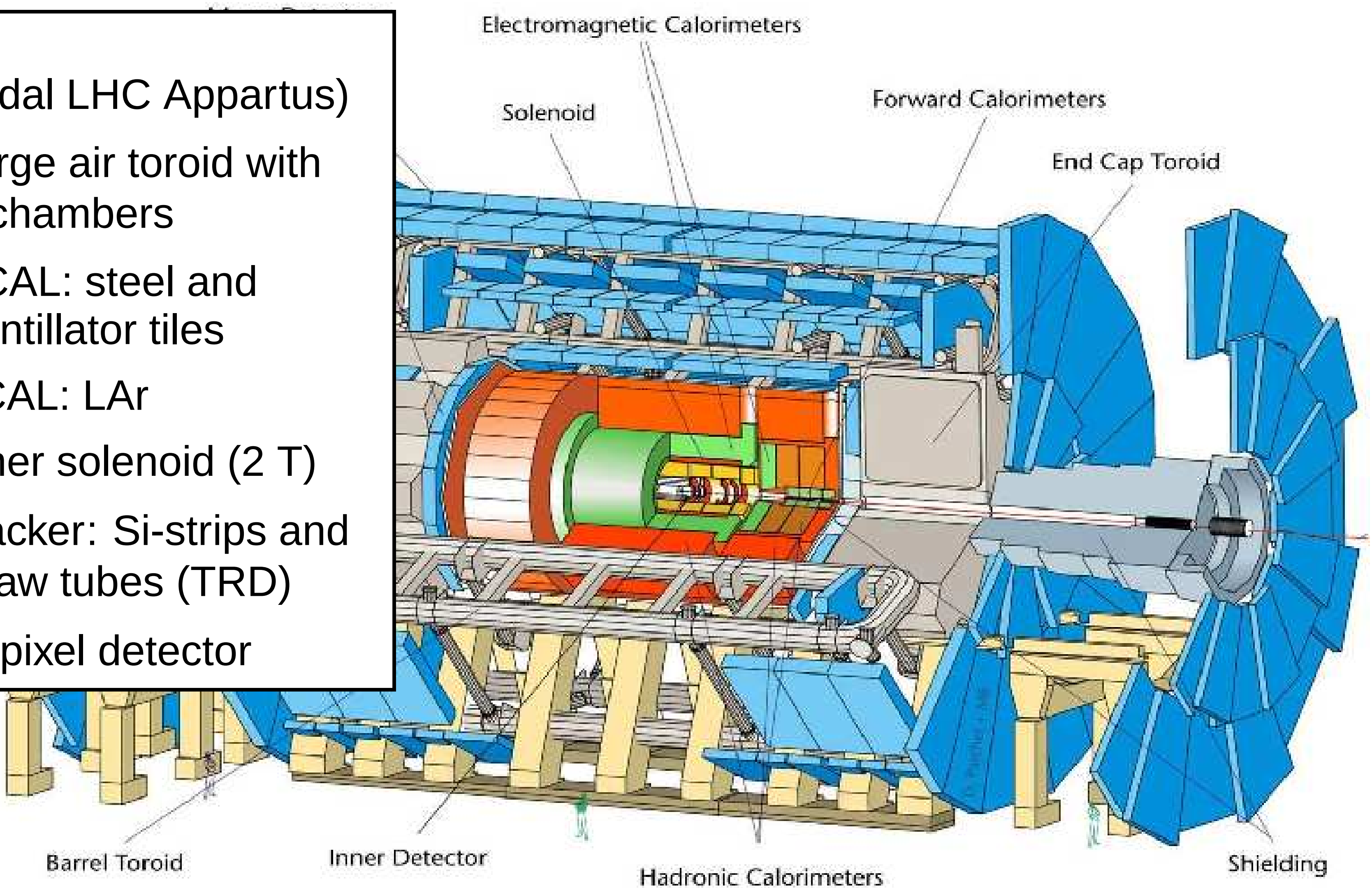
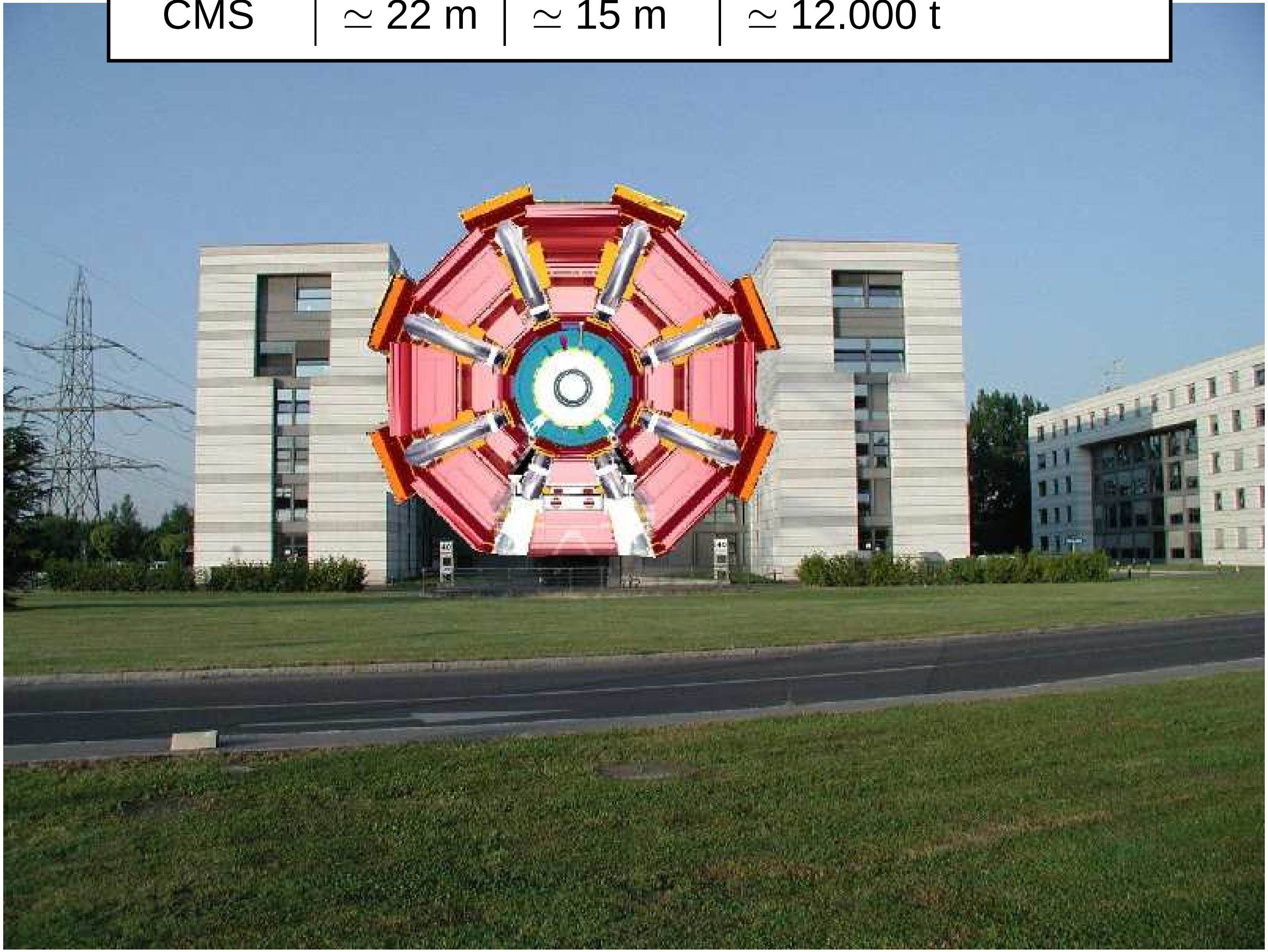


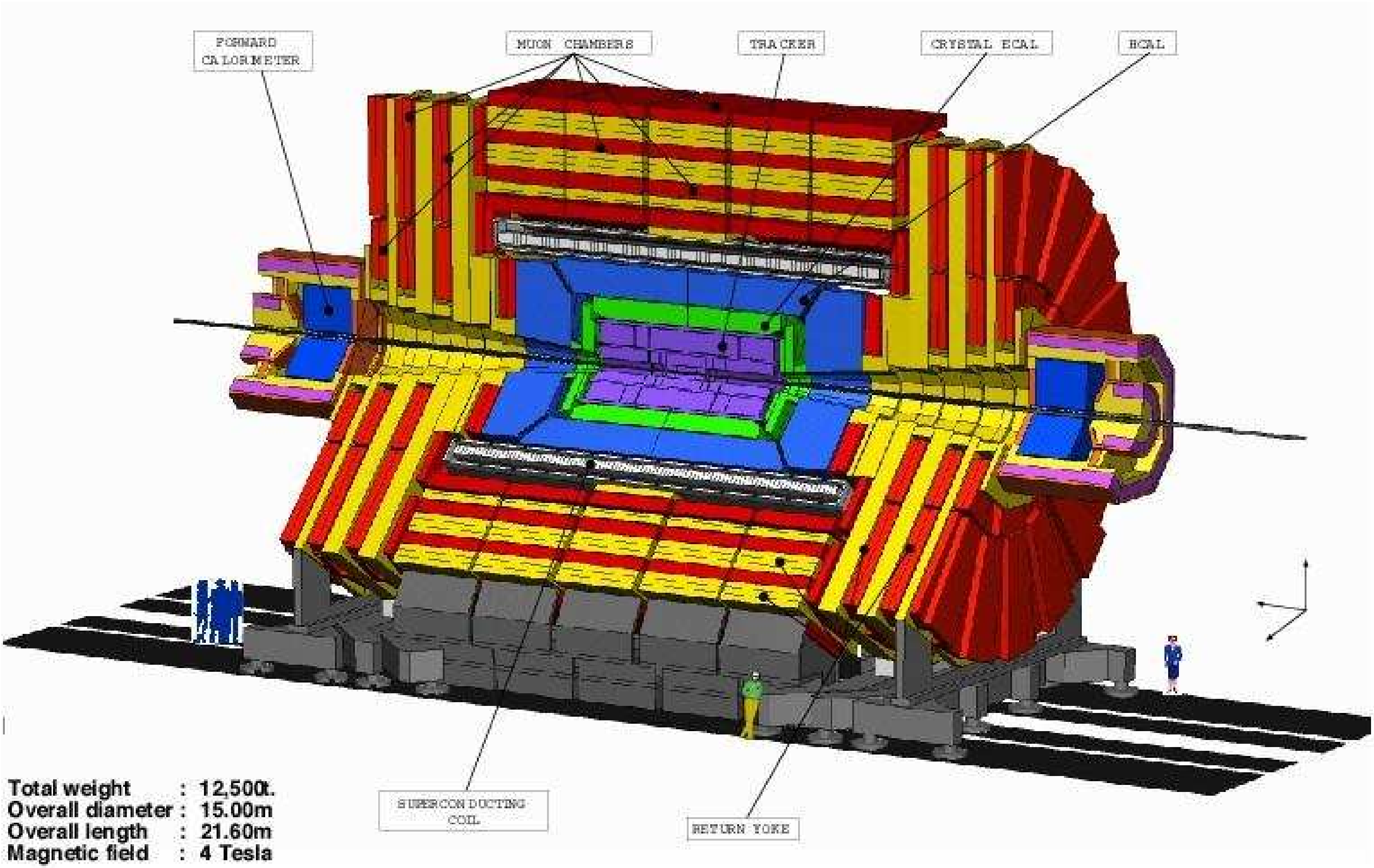
ATLAS
(A Toroidal LHC Appartus)

- Large air toroid with μ-chambers
- HCAL: steel and scintillator tiles
- ECAL: LAr
- Inner solenoid (2 T)
- Tracker: Si-strips and straw tubes (TRD)
- Si-pixel detector

	length	diameter	weight
ATLAS	$\simeq$ 46 m	$\simeq$ 25 m	$\simeq$ 7.000 t
CMS	$\simeq$ 22 m	$\simeq$ 15 m	$\simeq$ 12.000 t



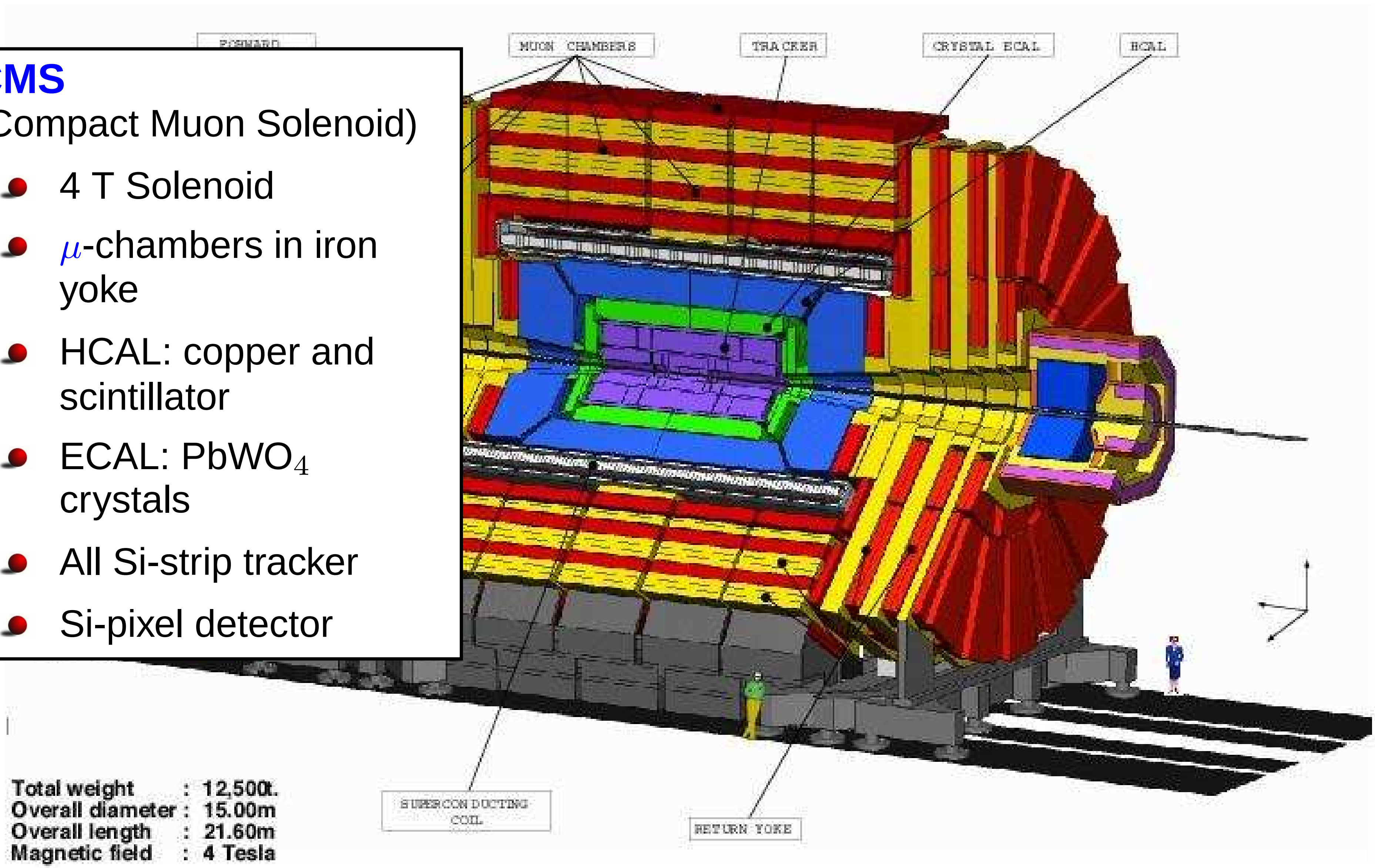
The CMS Detector



The CMS Detector

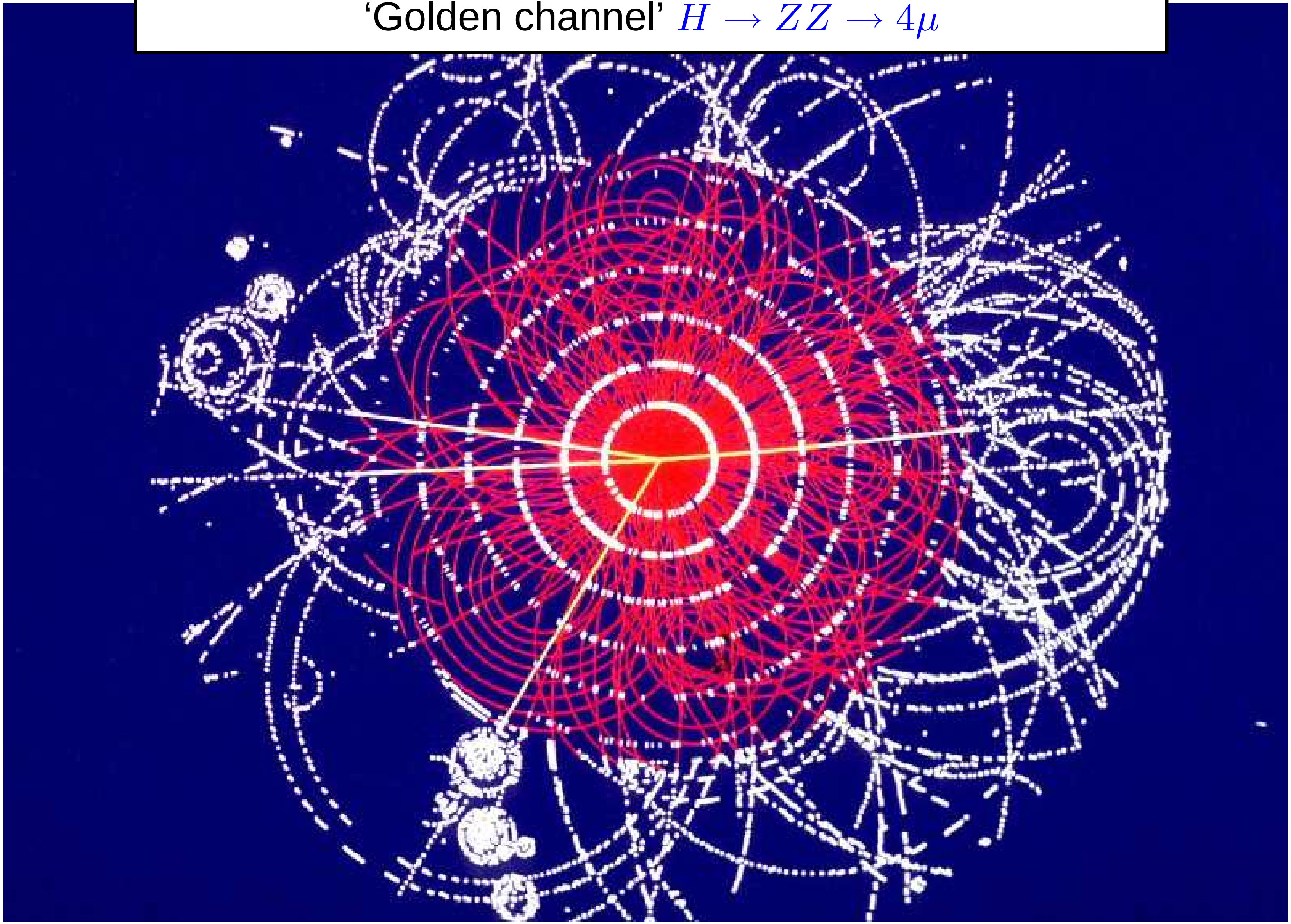
CMS
(Compact Muon Solenoid)

- 4 T Solenoid
- μ-chambers in iron yoke
- HCAL: copper and scintillator
- ECAL: $PbWO_4$ crystals
- All Si-strip tracker
- Si-pixel detector



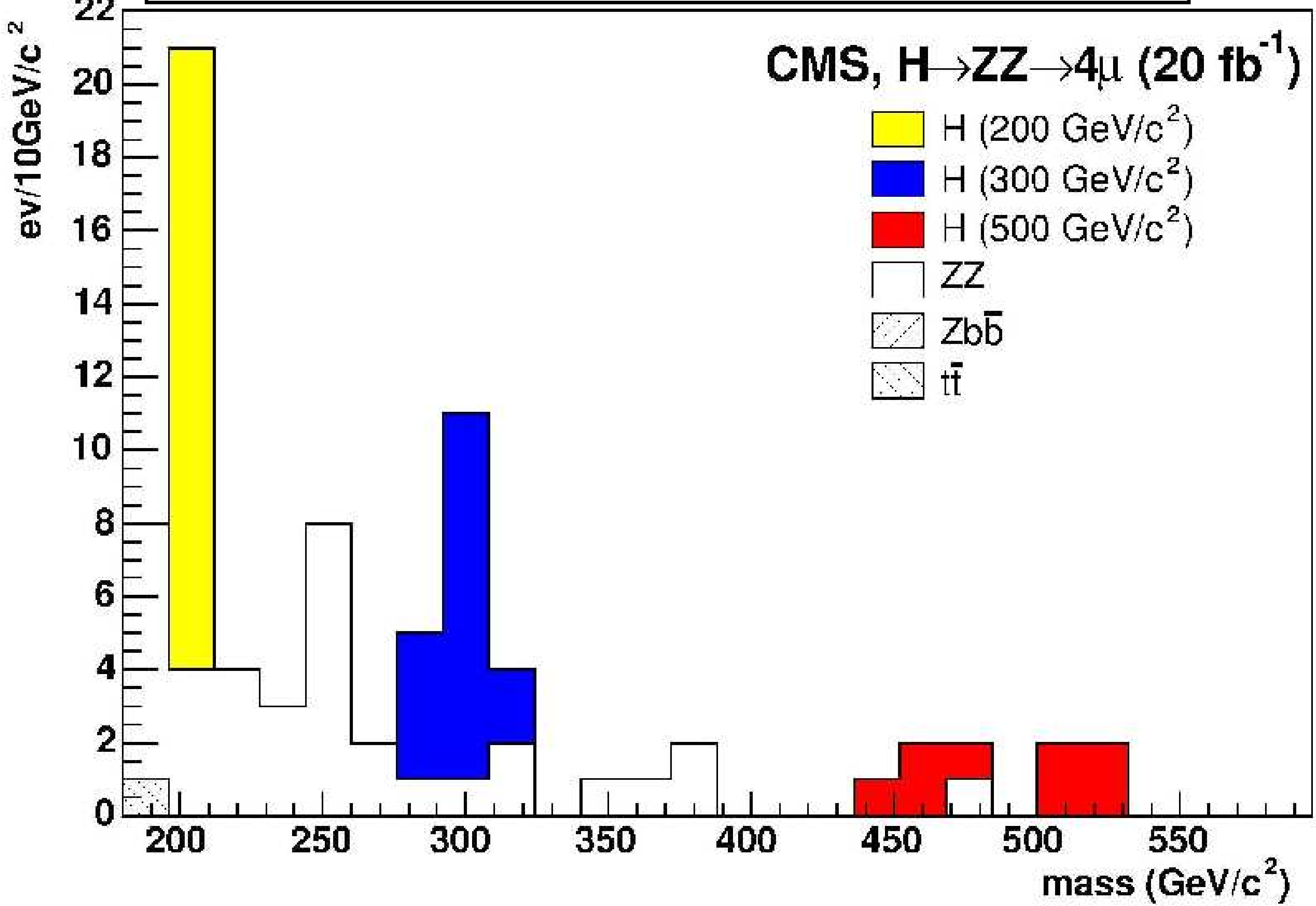
Higgs search

'Golden channel' $H \to ZZ \to 4\mu$



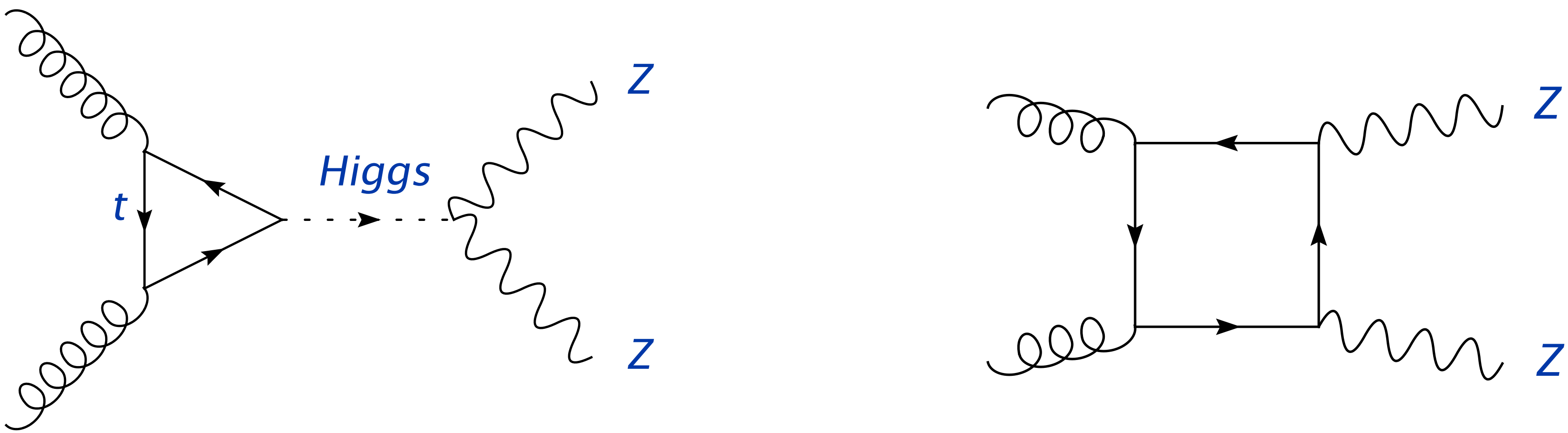
Higgs mass search

Sensitivity study for various Higgs masses



Quantum field theory

- Perturbative predictions for cross sections of scattering processes
  - series expansion in coupling constant



- Example: possible diagrams for $gg \to ZZ$
  - Higgs signal (left) and QCD background (right)

Perturbation theory in a nut shell

- Feynman diagrams for a given process (tensor integrals, Lorentz structure, . . .)
- Feynman integral (l-loops, n-propagators)

$$I(D;\nu_1,\dots,\nu_n) = \int d^Dp_1 \,\dots\, d^Dp_l \,\frac{1}{(p_1^2)^{\nu_1}\,\dots\,(p_n^2)^{\nu_n}}$$

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- Dimensional regularization
  Bollini, Giambiagi '72; Ashmore '72; Cicuta, Montaldi '72; 't Hooft, Veltman '72
  - all momenta and polarisation vectors in $D = 4 - 2\epsilon$ dimensions
  - manifest gauge invariance

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  - manifest gauge invariance
- Classification
  - topology, n-point function (number of loops, legs)
  - scales (number of non-vanishing scalar products of momenta and masses)

- Renormalization
  - finite results for physical observables O
  - bare quantities O^{bare} as Laurent series in ϵ
  - multiplicative renormalization $O^{\mathrm{bare}} = Z O^{\mathrm{ren}}$ ($\overline{\mathrm{MS}}$-scheme) with constants $Z = \sum_i \alpha^i \frac{1}{\epsilon^i} \gamma^{(i)}$ (schematically)

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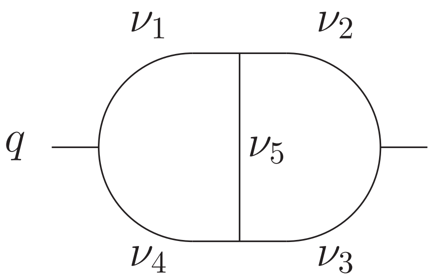
Difficulty

- Calculation of Feynman (loop) integrals

From Feynman integrals to sums

- Example: massless two-loop two-point function

$$\mathrm{T1} \;=\; \int d^D p_1 \, d^D p_2 \, \frac{1}{(p_1^2)^{\nu_1}\,(p_2^2)^{\nu_2}\,(p_3^2)^{\nu_3}\,(p_4^2)^{\nu_4}\,(p_5^2)^{\nu_5}}$$



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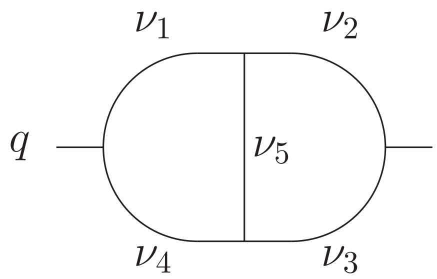

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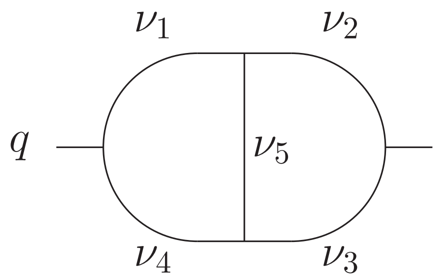

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- History (all powers of propagators $\nu_i = 1 + \epsilon$)
  - ϵ^3, ϵ^4 (ζ_6, ζ_7) Kazakov ‘84,Kazakov ‘85
  - ϵ^5 $(\zeta_8, \zeta_{6,2})$ Broadhurst ‘86
  - ϵ^6 (ζ_9) Barfoot, Broadhurst ‘88
  - ϵ^7,ϵ^8 $(\zeta_{10}, \zeta_{8,2}, \zeta_{11}, \zeta_{8,2,1})$ Broadhurst, Gracey, Kreimer ’96
  - ϵ^9 $(\zeta_{12}, \zeta_{10,2}, \zeta_{8,2,1,1})$ Broadhurst ’02

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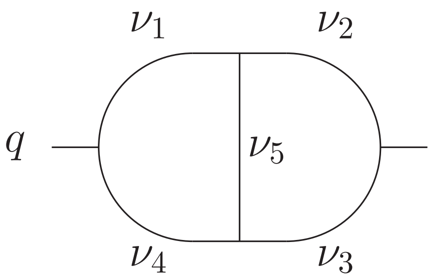

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  - ϵ^{10},ϵ^{11}, ϵ^{12},ϵ^{13} Vermaseren '03 [based on Bierenbaum, Weinzierl '03]

Solution

- All order solution: multiple zeta values are indeed sufficient for massless two-loop two-point function Bierenbaum, Weinzierl '03
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The two-loop integral can therefore be written as a sum of three terms,

$$\hat{I}^{(2,5)} = T_{(1)} + T_{(2)} + T_{(3)}, \tag{29}$$

where each term is obtained by taking the residues of one Gamma function from the set $\Gamma(-\sigma)$, $\Gamma(-\sigma+m-\varepsilon-\nu_{35})$ and $\Gamma(-\sigma-\tau-m+\varepsilon+\nu_{14})$. Explicitly,

$$T_{(k)} = c\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{1}{2\pi i}\int\limits_{\gamma_2-i\infty}^{\gamma_2+i\infty} d\tau\,\frac{\Gamma(-\tau)\Gamma(-\tau+m-\varepsilon-\nu_{25})\Gamma(\tau+m-\varepsilon-\nu_1)}{\Gamma(-\tau+\nu_1)}H_{(k)}, \tag{30}$$

with

$$\begin{aligned} H_{(1)} &= \frac{\Gamma(-n+m-\varepsilon-\nu_{35})\Gamma(n+m-\varepsilon-\nu_4)}{\Gamma(-n+\nu_4)} \\ &\quad\times\frac{\Gamma(-\tau-n-m+\varepsilon+\nu_{14})\Gamma(\tau+n-m+\varepsilon+\nu_{235})\Gamma(\tau+n+\nu_5)}{\Gamma(\tau+n+2m-2\varepsilon-\nu_{14})}, \\ H_{(2)} &= \frac{\Gamma(-n-m+\varepsilon+\nu_{35})\Gamma(n+2m-2\varepsilon-\nu_{345})}{\Gamma(-n-m+\varepsilon+\nu_{345})} \end{aligned}$$

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Symbolic Summation

Polynomial summation

- Examples

$$\sum_{i=0}^{n-1} i = \frac{1}{2} n(n-1)$$

$$\sum_{i=0}^{n-1} i^2 = \frac{1}{6} n(n-1)(2n-1)$$

$$\sum_{i=0}^{n-1} i^3 = \frac{1}{4} n^2 (n-1)^2$$

$$\sum_{i=0}^{n-1} i^4 = \frac{1}{30} n(n-1)(2n-1)(3n^2-3n-1)$$

Polynomial summation

- Define rising factorials as $f^{\overline{m}} = f(x)f(x+1)\dots f(x+m-1)$ (also known as Pochhammer symbols $(x)_m$)
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- Difference operator acts on falling factorials
  - Differential operator D acts in continuum as $D(x^m) = mx^{m-1}$

$$\Delta(x^{\underline{m}}) = mx^{\underline{m-1}} \qquad \sum_{i=0}^{n-1} i^{\underline{m}} = \frac{1}{m+1} n^{\underline{m+1}}$$

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- Conversion of polynomial powers x^m (decomposition with Stirling numbers of second kind $\left\{ \begin{matrix} m \\ i \end{matrix} \right\}$)

$$x^m = \sum_{i=0}^{m} \left\{ \begin{matrix} m \\ i \end{matrix} \right\} x^{\underline{i}}$$

  - Stirling numbers of second kind denote # of ways to partition n things in k non-empty sets

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$$\sum_{i=0}^{n-1} i = \sum_{i=0}^{n-1} i^{\underline{1}} = \frac{1}{2} n^{\underline{2}} = \frac{1}{2} n(n-1)$$

Hypergeometric summation

Definition

- Hypergeometric function ${}_mF_n$

$$ {}_mF_n\left(\begin{array}{c} a_1,\ldots,a_m \\ b_1,\ldots,b_n \end{array} \Bigg| z \right) = \sum_{i\geq 0} \frac{a_1^{\bar{i}}\ldots a_m^{\bar{i}}}{b_1^{\bar{i}}\ldots b_n^{\bar{i}}}\,\frac{z^i}{i!} $$

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Examples

$$ {}_0F_0\left(\, \middle| z \right) = \sum_{i\geq 0} \frac{z^i}{i!} = \mathrm{e}^z $$

$$ {}_2F_1\left(\begin{array}{c} a,1 \\ 1 \end{array} \middle| z \right) = \sum_{i\geq 0} a^{\overline{i}} \frac{z^i}{i!} = \frac{1}{(1-z)^a} $$

$$ {}_2F_1\left(\begin{array}{c} 1,1 \\ 2 \end{array} \middle| z \right) = z\sum_{i\geq 0} \frac{1^{\overline{i}}1^{\overline{i}}}{2^{\overline{i}}} \frac{z^i}{i!} = -\ln(1-z) $$

Ratios

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- Given a hypergeometric term g, is there hypergeometric term f such that $\Delta f = g$?

$$f_{n+1} - f_n = g_n$$

Gospers algorithm

- Gospers algorithm for indefinite hypergeometric summation determines f_n from a given recursion

$$f_n = f_{n-1} + g_{n-1} = f_{n-2} + g_{n-1} + g_{n-2} = \cdots = f_0 + \sum_{k=0}^{n-1} g_k$$

  - Idea: recursive algorithm; telescoping

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- WZ algorithm
  - definite hypergeometric summation
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Definite vs. indefinite summation

- Examples

$$\sum_k \binom{n}{k} = \sum_k \binom{n}{k} \qquad \sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Harmonic summation

- Harmonic sums $S_{m_1,\ldots,m_k}(n)$

  Gonzalez-Arroyo, Lopez, Ynduráin '79; Vermaseren '98; S.M., Uwer, Weinzierl '01

  - recursive definition $S_{\pm m_1,\ldots,m_k}(n) = \sum_{i=1}^{n} \frac{(\pm 1)^i}{i^{m_1}} S_{m_2,\ldots,m_k}(i)$

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- Particle physics
  - dimensional regularization $D = 4 - 2\epsilon$ requires expansion of the Gamma-function around positive integers values ($n \geq 0$)

$$\frac{\Gamma(n+1+\epsilon)}{\Gamma(1+\epsilon)} \quad = \quad \Gamma(n+1)\,\exp\left(-\sum_{k=1}^{\infty} \epsilon^k \,\frac{(-1)^k}{k}\, S_k(n)\right)$$

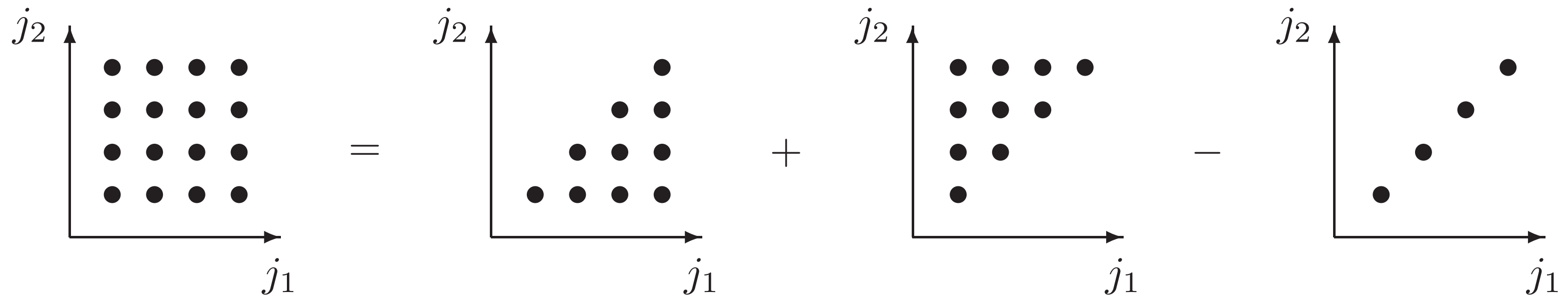
Algorithms for harmonic sums

- Multiplication (Hopf algebra)
  - basic formula (recursion)

$$
\begin{aligned}
S_{m_1,\ldots,m_k}(n) \times S_{m'_1,\ldots,m'_l}(n) \quad = \quad & \sum_{j_1=1}^{n} \frac{1}{j_1^{m_1}} S_{m_2,\ldots,m_k}(j_1) S_{m'_1,\ldots,m'_l}(j_1) \\
& + \sum_{j_2=1}^{n} \frac{1}{j_2^{m'_1}} S_{m_1,\ldots,m_k}(j_2) S_{m'_2,\ldots,m'_l}(j_2) \\
& - \sum_{j=1}^{n} \frac{1}{j^{m_1+m'_1}} S_{m_2,\ldots,m_k}(j) S_{m'_2,\ldots,m'_l}(j)
\end{aligned}
$$

- Proof uses decomposition

$$
\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \quad = \quad \sum_{i=1}^{n}\sum_{j=1}^{i} a_{ij} + \sum_{j=1}^{n}\sum_{i=1}^{j} a_{ij} - \sum_{i=1}^{n} a_{ii}
$$



Algorithms for harmonic sums (cont'd)

- Convolution (sum over $n-j$ and j)

$$\sum_{j=1}^{n-1} \frac{1}{j^{m_1}} S_{m_2,\ldots,m_k}(j) \frac{1}{(n-j)^{n_1}} S_{n_2,\ldots,n_l}(n-j)$$

- Conjugation

$$-\sum_{j=1}^{n} \binom{n}{j} (-1)^j \frac{1}{j^{m_1}} S_{m_2,\ldots,m_k}(j)$$

- Binomial convolution (sum over binomial, $n-j$ and j)

$$-\sum_{j=1}^{n-1} \binom{n}{j} (-1)^j \frac{1}{j^{m_1}} S_{m_2,\ldots,m_k}(j) \frac{1}{(n-j)^{n_1}} S_{n_2,\ldots,n_l}(n-j)$$

Multiple scales

- Generalized sums $S(n; m_1, ..., m_k; x_1, ..., x_k)$
  - recursive definition

$$S(n; m_1, ..., m_k; x_1, ..., x_k) = \sum_{i=1}^{n} \frac{x_1^i}{i^{m_1}} S(i; m_2, ..., m_k; x_2, ..., x_k)$$

  - multiple scales $x_1, ..., x_k$
  - depth k, weight $w = m_1 + ... + m_k$

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  - depth k, weight $w = m_1 + ... + m_k$
- Special cases

$S(\infty; m_1, ..., m_k; x_1, ..., x_k) \quad \to \quad \mathrm{Li}_{m_k,...,m_1}(x_k, ..., x_1)$

multiple polylogarithms

Goncharov '98; Borwein, Bradley, Broadhurst, Lisonek '99

$S(\infty; m_1, ..., m_k; x, 1, ..., 1) \quad \to \quad H_{m_1,...,m_k}(x)$

harmonic polylogarithms

Remiddi, Vermaseren '98

$S(n; m_1, ..., m_k; 1, ..., 1) \quad \to \quad S_{m_1,...,m_k}(n)$

Euler-Zagier sums

Algorithms for nested sums

- Same structures as for harmonic sums, in particular
  - multiplication

$$S(n; m_1, ...; x_1, ...) \times S(n; m_1', ...; x_1', ...)$$

  - convolution
  - conjugation
  - binomial convolution
- Recursive algorithms analogous to harmonic sums solve multiple nested sums

Higher transcendental functions

- Expansion of higher transcendental functions in small parameter
  - expansion parameter ϵ occurs in the rising factorials (Pochhammer symbols)
- Hypergeometric function

$$ {}_2F_1(a,b;c,x_0) = \sum_{i=0}^{\infty} \frac{a^{\overline{i}} b^{\overline{i}}}{c^{\overline{i}}} \frac{x_0^i}{i!} $$

- First Appell function

$$ F_1(a,b_1,b_2;c;x_1,x_2) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{a^{\overline{m_1+m_2}} b_1^{\overline{m_1}} b_2^{\overline{m_2}}}{c^{\overline{m_1+m_2}}} \frac{x_1^{m_1}}{m_1!} \frac{x_2^{m_2}}{m_2!} $$

- Second Appell function

$$ F_2(a,b_1,b_2;c_1,c_2;x_1,x_2) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{a^{\overline{m_1+m_2}} b_1^{\overline{m_1}} b_2^{\overline{m_2}}}{c_1^{\overline{m_1}} c_2^{\overline{m_2}}} \frac{x_1^{m_1}}{m_1!} \frac{x_2^{m_2}}{m_2!} $$

Harmonic polylogarithms

- Harmonic polylogarithms $H_{m_1,\ldots,m_k}(x)$
  Remiddi, Vermaseren '99
  - physical quantities in momentum (x)-space

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      $H_0(x) = \ln x\,, \quad H_1(x) = -\ln(1-x)\,, \quad H_{-1}(x) = \ln(1+x)$
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$$H_{m_1,\ldots,m_w}(x) = \int_0^x dz\ f_{m_1}(z)\ H_{m_2,\ldots,m_w}(z)$$

$$f_0(x) = \frac{1}{x}\,, \qquad f_1(x) = \frac{1}{1-x}\,, \qquad f_{-1}(x) = \frac{1}{1+x}$$

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      $$H_{m_1,\ldots,m_w}(x) = \int_0^x dz\; f_{m_1}(z)\; H_{m_2,\ldots,m_w}(z)$$
      $$f_0(x) = \frac{1}{x}\,, \qquad f_1(x) = \frac{1}{1-x}\,, \qquad f_{-1}(x) = \frac{1}{1+x}$$
- (Inverse) Mellin transformation
  $$\tilde{f}(N) \;=\; \int_0^1 dx\;x^{\,N}\,f(x)$$
  - unique mapping $\dfrac{H_{m_1,\ldots,m_w}(x)}{(1\pm x)} \longleftrightarrow S_{n_1,\ldots,n_{w+1}}(N)$

Beyond

Binomial and inverse binomial sums

- Feynman integrals with massive propagators lead to structures

$$\chi \quad = \quad \frac{1-\sqrt{1-4x}}{1+\sqrt{1-4x}}, \qquad x = \frac{\chi}{(1+\chi)^2}, \qquad \sqrt{1-4x} = \frac{1-\chi}{1+\chi}.$$

- Mellin-Barnes representations lead to new classes of sums
  - two-loop two-point functions Davydychev, Kalmykov '02

$$\sum_{k=1}^{\infty} \binom{2k}{k} \frac{x^k}{k^a}$$

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- Roots of unity $\exp\left(\frac{2\pi i p}{q}\right)$ from expansions of transcendental functions around rational values Weinzierl '04

Applications in particle physics

Feynman integrals from deep-inelastic scattering

- Two-scale problem
  - virtuality of the exchanged gauge boson Q^2
  - scalar product of boson's and nucleon's momenta, $2P \cdot Q$
  - Bjorken's variable $x = Q^2/(2P \cdot Q)$ with $0 \leq x \leq 1$ (dimensionless)

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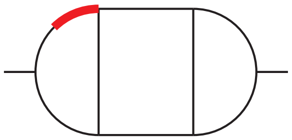

$$= \int \prod_{n}^{3} d^D p_n \, \frac{1}{(P - p_1)^2} \, \frac{1}{p_1^2 \ldots p_8^2}$$

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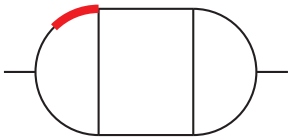
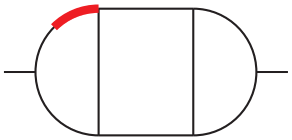


$$= \int \prod_n^3 d^D p_n \frac{1}{(P - p_1)^2} \frac{1}{p_1^2 \dots p_8^2}$$

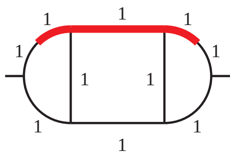
- N-th moment: coefficient of $(2P \cdot Q)^N$



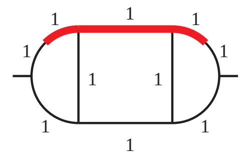
$$= \frac{(2\, P \cdot Q)^N}{(Q^2)^{N+\alpha}}\, C_N$$

Difference equations

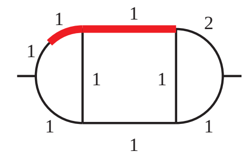
- Single-step difference equation in N
  - extremely simple example



$$= -\frac{N+3+3\varepsilon}{N+2}\,\frac{2\,p\cdot q}{q^2}$$



$$+\frac{2}{N+2}$$



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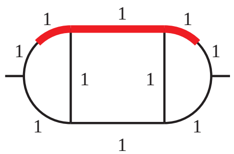

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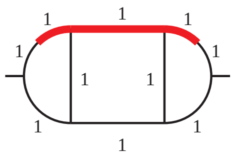

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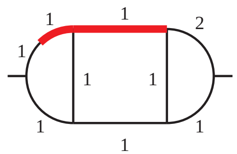

- Formal equation

$$\mathbf{I(N)} \quad = \quad -\frac{N+3+3\epsilon}{N+2}\,\mathbf{I(N-1)} + \frac{2}{N+2}\,\mathbf{G(N)}$$

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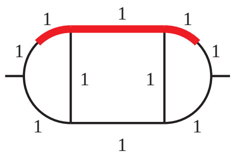
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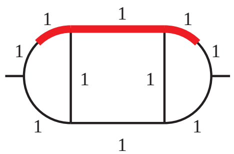
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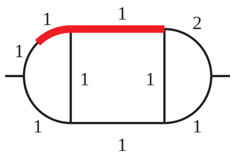

- Formal equation, formal solution

$$\mathbf{I(N)} \quad = \quad (-1)^N \frac{\prod_{j=1}^{N}(j+3+3\epsilon)}{\prod_{j=1}^{N}(j+2)}\mathbf{I(0)} + (-1)^N \sum_{\mathbf{i=1}}^{\mathbf{N}} (-1)^j \frac{\prod_{j=i+1}^{N}(j+3+3\epsilon)}{\prod_{j=i}^{N}(j+2)}\mathbf{G(i)}$$

Difference equations

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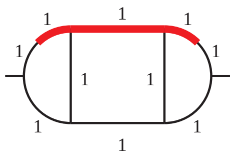
  - extremely simple example



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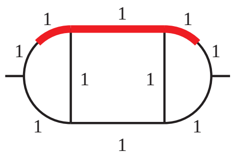

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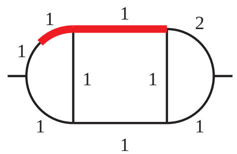

- Formal equation, formal solution, input to solution

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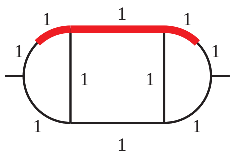
$$\mathbf{I(0)} = -\frac{2}{3}\frac{1}{\epsilon^2} + \frac{23}{3}\frac{1}{\epsilon} - 42$$

$$\mathbf{G(i)} = \frac{(-1)^i}{\epsilon^2}\frac{2}{3}\left(\frac{S_1(i+2)}{i+2} - \frac{S_{1,2}(i)}{2} - \frac{S_2(i+1)}{2(i+1)} - S_2(i) - \frac{1}{(i+1)^2} - \frac{1}{(i+2)^2}\right) + \dots$$

Difference equations

- Single-step difference equation in N
  - extremely simple example



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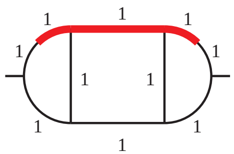

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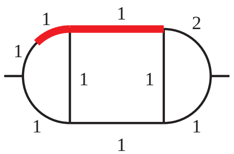

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- **Upshot**
  - automatic build-up of **nested sums**
  - efficient implementation in FORM

I(N) =

```
sign(N)*ep^-2 * ( 4/3*S(R(1),1 + N)*den(1 + N) + 8/3*S(R(1),1 + N)*den(
  1 + N)^2 + 4/3*S(R(1),2 + N)*den(2 + N) + 4/3*S(R(1),2 + N)*den(2 + N)^2 + 4/3*
  S(R(1),N) + 2/3*S(R(1,2),N) + 2/3*S(R(2),1 + N)*den(1 + N) + 2/3*S(R(2),2 + N)*
  den(2 + N) - 2*S(R(2),N) - 4/3*S(R(2),N)*N + 4*S(R(2,1),N) + 4/3*S(R(2,1),N)*N
   - 6*S(R(3),N) - 2*S(R(3),N)*N - 8/3*den(1 + N)^2 - 4*den(1 + N)^3 - 4/3*den(2
   + N)^2 - 2*den(2 + N)^3 )

+ sign(N)*ep^-1 * (  - 16*S(R(1),1 + N)*den(1 + N) - 88/3*S(R(1),1 + N)*
  den(1 + N)^2 - 20/3*S(R(1),1 + N)*den(1 + N)^3 - 16*S(R(1),2 + N)*den(2 + N) -
  44/3*S(R(1),2 + N)*den(2 + N)^2 - 10/3*S(R(1),2 + N)*den(2 + N)^3 - 20*S(R(1),N)
   + 8/3*S(R(1,1),1 + N)*den(1 + N) + 8/3*S(R(1,1),1 + N)*den(1 + N)^2 + 8/3*S(R(1
  ,1),2 + N)*den(2 + N) + 8/3*S(R(1,1),N) + 10/3*S(R(1,1,2),N) + 10/3*S(R(1,2),1
   + N)*den(1 + N) + 10/3*S(R(1,2),2 + N)*den(2 + N) - 16*S(R(1,2),N) - 4*S(R(1,2)
  ,N)*N + 14*S(R(1,2,1),N) + 4*S(R(1,2,1),N)*N - 24*S(R(1,3),N) - 6*S(R(1,3),N)*N
   - 58/3*S(R(2),1 + N)*den(1 + N) - 40/3*S(R(2),1 + N)*den(1 + N)^2 - 46/3*S(R(2)
  ,2 + N)*den(2 + N) - 6*S(R(2),2 + N)*den(2 + N)^2 + 56/3*S(R(2),N) + 20*S(R(2),N
  )*N + 10*S(R(2,1),1 + N)*den(1 + N) + 6*S(R(2,1),2 + N)*den(2 + N) - 134/3*S(R(2
  ,1),N) - 56/3*S(R(2,1),N)*N + 16/3*S(R(2,1,1),N) + 8/3*S(R(2,1,1),N)*N - 62/3*S(
  R(2,2),N) - 22/3*S(R(2,2),N)*N - 18*S(R(3),1 + N)*den(1 + N) - 12*S(R(3),2 + N)*
  den(2 + N) + 76*S(R(3),N) + 100/3*S(R(3),N)*N - 10*S(R(3,1),N) - 10/3*S(R(3,1),N
  )*N + 36*S(R(4),N) + 12*S(R(4),N)*N + 32*den(1 + N)^2 + 164/3*den(1 + N)^3 + 24*
  den(1 + N)^4 + 16*den(2 + N)^2 + 82/3*den(2 + N)^3 + 12*den(2 + N)^4 )

+ sign(N) * ( 100*S(R(1),1 + N)*den(1 + N) + 168*S(R(1),1 + N)*den(1 + N)
  ^2 + 268/3*S(R(1),1 + N)*den(1 + N)^3 - 16/3*S(R(1),1 + N)*den(1 + N)^4 + 100*S(
  R(1),2 + N)*den(2 + N) + 84*S(R(1),2 + N)*den(2 + N)^2 + 134/3*S(R(1),2 + N)*
  den(2 + N)^3 - 8/3*S(R(1),2 + N)*den(2 + N)^4 + 160*S(R(1),N) - 32*S(R(1,1),1 +
  N)*den(1 + N) - 80/3*S(R(1,1),1 + N)*den(1 + N)^2 - 20/3*S(R(1,1),1 + N)*den(1
   + N)^3 - 32*S(R(1,1),2 + N)*den(2 + N) - 4/3*S(R(1,1),2 + N)*den(2 + N)^2 - 10/
  3*S(R(1,1),2 + N)*den(2 + N)^3 - 40*S(R(1,1),N) + 4/3*S(R(1,1,1),1 + N)*den(1 +
  N) - 40/3*S(R(1,1,1),1 + N)*den(1 + N)^2 + 4/3*S(R(1,1,1),2 + N)*den(2 + N) - 44/
  3*S(R(1,1,1),2 + N)*den(2 + N)^2 + 4/3*S(R(1,1,1),N) + 38/3*S(R(1,1,1,2),N) + 38/
  3*S(R(1,1,2),1 + N)*den(1 + N) + 38/3*S(R(1,1,2),2 + N)*den(2 + N) - 68*S(R(1,1,
  2),N) - 12*S(R(1,1,2),N)*N + 42*S(R(1,1,2,1),N) + 12*S(R(1,1,2,1),N)*N - 76*S(R(
  1,1,3),N) - 18*S(R(1,1,3),N)*N - 170/3*S(R(1,2),1 + N)*den(1 + N) + 40/3*S(R(1,2
  ),1 + N)*den(1 + N)^2 - 134/3*S(R(1,2),2 + N)*den(2 + N) + 14*S(R(1,2),2 + N)*
  den(2 + N)^2 + 430/3*S(R(1,2),N) + 60*S(R(1,2),N)*N + 30*S(R(1,2,1),1 + N)*den(1
   + N) + 18*S(R(1,2,1),2 + N)*den(2 + N) - 452/3*S(R(1,2,1),N) - 56*S(R(1,2,1),N)
  *N + 74/3*S(R(1,2,1,1),N) + 8*S(R(1,2,1,1),N)*N - 248/3*S(R(1,2,2),N) - 22*S(R(1
  ,2,2),N)*N - 58*S(R(1,3),1 + N)*den(1 + N) - 40*S(R(1,3),2 + N)*den(2 + N) + 886/
3*S(R(1,3),N) + 100*S(R(1,3),N)*N - 116/3*S(R(1,3,1),N) - 10*S(R(1,3,1),N)*N +
410/3*S(R(1,4),N) + 36*S(R(1,4),N)*N + 186*S(R(2),1 + N)*den(1 + N) + 448/3*S(R(
2),1 + N)*den(1 + N)^2 + 160/3*S(R(2),1 + N)*den(1 + N)^3 + 138*S(R(2),2 + N)*
den(2 + N) + 206/3*S(R(2),2 + N)*den(2 + N)^2 + 80/3*S(R(2),2 + N)*den(2 + N)^3
 - 70*S(R(2),N) - 160*S(R(2),N)*N - 338/3*S(R(2,1),1 + N)*den(1 + N) - 64/3*S(R(
2,1),1 + N)*den(1 + N)^2 - 206/3*S(R(2,1),2 + N)*den(2 + N) - 10/3*S(R(2,1),2 +
N)*den(2 + N)^2 + 760/3*S(R(2,1),N) + 140*S(R(2,1),N)*N + 50/3*S(R(2,1,1),1 + N)
*den(1 + N) + 26/3*S(R(2,1,1),2 + N)*den(2 + N) - 170/3*S(R(2,1,1),N) - 100/3*S(
R(2,1,1),N)*N - 12*S(R(2,1,1,1),N) + 4/3*S(R(2,1,1,1),N)*N + 38/3*S(R(2,1,2),N)
 - 2/3*S(R(2,1,2),N)*N - 182/3*S(R(2,2),1 + N)*den(1 + N) - 116/3*S(R(2,2),2 + N
)*den(2 + N) + 676/3*S(R(2,2),N) + 308/3*S(R(2,2),N)*N - 118/3*S(R(2,2,1),N) -
18*S(R(2,2,1),N)*N + 296/3*S(R(2,3),N) + 36*S(R(2,3),N)*N + 694/3*S(R(3),1 + N)*
den(1 + N) + 188/3*S(R(3),1 + N)*den(1 + N)^2 + 448/3*S(R(3),2 + N)*den(2 + N)
 + 80/3*S(R(3),2 + N)*den(2 + N)^2 - 1454/3*S(R(3),N) - 290*S(R(3),N)*N - 86/3*
S(R(3,1),1 + N)*den(1 + N) - 56/3*S(R(3,1),2 + N)*den(2 + N) + 440/3*S(R(3,1),N)
 + 164/3*S(R(3,1),N)*N - 10*S(R(3,1,1),N) - 10/3*S(R(3,1,1),N)*N + 80*S(R(3,2),N
) + 80/3*S(R(3,2),N)*N + 302/3*S(R(4),1 + N)*den(1 + N) + 194/3*S(R(4),2 + N)*
den(2 + N) - 434*S(R(4),N) - 556/3*S(R(4),N)*N - 8*S(R(4,1),N) - 8/3*S(R(4,1),N)
*N - 150*S(R(5),N) - 50*S(R(5),N)*N - 200*den(1 + N)^2 - 380*den(1 + N)^3 - 896/
3*den(1 + N)^4 - 100*den(1 + N)^5 - 100*den(2 + N)^2 - 190*den(2 + N)^3 - 448/3*
den(2 + N)^4 - 50*den(2 + N)^5 );
```

I(N) =

```
sign(N)*ep^-2 * ( 4/3*S(R(1),1 + N)*den(1 + N) + 8/3*S(R(1),1 + N)*den(
  1 + N)^2 + 4/3*S(R(1),2 + N)*den(2 + N) + 4/3*S(R(1),2 + N)*den(2 + N)^2 + 4/3*
  S(R(1),N) + 2/3*S(R(1,2),N) + 2/3*S(R(2),1 + N)*den(1 + N) + 2/3*S(R(2),2 + N)*
  den(2 + N) - 2*S(R(2),N) - 4/3*S(R(2),N)*N + 4*S(R(2,1),N) + 4/3*S(R(2,1),N)*N
   - 6*S(R(3),N) - 2*S(R(3),N)*N - 8/3*den(1 + N)^2 - 4*den(1 + N)^3 - 4/3*den(2
   + N)^2 - 2*den(2 + N)^3 )

+ sign(N)*ep^-1 * (  - 16*S(R(1),1 + N)*den(1 + N) - 88/3*S(R(1),1 + N)*
  den(1 + N)^2 - 20/3*S(R(1),1 + N)*den(1 + N)^3 - 16*S(R(1),2 + N)*den(2 + N) -
  44/3*S(R(1),2 + N)*den(2 + N)^2 - 10/3*S(R(1),2 + N)*den(2 + N)^3 - 20*S(R(1),N)
   + 8/3*S(R(1,1),1 + N)*den(1 + N) + 8/3*S(R(1,1),1 + N)*den(1 + N)^2 + 8/3*S(R(1
  ,1),2 + N)*den(2 + N) + 8/3*S(R(1,1),N) + 10/3*S(R(1,1,2),N) + 10/3*S(R(1,2),1
   + N)*den(1 + N) + 10/3*S(R(1,2),2 + N)*den(2 + N) - 16*S(R(1,2),N) - 4*S(R(1,2)
  ,N)*N + 14*S(R(1,2,1),N) + 4*S(R(1,2,1),N)*N - 24*S(R(1,3),N) - 6*S(R(1,3),N)*N
   - 58/3*S(R(2),1 + N)*den(1 + N) - 40/3*S(R(2),1 + N)*den(1 + N)^2 - 46/3*S(R(2)
  ,2 + N)*den(2 + N) - 6*S(R(2),2 + N)*den(2 + N)^2 + 56/3*S(R(2),N) + 20*S(R(2),N
  )*N + 10*S(R(2,1),1 + N)*den(1 + N) + 6*S(R(2,1),2 + N)*den(2 + N) - 134/3*S(R(2
  ,1),N) - 56/3*S(R(2,1),N)*N + 16/3*S(R(2,1,1),N) + 8/3*S(R(2,1,1),N)*N - 62/3*S(
  R(2,2),N) - 22/3*S(R(2,2),N)*N - 18*S(R(3),1 + N)*den(1 + N) - 12*S(R(3),2 + N)*
  den(2 + N) + 76*S(R(3),N) + 100/3*S(R(3),N)*N - 10*S(R(3,1),N) - 10/3*S(R(3,1),N
  )*N + 36*S(R(4),N) + 12*S(R(4),N)*N + 32*den(1 + N)^2 + 164/3*den(1 + N)^3 + 24*
  den(1 + N)^4 + 16*den(2 + N)^2 + 82/3*den(2 + N)^3 + 12*den(2 + N)^4 )

+ sign(N) * ( 100*S(R(1),1 + N)*den(1 + N) + 168*S(R(1),1 + N)*den(1 + N)
  ^2 + 268/3*S(R(1),1 + N)*den(1 + N)^3 - 16/3*S(R(1),1 + N)*den(1 + N)^4 + 100*S(
  R(1),2 + N)*den(2 + N) + 84*S(R(1),2 + N)*den(2 + N)^2 + 134/3*S(R(1),2 + N)*
  den(2 + N)^3 - 8/3*S(R(1),2 + N)*den(2 + N)^4 + 160*S(R(1),N) - 32*S(R(1,1),1 +
  N)*den(1 + N) - 80/3*S(R(1,1),1 + N)*den(1 + N)^2 - 20/3*S(R(1,1),1 + N)*den(1
   + N)^3 - 32*S(R(1,1),2 + N)*den(2 + N) - 4/3*S(R(1,1),2 + N)*den(2 + N)^2 - 10/
  3*S(R(1,1),2 + N)*den(2 + N)^3 - 40*S(R(1,1),N) + 4/3*S(R(1,1,1),1 + N)*den(1 +
  N) - 40/3*S(R(1,1,1),1 + N)*den(1 + N)^2 + 4/3*S(R(1,1,1),2 + N)*den(2 + N) - 44/
  3*S(R(1,1,1),2 + N)*den(2 + N)^2 + 4/3*S(R(1,1,1),N) + 38/3*S(R(1,1,1,2),N) + 38/
  3*S(R(1,1,2),1 + N)*den(1 + N) + 38/3*S(R(1,1,2),2 + N)*den(2 + N) - 68*S(R(1,1,
  2),N) - 12*S(R(1,1,2),N)*N + 42*S(R(1,1,2,1),N) + 12*S(R(1,1,2,1),N)*N - 76*S(R(
  1,1,3),N) - 18*S(R(1,1,3),N)*N - 170/3*S(R(1,2),1 + N)*den(1 + N) + 40/3*S(R(1,2
  ),1 + N)*den(1 + N)^2 - 134/3*S(R(1,2),2 + N)*den(2 + N) + 14*S(R(1,2),2 + N)*
  den(2 + N)^2 + 430/3*S(R(1,2),N) + 60*S(R(1,2),N)*N + 30*S(R(1,2,1),1 + N)*den(1
   + N) + 18*S(R(1,2,1),2 + N)*den(2 + N) - 452/3*S(R(1,2,1),N) - 56*S(R(1,2,1),N)
  *N + 74/3*S(R(1,2,1,1),N) + 8*S(R(1,2,1,1),N)*N - 248/3*S(R(1,2,2),N) - 22*S(R(1
  ,2,2),N)*N - 58*S(R(1,3),1 + N)*den(1 + N) - 40*S(R(1,3),2 + N)*den(2 + N) + 886/
3*S(R(1,3),N) + 100*S(R(1,3),N)*N - 116/3*S(R(1,3,1),N) - 10*S(R(1,3,1),N)*N +
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2),1 + N)*den(1 + N)^2 + 160/3*S(R(2),1 + N)*den(1 + N)^3 + 138*S(R(2),2 + N)*
den(2 + N) + 206/3*S(R(2),2 + N)*den(2 + N)^2 + 80/3*S(R(2),2 + N)*den(2 + N)^3
 - 70*S(R(2),N) - 160*S(R(2),N)*N - 338/3*S(R(2,1),1 + N)*den(1 + N) - 64/3*S(R(
2,1),1 + N)*den(1 + N)^2 - 206/3*S(R(2,1),2 + N)*den(2 + N) - 10/3*S(R(2,1),2 +
N)*den(2 + N)^2 + 760/3*S(R(2,1),N) + 140*S(R(2,1),N)*N + 50/3*S(R(2,1,1),1 + N)
*den(1 + N) + 26/3*S(R(2,1,1),2 + N)*den(2 + N) - 170/3*S(R(2,1,1),N) - 100/3*S(
R(2,1,1),N)*N - 12*S(R(2,1,1,1),N) + 4/3*S(R(2,1,1,1),N)*N + 38/3*S(R(2,1,2),N)
 - 2/3*S(R(2,1,2),N)*N - 182/3*S(R(2,2),1 + N)*den(1 + N) - 116/3*S(R(2,2),2 + N
)*den(2 + N) + 676/3*S(R(2,2),N) + 308/3*S(R(2,2),N)*N - 118/3*S(R(2,2,1),N) -
18*S(R(2,2,1),N)*N + 296/3*S(R(2,3),N) + 36*S(R(2,3),N)*N + 694/3*S(R(3),1 + N)*
den(1 + N) + 188/3*S(R(3),1 + N)*den(1 + N)^2 + 448/3*S(R(3),2 + N)*den(2 + N)
 + 80/3*S(R(3),2 + N)*den(2 + N)^2 - 1454/3*S(R(3),N) - 290*S(R(3),N)*N - 86/3*
S(R(3,1),1 + N)*den(1 + N) - 56/3*S(R(3,1),2 + N)*den(2 + N) + 440/3*S(R(3,1),N)
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) + 80/3*S(R(3,2),N)*N + 302/3*S(R(4),1 + N)*den(1 + N) + 194/3*S(R(4),2 + N)*
den(2 + N) - 434*S(R(4),N) - 556/3*S(R(4),N)*N - 8*S(R(4,1),N) - 8/3*S(R(4,1),N)
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3*den(1 + N)^4 - 100*den(1 + N)^5 - 100*den(2 + N)^2 - 190*den(2 + N)^3 - 448/3*
den(2 + N)^4 - 50*den(2 + N)^5 );
```

- Result for $\mathbf{I}(\mathbf{N})$ in the G-scheme

Splitting functions in a nut shell

- Calculate anomalous dimensions (Mellin moments of splitting functions) $\longrightarrow$ divergence of Feynman diagrams in dimensional regularization $D = 4 - 2\epsilon$

$$\gamma_{\rm ij}^{(n)}(N) \;=\; -\int_0^1 dx\; x^{\,N-1}\, P_{\rm ij}^{(n)}(x)$$

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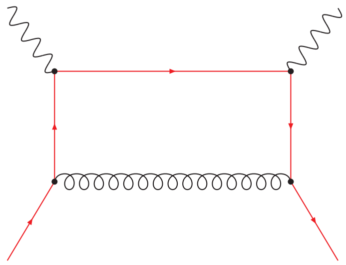
- **One-loop** Feynman diagrams
  $\longrightarrow$ in total 18 for $\gamma_{\rm ij}^{(0)}$ / $P_{\rm ij}^{(0)}$
  (pencil + paper)



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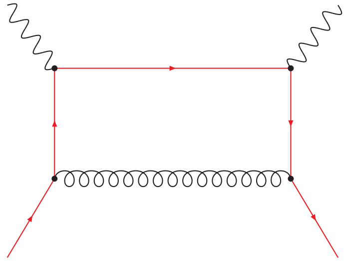
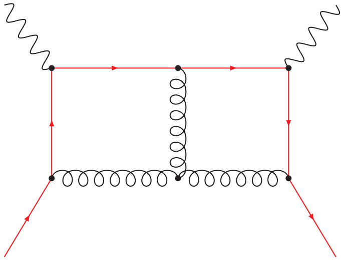


- **Two-loop** Feynman diagrams
  $\longrightarrow$ in total 350 for $\gamma_{\rm ij}^{(1)}$ / $P_{\rm ij}^{(1)}$
  (simple computer algebra)



Splitting functions in a nut shell

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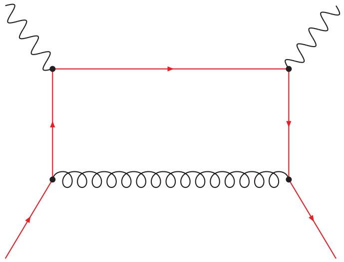
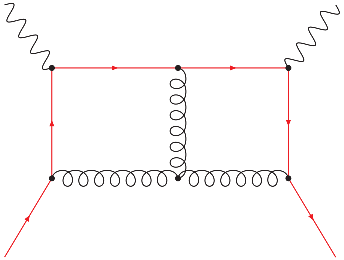
$$\gamma_{ij}^{(n)}(N) \;=\; -\int_0^1 dx\; x^{\,N-1}\, P_{ij}^{(n)}(x)$$

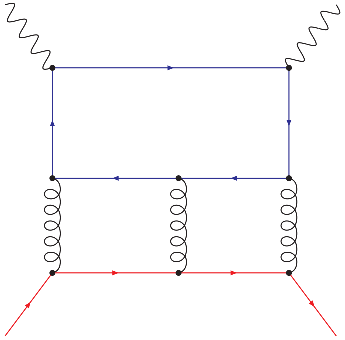
- **One-loop** Feynman diagrams
  $\longrightarrow$ in total 18 for $\gamma_{ij}^{(0)}$ / $P_{ij}^{(0)}$
  (pencil + paper)



- **Two-loop** Feynman diagrams
  $\longrightarrow$ in total 350 for $\gamma_{ij}^{(1)}$ / $P_{ij}^{(1)}$
  (simple computer algebra)



- **Three-loop** Feynman diagrams
  $\longrightarrow$ in total 9607 for $\gamma_{ij}^{(2)}$ / $P_{ij}^{(2)}$
  (cutting edge technology $\longrightarrow$ computer algebra system FORM Vermaseren '89-'05)



LO and NLO splitting functions

$$
\begin{aligned}
P^{(0)}_{\rm ns}(x) &= C_F\,(2p_{\rm qq}(x)+3\delta(1-x))\\
P^{(0)}_{\rm ps}(x) &= 0\\
P^{(0)}_{\rm qg}(x) &= 2n_f\,p_{\rm qg}(x)\\
P^{(0)}_{\rm gq}(x) &= 2C_F\,p_{\rm gq}(x)\\
P^{(0)}_{\rm gg}(x) &= C_A\Big(4p_{\rm gg}(x)+\frac{11}{3}\delta(1-x)\Big)-\frac{2}{3}n_f\,\delta(1-x)
\end{aligned}
$$

$$
\begin{aligned}
P^{(1)+}_{\rm ns}(x) &= 4C_AC_F\Big(p_{\rm qq}(x)\Big[\frac{67}{18}-\zeta_2+\frac{11}{6}\mathrm{H}_0+\mathrm{H}_{0,0}\Big]+p_{\rm qq}(-x)\Big[\zeta_2+2\mathrm{H}_{-1,0}-\mathrm{H}_{0,0}\Big]\\
&\quad+\frac{14}{3}(1-x)+\delta(1-x)\Big[\frac{17}{24}+\frac{11}{3}\zeta_2-3\zeta_3\Big]\Big)-4C_Fn_f\Big(p_{\rm qq}(x)\Big[\frac{5}{9}+\frac{1}{3}\mathrm{H}_0\Big]+\frac{2}{3}(1-x)\\
&\quad+\delta(1-x)\Big[\frac{1}{12}+\frac{2}{3}\zeta_2\Big]\Big)+4{C_F}^2\Big(2p_{\rm qq}(x)\Big[\mathrm{H}_{1,0}-\frac{3}{4}\mathrm{H}_0+\mathrm{H}_2\Big]-2p_{\rm qq}(-x)\Big[\zeta_2+2\mathrm{H}_{-1,0}\\
&\quad-\mathrm{H}_{0,0}\Big]-(1-x)\Big[1-\frac{3}{2}\mathrm{H}_0\Big]-\mathrm{H}_0-(1+x)\mathrm{H}_{0,0}+\delta(1-x)\Big[\frac{3}{8}-3\zeta_2+6\zeta_3\Big]\Big)\\
P^{(1)-}_{\rm ns}(x) &= P^{(1)+}_{\rm ns}(x)+16C_F\Big(C_F-\frac{C_A}{2}\Big)\Big(p_{\rm qq}(-x)\Big[\zeta_2+2\mathrm{H}_{-1,0}-\mathrm{H}_{0,0}\Big]-2(1-x)\\
&\quad-(1+x)\mathrm{H}_0\Big)
\end{aligned}
$$

$$
P^{(1)}_{\rm ps}(x) = 4C_Fn_f\Big(\frac{20}{9}\frac{1}{x}-2+6x-4\mathrm{H}_0+x^2\Big[\frac{8}{3}\mathrm{H}_0-\frac{56}{9}\Big]+(1+x)\Big[5\mathrm{H}_0-2\mathrm{H}_{0,0}\Big]\Big)
$$

$$
\begin{aligned}
P^{(1)}_{\rm qg}(x) &= 4C_An_f\Big(\frac{20}{9}\frac{1}{x}-2+25x-2p_{\rm qg}(-x)\mathrm{H}_{-1,0}-2p_{\rm qg}(x)\mathrm{H}_{1,1}+x^2\Big[\frac{44}{3}\mathrm{H}_0-\frac{218}{9}\Big]\\
&+4(1-x)\Big[\mathrm{H}_{0,0}-2\mathrm{H}_0+x\mathrm{H}_1\Big]-4\zeta_2x-6\mathrm{H}_{0,0}+9\mathrm{H}_0\Big)+4C_Fn_f\Big(2p_{\rm qg}(x)\Big[\mathrm{H}_{1,0}+\mathrm{H}_{1,1}+\mathrm{H}_2\\
&-\zeta_2\Big]+4x^2\Big[\mathrm{H}_0+\mathrm{H}_{0,0}+\frac{5}{2}\Big]+2(1-x)\Big[\mathrm{H}_0+\mathrm{H}_{0,0}-2x\mathrm{H}_1+\frac{29}{4}\Big]-\frac{15}{2}-\mathrm{H}_{0,0}-\frac{1}{2}\mathrm{H}_0\Big)
\end{aligned}
$$

$$
\begin{aligned}
P^{(1)}_{\rm gq}(x) &= 4C_AC_F\Big(\frac{1}{x}+2p_{\rm gq}(x)\Big[\mathrm{H}_{1,0}+\mathrm{H}_{1,1}+\mathrm{H}_2-\frac{11}{6}\mathrm{H}_1\Big]-x^2\Big[\frac{8}{3}\mathrm{H}_0-\frac{44}{9}\Big]+4\zeta_2-2\\
&-7\mathrm{H}_0+2\mathrm{H}_{0,0}-2\mathrm{H}_1x+(1+x)\Big[2\mathrm{H}_{0,0}-5\mathrm{H}_0+\frac{37}{9}\Big]-2p_{\rm gq}(-x)\mathrm{H}_{-1,0}\Big)-4C_Fn_f\Big(\frac{2}{3}x\\
&-p_{\rm gq}(x)\Big[\frac{2}{3}\mathrm{H}_1-\frac{10}{9}\Big]\Big)+4{C_F}^2\Big(p_{\rm gq}(x)\Big[3\mathrm{H}_1-2\mathrm{H}_{1,1}\Big]+(1+x)\Big[\mathrm{H}_{0,0}-\frac{7}{2}+\frac{7}{2}\mathrm{H}_0\Big]-3\mathrm{H}_{0,0}\\
&+1-\frac{3}{2}\mathrm{H}_0+2\mathrm{H}_1x\Big)
\end{aligned}
$$

$$
\begin{aligned}
P^{(1)}_{\rm gg}(x) &= 4C_An_f\Big(1-x-\frac{10}{9}p_{\rm gg}(x)-\frac{13}{9}\Big(\frac{1}{x}-x^2\Big)-\frac{2}{3}(1+x)\mathrm{H}_0-\frac{2}{3}\delta(1-x)\Big)+4{C_A}^2\Big(27\\
&+(1+x)\Big[\frac{11}{3}\mathrm{H}_0+8\mathrm{H}_{0,0}-\frac{27}{2}\Big]+2p_{\rm gg}(-x)\Big[\mathrm{H}_{0,0}-2\mathrm{H}_{-1,0}-\zeta_2\Big]-\frac{67}{9}\Big(\frac{1}{x}-x^2\Big)-12\mathrm{H}_0\\
&-\frac{44}{3}x^2\mathrm{H}_0+2p_{\rm gg}(x)\Big[\frac{67}{18}-\zeta_2+\mathrm{H}_{0,0}+2\mathrm{H}_{1,0}+2\mathrm{H}_2\Big]+\delta(1-x)\Big[\frac{8}{3}+3\zeta_3\Big]\Big)+4C_Fn_f\Big(2\mathrm{H}_0\\
&+\frac{2}{3}\frac{1}{x}+\frac{10}{3}x^2-12+(1+x)\Big[4-5\mathrm{H}_0-2\mathrm{H}_{0,0}\Big]-\frac{1}{2}\delta(1-x)\Big)\,.
\end{aligned}
$$

NNLO non-singlet splitting functions

S.M., Vermaseren, Vogt ‘04

$$
\begin{aligned}
P_{\rm ns}^{(2)+}(x) \;=\; & 16 C_A C_F n_f \Big(\frac{1}{6} p_{\rm qq}(x) \Big[\frac{10}{3}\zeta_2 - \frac{209}{36} - 9\zeta_3 - \frac{167}{18}H_0 + 2H_0\zeta_2 - 7H_{0,0} - 2H_{0,0,0} \\
& + 3H_{1,0,0} - H_3 \Big] + \frac{1}{3} p_{\rm qq}(-x) \Big[\frac{3}{2}\zeta_3 - \frac{5}{3}\zeta_2 - H_{-2,0} - 2H_{-1}\zeta_2 - \frac{10}{3}H_{-1,0} - H_{-1,0,0} \\
& + 2H_{-1,2} + \frac{1}{2}H_0\zeta_2 + \frac{5}{3}H_{0,0} + H_{0,0,0} - H_3 \Big] + (1-x)\Big[\frac{1}{6}\zeta_2 - \frac{257}{54} - \frac{43}{18}H_0 - \frac{1}{6}H_{0,0} - H_1 \Big] \\
& - (1+x)\Big[\frac{2}{3}H_{-1,0} + \frac{1}{2}H_2 \Big] + \frac{1}{3}\zeta_2 + H_0 + \frac{1}{6}H_{0,0} + \delta(1-x)\Big[\frac{5}{4} - \frac{167}{54}\zeta_2 + \frac{1}{20}{\zeta_2}^2 + \frac{25}{18}\zeta_3 \Big] \Big) \\
& + 16 C_A {C_F}^2 \Big(p_{\rm qq}(x) \Big[\frac{5}{6}\zeta_3 - \frac{69}{20}{\zeta_2}^2 - H_{-3,0} - 3H_{-2}\zeta_2 - 14H_{-2,-1,0} + 3H_{-2,0} + 5H_{-2,0,0} \\
& - 4H_{-2,2} - \frac{151}{48}H_0 + \frac{41}{12}H_0\zeta_2 - \frac{17}{2}H_0\zeta_3 - \frac{13}{4}H_{0,0} - 4H_{0,0}\zeta_2 - \frac{23}{12}H_{0,0,0} + 5H_{0,0,0,0} + \frac{2}{3}H_3 \\
& - 24H_1\zeta_3 - 16H_{1,-2,0} + \frac{67}{9}H_{1,0} - 2H_{1,0}\zeta_2 + \frac{31}{3}H_{1,0,0} + 11H_{1,0,0,0} + 8H_{1,1,0,0} - 8H_{1,3} + H_4 \\
& + \frac{67}{9}H_2 - 2H_2\zeta_2 + \frac{11}{3}H_{2,0} + 5H_{2,0,0} + H_{3,0} \Big] + p_{\rm qq}(-x)\Big[\frac{1}{4}{\zeta_2}^2 - \frac{67}{9}\zeta_2 + \frac{31}{4}\zeta_3 + 5H_{-3,0} \\
& - 32H_{-2}\zeta_2 - 4H_{-2,-1,0} - \frac{31}{6}H_{-2,0} + 21H_{-2,0,0} + 30H_{-2,2} - \frac{31}{3}H_{-1}\zeta_2 - 42H_{-1}\zeta_3 + \frac{9}{4}H_0 \\
& - 4H_{-1,-2,0} + 56H_{-1,-1}\zeta_2 - 36H_{-1,-1,0,0} - 56H_{-1,-1,2} - \frac{134}{9}H_{-1,0} - 42H_{-1,0}\zeta_2 - H_{3,0} \\
& + 32H_{-1,3} - \frac{31}{6}H_{-1,0,0} + 17H_{-1,0,0,0} + \frac{31}{3}H_{-1,2} + 2H_{-1,2,0} + \frac{13}{12}H_0\zeta_2 + \frac{29}{2}H_0\zeta_3 + \frac{67}{9}H_{0,0} \\
& + 13H_{0,0}\zeta_2 + \frac{89}{12}H_{0,0,0} - 5H_{0,0,0,0} - 7H_2\zeta_2 - \frac{31}{6}H_3 - 10H_4 \Big] + (1-x)\Big[\frac{133}{36} + 4H_{0,0,0,0} \\
& - \frac{167}{4}\zeta_3 - 2H_0\zeta_3 - 2H_{-3,0} + H_{-2}\zeta_2 + 2H_{-2,-1,0} - 3H_{-2,0,0} + \frac{77}{4}H_{0,0,0} - \frac{209}{6}H_1 - 7H_1\zeta_2 \\
& + 4H_{1,0,0} + \frac{14}{3}H_{1,0} \Big] + (1+x)\Big[\frac{43}{2}\zeta_2 - 3{\zeta_2}^2 + \frac{25}{2}H_{-2,0} - 31H_{-1}\zeta_2 - 14H_{-1,-1,0} - \frac{13}{3}H_{-1,0} \\
& + 24H_{-1,2} + 23H_{-1,0,0} + \frac{55}{2}H_0\zeta_2 + 5H_{0,0}\zeta_2 + \frac{1457}{48}H_0 - \frac{1025}{36}H_{0,0} - \frac{155}{6}H_2 + H_2\zeta_2 - 15H_3 \\
& + 2H_{2,0,0} - 3H_4 \Big] - 5\zeta_2 - \frac{1}{2}{\zeta_2}^2 + 50\zeta_3 - 2H_{-3,0} - 7H_{-2,0} - H_0\zeta_3 - \frac{37}{2}H_0\zeta_2 - \frac{242}{9}H_0 \\
& - 2H_{0,0}\zeta_2 + \frac{185}{6}H_{0,0} - 22H_{0,0,0} - 4H_{0,0,0,0} + \frac{28}{3}H_2 + 6H_3 + \delta(1-x)\Big[\frac{151}{64} + \zeta_2\zeta_3 - \frac{205}{24}\zeta_2 \\
& - \frac{247}{60}{\zeta_2}^2 + \frac{211}{12}\zeta_3 + \frac{15}{2}\zeta_5 \Big] \Big) + 16 {C_A}^2 C_F \Big(p_{\rm qq}(x)\Big[\frac{245}{48} - \frac{67}{18}\zeta_2 + \frac{12}{5}{\zeta_2}^2 + \frac{1}{2}\zeta_3 + \frac{1043}{216}H_0 \\
& + H_{-3,0} + 4H_{-2,-1,0} - \frac{3}{2}H_{-2,0} - H_{-2,0,0} + 2H_{-2,2} - \frac{31}{12}H_0\zeta_2 + 4H_0\zeta_3 + \frac{389}{72}H_{0,0} - 2H_{2,0,0} \\
& - H_{0,0,0,0} + 9H_1\zeta_3 + 6H_{1,-2,0} - H_{1,0}\zeta_2 - \frac{11}{4}H_{1,0,0} - 3H_{1,0,0,0} - 4H_{1,1,0,0} + 4H_{1,3} + \frac{31}{12}H_{0,0,0} \\
& + \frac{11}{12}H_3 + H_4 \Big] + p_{\rm qq}(-x)\Big[\frac{67}{18}\zeta_2 - {\zeta_2}^2 - \frac{11}{4}\zeta_3 - H_{-3,0} + 8H_{-2}\zeta_2 + \frac{11}{6}H_{-2,0} - 4H_{-2,0,0} \\
& - 3H_{-1,0,0,0} + \frac{11}{3}H_{-1}\zeta_2 + 12H_{-1}\zeta_3 - 16H_{-1,-1}\zeta_2 + 8H_{-1,-1,0,0} + 16H_{-1,-1,2} + \frac{67}{9}H_{-1,0} \\
& - 8H_{-2,2} + 11H_{-1,0}\zeta_2 + \frac{11}{6}H_{-1,0,0} - \frac{11}{3}H_{-1,2} - 8H_{-1,3} - \frac{3}{4}H_0 - \frac{1}{6}H_0\zeta_2 - 4H_0\zeta_3 - \frac{67}{18}H_{0,0} \\
& - 3H_{0,0}\zeta_2 - \frac{31}{12}H_{0,0,0} + H_{0,0,0,0} + 2H_2\zeta_2 + \frac{11}{6}H_3 + 2H_4 \Big] + (1-x)\Big[\frac{1883}{108} - \frac{1}{2}H_{0,0,0,0} + 11H_1 \\
& - H_{-2,-1,0} + \frac{1}{2}H_{-3,0} - \frac{1}{2}H_{-2}\zeta_2 + \frac{1}{2}H_{-2,0,0} + \frac{523}{36}H_0 + H_0\zeta_3 - \frac{13}{3}H_{0,0} - \frac{5}{2}H_{0,0,0} + 2H_1\zeta_2 \\
& - 2H_{1,0,0} \Big] + (1+x)\Big[8H_{-1}\zeta_2 + 4H_{-1,-1,0} + \frac{8}{3}H_{-1,0} - 5H_{-1,0,0} - 6H_{-1,2} - \frac{13}{3}\zeta_2 + \frac{3}{8}{\zeta_2}^2 \\
& - \frac{43}{4}\zeta_3 - \frac{5}{2}H_{-2,0} - \frac{11}{2}H_0\zeta_2 - \frac{1}{2}H_2\zeta_2 - \frac{5}{4}H_{0,0}\zeta_2 + 7H_2 - \frac{1}{4}H_{2,0,0} + 3H_3 + \frac{3}{4}H_4 \Big] + \frac{1}{2}H_{0,0}\zeta_2 \\
& + \frac{1}{4}{\zeta_2}^2 - \frac{8}{3}\zeta_2 + \frac{17}{2}\zeta_3 + H_{-2,0} - \frac{19}{2}H_0 + \frac{5}{2}H_0\zeta_2 - H_0\zeta_3 + \frac{13}{3}H_{0,0} + \frac{5}{2}H_{0,0,0} + \frac{1}{2}H_{0,0,0,0} \\
& - \delta(1-x)\Big[\frac{1657}{576} - \frac{281}{27}\zeta_2 + \frac{1}{8}{\zeta_2}^2 + \frac{97}{9}\zeta_3 - \frac{5}{2}\zeta_5 \Big] \Big) + 16 C_F n_f^2 \Big(\frac{1}{18} p_{\rm qq}(x)\Big[H_{0,0} - \frac{1}{3} + \frac{5}{3}H_0 \Big] \\
& + (1-x)\Big[\frac{13}{54} + \frac{1}{9}H_0 \Big] - \delta(1-x)\Big[\frac{17}{144} - \frac{5}{27}\zeta_2 + \frac{1}{9}\zeta_3 \Big] \Big) + 16 {C_F}^2 n_f \Big(\frac{1}{3} p_{\rm qq}(x)\Big[5\zeta_3 - 4H_{1,0,0} \\
& - \frac{55}{16} + \frac{5}{8}H_0 + H_0\zeta_2 + \frac{3}{2}H_{0,0} - H_{0,0,0} - \frac{10}{3}H_{1,0} - \frac{10}{3}H_2 - 2H_{2,0} - 2H_3 \Big] + \frac{2}{3} p_{\rm qq}(-x)\Big[\frac{5}{3}\zeta_2 \\
& - \frac{3}{2}\zeta_3 + H_{-2,0} + 2H_{-1}\zeta_2 + \frac{10}{3}H_{-1,0} + H_{-1,0,0} - 2H_{-1,2} - \frac{1}{2}H_0\zeta_2 - \frac{5}{3}H_{0,0} - H_{0,0,0} + H_3 \Big] \\
& - (1-x)\Big[\frac{10}{9} + \frac{19}{18}H_{0,0} - \frac{4}{3}H_1 + \frac{2}{3}H_{1,0} + \frac{4}{3}H_2 \Big] + (1+x)\Big[\frac{4}{3}H_{-1,0} - \frac{25}{24}H_0 + \frac{1}{2}H_{0,0,0} \Big] + \frac{2}{9}H_0 \\
& + \frac{7}{9}H_{0,0} + \frac{4}{3}H_2 - \delta(1-x)\Big[\frac{23}{16} - \frac{5}{12}\zeta_2 - \frac{29}{30}{\zeta_2}^2 + \frac{17}{6}\zeta_3 \Big] \Big) + 16 {C_F}^3 \Big(p_{\rm qq}(x)\Big[\frac{9}{10}{\zeta_2}^2 - 2H_{-3,0} \\
& + 6H_{-2}\zeta_2 + 12H_{-2,-1,0} - 6H_{-2,0,0} - \frac{3}{16}H_0 - \frac{3}{2}H_0\zeta_2 + H_0\zeta_3 + \frac{13}{8}H_{0,0} - 2H_{0,0,0,0} + 8H_{1,3} \\
& + 12H_1\zeta_3 + 8H_{1,-2,0} - 6H_{1,0,0} - 4H_{1,0,0,0} + 4H_{1,2,0} - 3H_{2,0} + 2H_{2,0,0} + 4H_{2,1,0} + 4H_{2,2} \\
& + 4H_{3,0} + 4H_{3,1} + 2H_4 \Big] + p_{\rm qq}(-x)\Big[\frac{7}{2}{\zeta_2}^2 - \frac{9}{2}\zeta_3 - 6H_{-3,0} + 32H_{-2}\zeta_2 + 8H_{-2,-1,0} + 3H_{-2,0} \\
& - 26H_{-2,0,0} - 28H_{-2,2} + 6H_{-1}\zeta_2 + 36H_{-1}\zeta_3 + 8H_{-1,-2,0} - 48H_{-1,-1}\zeta_2 + 40H_{-1,-1,0,0} \\
& + 48H_{-1,-1,2} + 40H_{-1,0}\zeta_2 + 3H_{-1,0,0} - 22H_{-1,0,0,0} - 6H_{-1,2} - 4H_{-1,2,0} - 32H_{-1,3} - \frac{3}{2}H_0 \\
& - \frac{3}{2}H_0\zeta_2 - 13H_0\zeta_3 - 14H_{0,0}\zeta_2 - \frac{9}{2}H_{0,0,0} + 6H_{0,0,0,0} + 6H_2\zeta_2 + 3H_3 + 2H_{3,0} + 12H_4 \Big] \\
& + (1-x)\Big[2H_{-3,0} - \frac{31}{8} + 4H_{-2,0,0} + H_{0,0}\zeta_2 - 3H_{0,0,0,0} + 35H_1 + 6H_1\zeta_2 - H_{1,0} + \frac{5}{2}H_{2,0} \Big] \\
& + (1+x)\Big[\frac{37}{10}{\zeta_2}^2 - \frac{93}{4}\zeta_2 - \frac{81}{2}\zeta_3 - 15H_{-2,0} + 30H_{-1}\zeta_2 + 12H_{-1,-1,0} - 2H_{-1,0} - 26H_{-1,0,0} \\
& - 24H_{-1,2} - \frac{539}{16}H_0 - 28H_0\zeta_2 + \frac{191}{8}H_{0,0} + 20H_{0,0,0} + \frac{85}{4}H_2 - 3H_{2,0,0} - 2H_{3,0} + 13H_3 \\
& - H_4 \Big] + 4\zeta_2 + 33\zeta_3 + 4H_{-3,0} + 10H_{-2,0} + \frac{67}{2}H_0 + 6H_0\zeta_3 + 19H_0\zeta_2 - 25H_{0,0} - 17H_{0,0,0} \\
& - 2H_2 - H_{2,0} - 4H_3 + \delta(1-x)\Big[\frac{29}{32} - 2\zeta_2\zeta_3 + \frac{9}{8}\zeta_2 + \frac{18}{5}{\zeta_2}^2 + \frac{17}{4}\zeta_3 - 15\zeta_5 \Big] \Big) \; .
\end{aligned}
$$

$$
\begin{aligned}
P_{\rm ns}^{(2)-}(x) \;=\; & P_{\rm ns}^{(2)+}(x) + 16 C_A C_F \Big(C_F - \frac{C_A}{2}\Big) \Big(p_{\rm qq}(-x)\Big[\frac{134}{9}\zeta_2 - 4{\zeta_2}^2 - 11\zeta_3 - 4H_{-3,0} \\
& + 32H_{-2}\zeta_2 + \frac{22}{3}H_{-2,0} - 16H_{-2,0,0} - 32H_{-2,2} + \frac{44}{3}H_{-1}\zeta_2 + 48H_{-1}\zeta_3 - 64H_{-1,-1}\zeta_2 \\
& + 32H_{-1,-1,0,0} + 64H_{-1,-1,2} + \frac{268}{9}H_{-1,0} + 44H_{-1,0}\zeta_2 + \frac{22}{3}H_{-1,0,0} - 12H_{-1,0,0,0} - \frac{44}{3}H_{-1,2} \\
& - 32H_{-1,3} - 3H_0 - \frac{2}{3}H_0\zeta_2 - 16H_0\zeta_3 - \frac{134}{9}H_{0,0} - 12H_{0,0}\zeta_2 - \frac{31}{3}H_{0,0,0} + 4H_{0,0,0,0} + 8H_2\zeta_2 \\
& + \frac{22}{3}H_3 + 8H_4 \Big] + (1-x)\Big[\frac{367}{18} + \frac{1}{2}{\zeta_2}^2 + 2H_{-3,0} - 2H_{-2}\zeta_2 - 4H_{-2,-1,0} - 10H_{-2,0} - 2H_{0,0} \\
& + 2H_{-2,0,0} + 2H_0\zeta_3 + H_{0,0}\zeta_2 - H_{0,0,0,0} + 8H_1\zeta_2 + \frac{140}{3}H_1 \Big] + (1+x)\Big[32H_{-1}\zeta_2 - 18\zeta_2 \\
& - 23\zeta_3 + \frac{26}{3}H_{-1,0} - 16H_{-1,0,0} - 32H_{-1,2} - \frac{481}{18}H_0 - 29H_0\zeta_2 + 5H_{0,0,0} + 24H_3 + \frac{70}{3}H_2 \Big] \\
& - 2\zeta_2 - 2\zeta_3 + 32H_0 + 14H_0\zeta_2 + 2H_{0,0,0} - 16H_3 \Big) + 16 C_F n_f \Big(C_F - \frac{C_A}{2}\Big) \Big(p_{\rm qq}(-x)\Big[2\zeta_3 \\
& - \frac{20}{9}\zeta_2 - \frac{4}{3}H_{-2,0} - \frac{8}{3}H_{-1}\zeta_2 - \frac{40}{9}H_{-1,0} - \frac{4}{3}H_{-1,0,0} + \frac{8}{3}H_{-1,2} + \frac{2}{3}H_0\zeta_2 + \frac{20}{9}H_{0,0} + \frac{4}{3}H_{0,0,0} \\
& - \frac{4}{3}H_3 \Big] + (1-x)\Big[\frac{61}{9} - \frac{8}{3}H_1 \Big] + (1+x)\Big[2H_{0,0} - \frac{8}{3}H_{-1,0} + \frac{41}{9}H_0 - \frac{4}{3}H_2 \Big] \Big) \\
& + 16 {C_F}^2 \Big(C_F - \frac{C_A}{2}\Big) \Big(p_{\rm qq}(-x)\Big[9\zeta_3 - 7{\zeta_2}^2 + 12H_{-3,0} - 64H_{-2}\zeta_2 - 16H_{-2,-1,0} - 6H_{-2,0} \\
& + 52H_{-2,0,0} + 56H_{-2,2} - 12H_{-1}\zeta_2 - 72H_{-1}\zeta_3 - 16H_{-1,-2,0} + 96H_{-1,-1}\zeta_2 - 80H_{-1,-1,0,0} \\
& - 96H_{-1,-1,2} - 80H_{-1,0}\zeta_2 - 6H_{-1,0,0} + 44H_{-1,0,0,0} + 12H_{-1,2} + 8H_{-1,2,0} + 64H_{-1,3} + 3H_0 \\
& + 3H_0\zeta_2 + 26H_0\zeta_3 + 28H_{0,0}\zeta_2 + 9H_{0,0,0} - 12H_{0,0,0,0} - 12H_2\zeta_2 - 6H_3 - 4H_{3,0} - 24H_4 \Big] \\
& - (1-x)\Big[15 + 8H_{-3,0} + 8H_{-2,0,0} + 61H_0 + 6H_0\zeta_3 + 2H_{0,0}\zeta_2 - 6H_{0,0,0,0} + 12H_1\zeta_2 + 60H_1 \\
& + 8H_{1,0} \Big] + (1+x)\Big[24\zeta_2 + 57\zeta_3 + 10H_{-2,0} - 48H_{-1}\zeta_2 - 4H_{-1,0} + 40H_{-1,0,0} + 48H_{-1,2} \\
& + 59H_0\zeta_2 - 22H_{0,0} - 35H_{0,0,0} - 22H_2 - 4H_{2,0} - 44H_3 \Big] + 8\zeta_2 - 42\zeta_3 - 4H_{-2,0} + 42H_0 \\
& - 38H_0\zeta_2 + 14H_{0,0} - 16H_2 + 26H_{0,0,0} + 24H_3 \Big) \; .
\end{aligned}
$$

$$
\begin{aligned}
P_{\rm ns}^{(2)s}(x) \;=\; & 16 n_f \frac{d^{abc}d_{abc}}{n_c} \Big(\frac{1}{2}(1-x)\Big[\frac{50}{3} + \frac{41}{12}\zeta_2 - \frac{5}{4}{\zeta_2}^2 - H_{-3,0} + H_{-2}\zeta_2 - H_{-2,0,0} + \frac{9}{4}H_3 \\
& + 2H_{-2,-1,0} + \frac{3}{2}H_0\zeta_2 - \frac{1}{2}H_1\zeta_2 - \frac{3}{4}H_{1,0,0} + \frac{91}{12}H_1 \Big] + \frac{1}{2}(1+x)\Big[H_{-1,-1,0} - \frac{3}{2}H_{-1}\zeta_2 + \frac{3}{4}H_0 \\
& - \frac{13}{6}H_{-1,0} + \frac{1}{2}H_{-1,0,0} + 2H_{-1,2} - \frac{3}{2}H_{-2,0} + \frac{9}{4}H_0\zeta_2 + \frac{29}{12}H_{0,0} + \frac{41}{12}H_2 - H_2\zeta_2 - \frac{1}{2}H_{2,0,0} \\
& + \frac{3}{2}H_4 \Big] - \frac{1}{3}\Big(\frac{1}{x} + x^2\Big)\Big[3H_{-1}\zeta_2 + 2H_{-1,-1,0} - 2H_{-1,0,0} - 2H_{-1,2} + H_1\zeta_2 \Big] + \frac{1}{3}x^2\Big[5\zeta_3 - 2H_3 \\
& + 2H_{-2,0} + 4H_0\zeta_2 - 2H_{0,0,0} + 2H_1\zeta_2 \Big] + \frac{91}{24}H_0 + \zeta_3 - \frac{9}{2}\zeta_2 + {\zeta_2}^2 - H_0\zeta_3 - H_0\zeta_2 - 2H_{0,0}\zeta_2 \\
& + \frac{3}{8}H_{0,0} - \frac{1}{4}H_{0,0,0} + \frac{1}{2}H_{0,0,0,0} + H_{-2,0} - H_3 \Big) \; .
\end{aligned}
$$

NNLO singlet splitting functions

S.M., Vermaseren, Vogt '04

Summary

Quantum field theory

- Precision predictions for particle physics at colliders
- Efficient method for scattering cross sections
- Improvements in understanding of quantum field theory

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- Polynomial summation $\longrightarrow$ solved in *Mathematica*, *Maple*, ...
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Riemann zeta values

- Basis of (transcendental) numbers for Feynman diagram calculations
- Generalization to cases of many variables $\longrightarrow$ multiple polylogarithms

Computer algebra implementations

- Symbolic summation for harmonic sums $S_{m_1,\ldots,m_k}(n)$
  - SUMMER package (using FORM) Vermaseren '98
- Symbolic summation for generalized sums $S(n; m_1, ..., m_k; x_1, ..., x_k)$
  - XSUMMER package (using FORM) S.M., Uwer '05
  - *nestedsums* (using GiNaC) Weinzierl '02