

Groebner bases under composition and multivariate matrix factorization

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Contents of the talk

In this talk, I will introduce some of my work related to Groebner bases.

- ◆ I. Behavior of a Groebner basis under composition — New Results;
- ◆ II. Multivariate polynomial matrix factorization — New progress.



PART(I)

Groebner bases under composition

Groebner bases under composition

Hoon Hong initiated the study of
Groebner bases under composition

Groebner bases under composition

Early papers:

- ◆ (i). H.Hong, *Groebner bases under composition II*, ISSAC, 2006.
- ◆ (ii) H.Hong, *Groebner basis under composition I*, JSC (1998), 25, 647-662.
- ◆ (iii) Gutierrez , J. Miguel, R, *Reduced Groebner basis under composition*, JSC(1998), 26, 433-444.

Basic research Problem

Let us consider the polynomial ring in n variables: $K[x] = k[x_1, \dots, x_n]$, and a given term ordering $>$.

The basic problem can be described as follows:

For an endomorphism U of $k[x]$, and a finite subset G of $k[x]$, how can we compute a Groebner basis of $U(G)$ under the term ordering $>$ by means of a Groebner basis of G and U ?

Hong's Theorem

For any Grobner basis G of $k[X]$, $U(G)$ is a Groebner basis if and only if

- ◆ (a) for any terms p and q , $p < q$ implies $\text{lt}(U(p)) < \text{lt}(U(q))$;
- ◆ (b) $\text{lt}(U)$ is a permuted powering, that is, every component of U has a power of one variable as the leading term, where $U = (u_1, \dots, u_n)$: $x_j \rightarrow u_j$. and $\text{lt}(U) = (\text{lt}(u_1), \dots, \text{lt}(u_n))$.

Result of Gutierrez etc.

Gutierrez etc. have proved a result on reduced case. Their result is:

For any reduced Groebner basis G , $U(G)$ is a reduced Groebner basis if and only if

- ◆ (a) for any terms p, q , $p < q$ implies $\text{lt}(U(p)) < \text{lt}(U(q))$.
- ◆ (b) every component of U is a polynomial in one variable, and different polynomial involves in different variable.

A generalization

First, We establish a more general framework:

Let U be a homomorphism from $k[x] = K[x_1, \dots, x_n]$ to $k[y] = k[y_1, \dots, y_m]$ s.t. $n < m$ or $n = m$.

Given two term orderings $>_1$ on $k[x]$ and $>_2$ on $K[y]$. In this case uj is in $k[y]$.

A generalization

Under this new framework, Hong's theorem can be **generalized** as follows:

For any Groebner basis G with respect to $<_1$, $U(G)$ is a Groebner basis with respect to $<_2$ if and only if

- ◆ (1) for any terms p, q , $p <_1 q$ implies that $\text{lt}(U(p)) <_2 \text{lt}(U(q))$;
- ◆ (2) $\text{lt}(u_i)$ and $\text{lt}(u_j)$ are pairwise coprime for different i and j .

A generalization

But in reduced case, it seems to lack a sufficient and necessary condition in this generalization case.

Details can be found in:

Remarks on Groebner basis for ideals under composition, ISSAC, 2001.

Check the term ordering conditions

From the above theorems, we need to solve **the term ordering compatible problem** in order to apply these theorems:

Find an efficient algorithm to check if for any terms p and q , $p <_1 q$ implies $lt(U(p)) <_2 lt(U(q))$? (Hong)

Continue

Let T be the matrix corresponding to exponent vectors of leading terms of u_j , where $U=(u_1, \dots, u_n)$

Let term ordering $>_1$ be represented by a $n \times n$ matrix A , and $>_2$ be represented by matrix B .

Thus our method is using rational elementary transformation to A and TB simultaneously to obtain some standard form in some sense, then check if these standard form are the “same” which is up to a positive number multiple.

Continue

So called elementary rational transformation for the real matrices we mean:

- ◆ I. Multiplying a row of a matrix by a non-zero rational number;
- ◆ II. Interchanging any two rows;
- ◆ III. Adding a rational multiple of one row to another row

See JSC(2003), vol.35.

Homogeneous case

An interesting problem is:

Under what conditions that for any homogeneous groebner bases G , $U(G)$ is homogeneous greobner bases?

Journal of Algebra, computational algebra section, In press.

Homogeneous case

We have provided a complete answer:

For any homogeneous Groebner bases G , $U(G)$ is a homogeneous Groebner bases if and only if

- ◆ (I) for any terms p, q with $\deg(p) = \deg(q)$, $p < q$ implies $\text{lt}(u(p)) < \text{lt}(u(q))$;
- ◆ (ii) $\text{lt}(U)$ is a permuted powering, and every u_i has the same degree.

Further results

The problems can be further extended more general case. Let L be an arbitrary grading on $k[x]$, we may ask:

Under what conditions that for any L - homogeneous groebner bases G , $U(G)$ is Groebner bases?

An answer

For any L -homogeneous $G \in \mathcal{G}$, $U(G)$ is GB if and only if

- ◆ (I) for any terms $p > q$, $L(p) = L(q)$ implies $lt(U(p)) > lt(U(q))$;
- ◆ (II) $lt(U)$ is a permuted powering;

Remarks

This new result unifies all the previous results. Because if $L(x_i)=0$ for every I , we get Hong's theorem; if $L(x_i)=1$ for every I , we get the usual homogeneous result.

The proof is slightly hard; submitted to Journal of Algebra, CA section.

Summary

- ◆ From above descriptions, we see that only universal cases are considered.
- ◆ For the basic problem: Given a Groebner basis G , and a homomorphism U , How can we compute the Groebner basis of $U(G)$ by means of G and U ?
- ◆ We have not solved the basic problem! It seems to have some difficulties.

Summary

- ◆ There are a lot of related work in Resultant, subresultant, Sagbi bases, non-commutative Groebner bases etc.
- ◆ See related papers.



PART(II)

Multivariate polynomial matrix factorization



Basic Problem

Given a $n \times m$ polynomial matrix F with full row rank of size (l, m) .

Let d be the gcd of all the l -minors, $f|d$, whether or not there exists square matrix G and matrix F_1 , such that

$$F = G \times F_1 \text{ with } \det(G) = f?$$

The History

- ◆ When $n=1$, this is solved completely using Gauss elimination, and play an essential role in linear systems and control theory, convolutional codes.
- ◆ When $n=2$, this is solved using pseudodivision by N. K. Bose and others, used in 2-D systems and signal processing.
- ◆ When $n \geq 3$, this is thought as very hard! Because previous methods can not be generalized to this case.

Motivation for research

Why we concern about this problem?

- ◆ (i). Generalize linear system theory to multidimensional systems theory since 1976.
- ◆ (ii). Many problems in multidimensional systems and signal processing can be formulated in multivariate matrix problems.
- ◆ (iii). May be useful in multidimensional convolutional codes.
- ◆ (iv). Possibly other applications.

Reduced minors

In order to introduce Lin-Bose Problem, we give a basic definition as follows:

Let F be a full row rank matrix of size (l, m) , let a_1, \dots, a_k be all the l -minors of F , d be the g.c.d of a_1, \dots, a_k , and let b_j such that $a_j = d x b_j$. b_1, \dots, b_k is said to be the **reduced minors** of F .

Some research papers

There are many papers concerning this problems, we just mention:

- ◆ C. Charoenlarnopparut, N. K. Bose, Multidimensional FIR filter Bank Design using Groebner basis, IEEE Trans. Circuits Systems II Vol 46, 1999, pp1475-1486.
- ◆ Z. Lin, Notes on nD polynomial matrix factorization, Multidimensional system and Signal Processing, Vol. 10, 1999, 379-393.

Research papers

- ◆ Z. Lin, Further results on nD polynomial matrix factorization, Multidimensional system and system processing, Vol 12, 2001, pp199-208.
- ◆ M. Wang, C.P.Kwong, Computing GCLF using Syzygy algorithm, Proceedings of ICMS, 2002.

Research papers

- ◆ M. Wang, D. Feng, On Lin-Bose problem, Linear algebra and application, vol. 290, 2004;
- ◆ M. Wang, C.P. Kwong, On multivariate matrix factorization problems. Mathematics of Control, signals and systems. 2005.

Zero prime factorization

A full row rank matrix is **zero prime** if all its maximal order minors generate the unit ideal.

Lin-Bose propose the following problem:

Given F , full row rank of size $l \times m$, if all the reduced minors generate the unit ideal, then whether or not F can be factorized as $F = G \times F_1$, with $\det(G) = d$, d the gcd, and F_1 being zero prime?

Zero prime factorization

J.F.Pommaret gave a proof using algebraic analysis method in 2001 Euro control conference, his method only holds for the **complex** number field.

We give a full proof for **any field**.
Linear algebra and applications, 2004

Minor prime factorization

Let F be as above, d the gcd of all the l -minors of F . Whether or not there exists G such that $F = G \times F_1$ and $\det(G) = d$?

If such factorization exists, we call it **minor prime factorization** of F .

Several people gave counter-examples to showed that the above problem may have no solution in 1976-1979;

However, **no one** propose a sufficient and necessary condition!

Minor prime factorization

We first found a **sufficient and necessary** condition.

Let K be submodule generated by all the rows of F . There exists a minor prime factorization if and only if the colon submodule $K:d$ is a free module of rank l .

Remarks to the proof

The proof of the above result relies on a characterization of so-called **minor left prime**(MLP) matrix.

A full row rank matrix F of size $l \times m$ is said to be a MLP matrix if all the l -minors have only trivial common divisors.

The result just holds for any field. See *Mathematics of control, signals and systems, 2005* for details.

Remarks to proof

We first prove, a matrix F is a MLP if and only if $K:d=K$, where K =submodule generated by all the rows of F , d the g.c.d of all the $|x|$ minors of F .

Then using a lifting to the linear mapping, we get the factor.

Note that when F is not full rank, this result will not hold, and above result does not hold either.

An algorithm

Based on above theorem, we propose an algorithm as follows:

Step 1 Let F as above, d the g.c.d of all l times l minors of F . Compute the syzygy module of row vectors of F and $-dxI_m$. If this syzygy module is free of rank l , then form a generating matrix $[G|H]$ such that G is a $l \times l$ matrix, thus dG^{-1} is the desired matrix factor;

Step 2 If this syzygy module is not free, then there do not exist such a matrix factor.

Continue

We can easily check if a submodule is free using Fitting ideals.

Factor prime matrix

F is said to be factor prime, if for any square matrix G , $F = G \times F^{-1}$, implies that $\det(G)$ is a non-zero constant.

A basic problem is to check if F is factor prime matrix .

Factor prime matrix

Checking if a matrix is factor prime is a long-standing open problem since the concept was proposed in 1979.

D.C.Youla, G,Gnavi, Notes on n-Dimensional systems theory, IEEE circuits and systems, vol. CAS-26, 1979.

We have also obtained some partial answers!

Remarks for checking of FLP

No any other methods available to attack deciding problem of factor prime matrix;

This is the first result concerning factor prime matrix.

Full solutions can not obtained at this time, to be submit.

Remarks to the algorithm

Major problem is : how to extract a system of generators with l elements from a system of generators of syzygy module?

Book “Computational Commutative algebra I” by Martin Kreuzer and Lorenzo Robbiano provides a partial answer in Corollary 3.1.12 which was implemented in CoCoA.

Compare with other related work

- ◆ Previous papers only deal with very special cases.
- ◆ Above results are true for any field, that is, without restriction to characteristic zero and algebraically closed.
- ◆ Only Groebner bases theory is available to attack matrix factorization problems currently.

Some problems

I'd like to list some problems which deserve to be considered :

- ◆ Problem 1: if there are other efficient methods to deal with this problem?
- ◆ Problem 2: How to find a minimal generating system from a given generating system?
- ◆ Problem 3: If there are other criteria to assure that the existence of a generating system of l elements? ($l = \text{rank}$)

Some problems

- ◆ Problem 4: If elementary transformations for multivariate polynomial matrices can be used to find matrix factor?
- ◆ Problem 5: Find other characterizations for the existence of the matrix factor!
- ◆ Problem 6: Find the methods for dealing with non-full rank cases.

Conclusion

- ◆ Matrix factorization problems connects with many problems in system theory, for example, rational matrix fraction description.
- ◆ Generalization of convolutional codes in one variable to multidimension also needs to develop multivariate matrix theory.
- ◆ It needs to develop new algorithmic methods!



Thank you for your attention!