Coherent Configurations and Association Schemes. Part I

Definitions, examples, simple facts

Mikhail Klin
Department of Mathematics
Ben Gurion University of the Negev
Beer Sheva, Israel

Special Semester on Gröbner bases and Related Methods,
Linz, May 2006

Contents

1. Graphs, relations and matrices

- *A (directed) graph* $\Gamma = (X, R)$, where X is a finite set of vertices of cardinality n, and $R \subseteq X^2$ is a set of arcs (a binary relation). Usually, $X = \{1, 2, \ldots, n\} = [1, n]$ or $X = [0, n-1]$.

- *A (complete) colour graph* $\left(X, \{R_l\}_{1 \le l \le r}\right)$ is defined by a partition of X^2 into r binary relations. Usually each index, i, is regarded as a *colour*.

- *Adjacency matrix* $A_l = A(\Gamma_l)$ for $\Gamma_l = (X, R_l)$ is a $(0,1)$-matrix $A = (a_{ij})$ such that

$$a_{ij} = 1 \iff (i, j) \in R_l$$

  In this case, R_l is called *support* of A_l.

- Let H be a group, $S \subseteq H$.
  A graph $\Gamma = Cay(H,S)$ is defined as follows:
  $\Gamma = (H,R)$, where
  $$R = \{(x, x+h) | x \in H, h \in S\}$$
  Γ is called *Cayley graph* over H with a *connection set* S.
  No loops in $\Gamma \iff 0 \notin S$.
  Γ is undirected $\iff S = -S$, where
  $-S = \{-x | x \in S\}$.

- A partition $\{S_0, S_1, \ldots, S_d\}$ of H such that $S_0 = \{0\}$ defines a complete colour Cayley graph over H.

- Let $R, S \subseteq X^2$, we can define $R \circ S$ (*product* of two relations) as a multi-subset of X^2:
  A pair (a, b) appears in $R \circ S$ with the multiplicity $m(a, b)$ where

$$m(a,b) = |\{c \in X | (a,c) \in R \wedge (c,b) \in S\}|$$

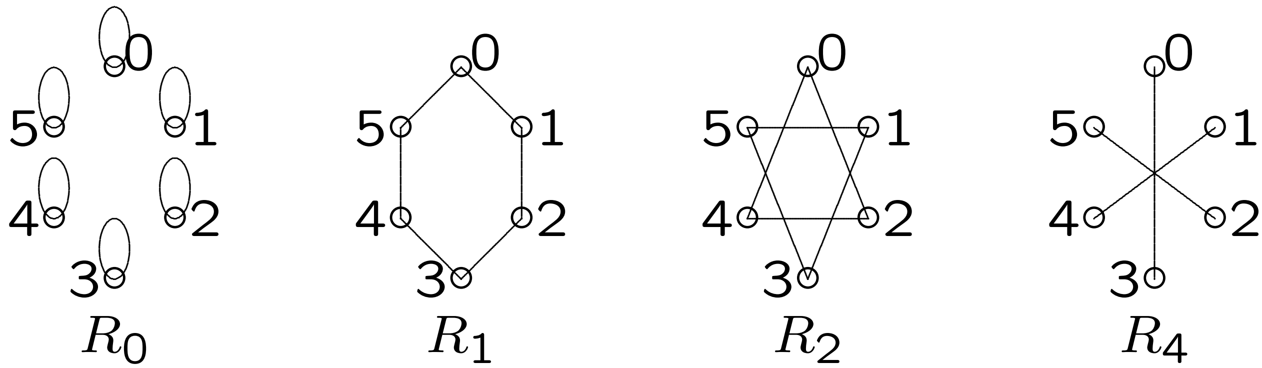
- This multi-subset, $R \circ S$, may be regarded as a colour graph over X.

- Clearly in this case we get for $\Gamma_1 = (X, R)$, $\Gamma_2 = (X, S)$,

$$A(\Gamma_1) \cdot A(\Gamma_2) = B$$

  where B is the linear combination of the adjacency matrices of "partial" relations in $R \circ S$.

Example 1. *a) Graphs:*



b) Connection sets of Cayley graphs over $\mathbb{Z}_6$*:*

$$S_0 = \{0\}, \quad S_1 = \{1,5\}, \quad S_2 = \{2,4\}, \quad S_3 = \{3\}$$

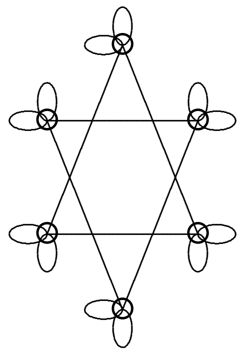
*c) Matrices (*0 *is substituted by* 6*)*

$$A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix} A_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Multiplication:

(i)



$R_1 \circ R_1$ $\qquad$ $A_1^2 = 2A_0 + A_2$

$$\begin{aligned} \underline{S_1} \circ \underline{S_1} &= \underline{1,5} \circ \underline{1,5} = \\ &= \underline{1+1, 1+5, 5+1, 5+5} = \\ &= \underline{2,0,0,4} = 2 \cdot \underline{0} + \underline{2,4} = \\ &= 2\underline{S_0} + \underline{S_2} \end{aligned}$$

(ii) $A_1 \cdot A_2 = A_1 + 2A_3$

$$\begin{aligned} \underline{S_1} \circ \underline{S_2} &= \underline{1,5} \circ \underline{2,4} = \\ &= \underline{3,5,1,3} = \underline{S_1} + 2\underline{S_3} \end{aligned}$$

Here $\underline{x_1, \ldots, x_k}$ *is element of group algebra over group* H *for* $x_1, \ldots, x_k \in H$*. (More about this in part II).*

2. 2-orbits of permutation groups

- *Permutation* g of a finite set X is a bijection on set X.
  For $x \in X$ and permutation g, x^g denotes image of x under g.
  For permutations g_1, g_2 on X, their *composition* (product) $g = g_1 g_2$ is $x^g = x^{g_1 g_2} = (x^{g_1})^{g_2}$ for $x \in X$.
  $S(X)$ is *Symmetric group* of X, if $|X| = n$, we denote it by S_n.

- A *permutation group* (G, X) of degree n is a subgroup G of the symmetric group $S(X)$; G *acts* on set X.

- A subset $G \subseteq S(X)$ is a permutation group $\iff$ G is closed under composition.

- For $x \in X$ and permutation group (G, X), the *orbit* $Orb(x)$ is $Orb(x) = x^G = \{x^g | g \in G\}$.
  $Orb(G, X)$ is the set of all orbits of (G, X).
  Transitive group $(G, X) \iff Orb(G, X) = \{X\}$.

- For (G, X) we consider induced action (G, X^2):
  For $(a, b) \in X^2$, $(a, b)^g = (a^g, b^g)$.
  Orbits of this induced action are called (following H. Wielandt) *2-orbits* of (G, X).
  $2 - Orb(G, X)$ is the set of all 2-orbits.

- Pair $(X, 2 - Orb(G, X))$ is a complete colour graph with the vertex set X.

Example 1. *(Revisited) Let D_6 be the dihedral group of degree 6 and order 12,*

$$D_6 =< (0,1,2,3,4,5),(1,5)(2,4) >$$

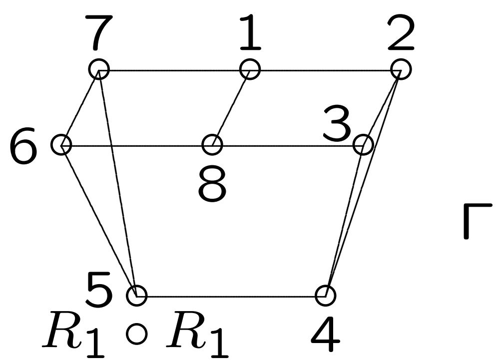
Clearly, D_6 is transitive. Then,

$$2 - orb(D_6, [0,5]) = \{R_0, R_1, R_2, R_3\}$$

*Here all graphs defined by the 2-orbits (*orbital *graphs in alternative terminology) are regular undirected graphs. The rank r of a permutation group is $|2 - Orb(G,X)|$, thus*

$$r(D_6, [0,5]) = 4$$

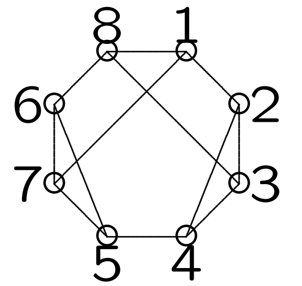
Example 2.



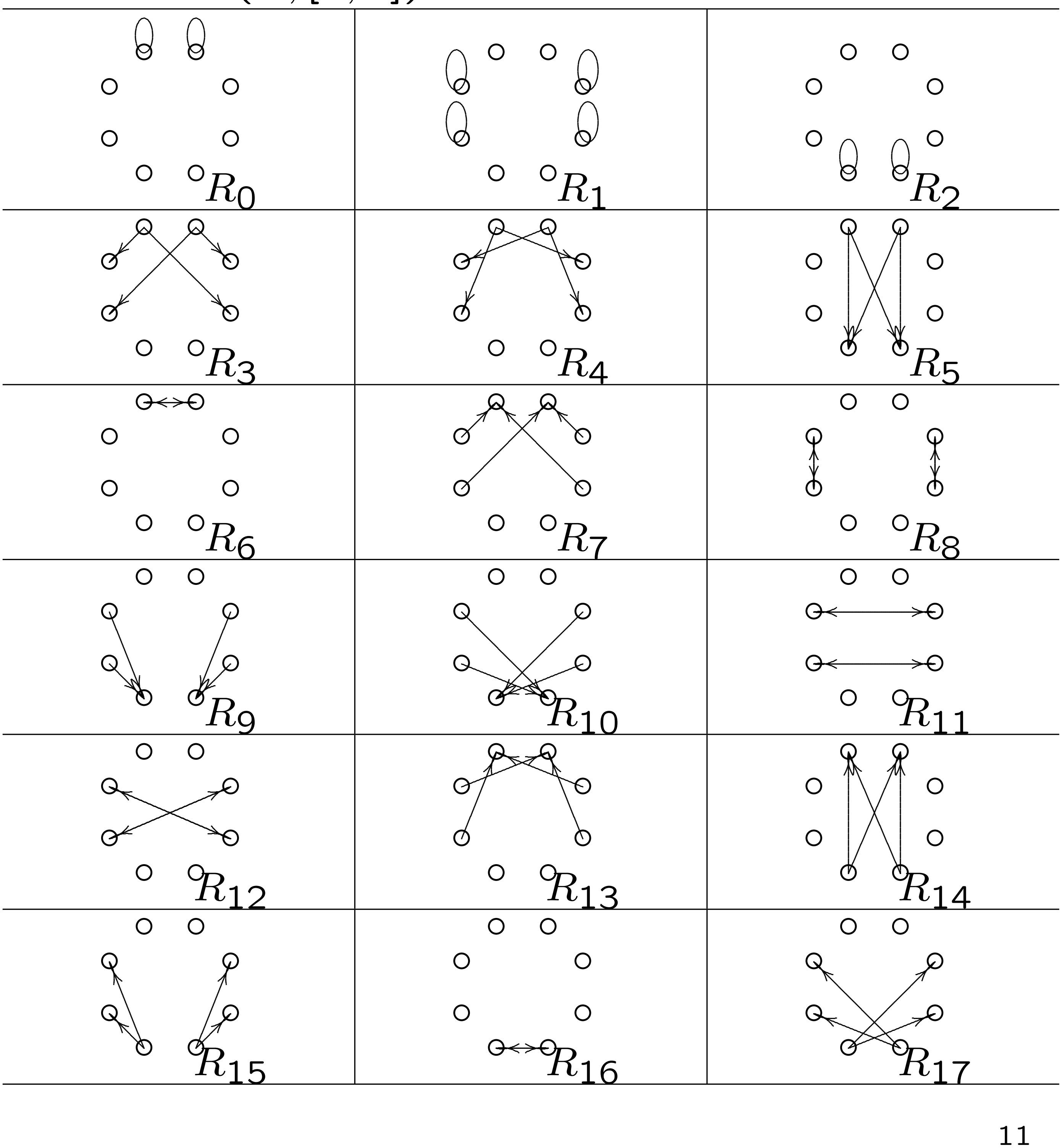
*"Chemical" graph cuneane=wedge=*КЛИН *(determines a synthesized compound).*

$G = Aut(\Gamma) = \{e, g_2, g_3, g_4\}$, *where* e *is the identity permutation,* $g_2 = (1,8)(2,3)(6,7)$, $g_3 = (2,7)(3,6)(4,5)$, $g_4 = (1,8)(2,6)(3,7)(4,5)$. $Orb(G, [1,8]) = \{\{1,8\}, \{2,3,6,7\}, \{4,5\}\}$. *Using the orbit counting lemma we get*

$$r(G, [1,8]) = \frac{1}{4}(8^2 + 2 \cdot 2^2 + 0^2) = 18$$

Cuneane again: 

2-orbits of $(G, [1, 8])$:



- Let $g \in S(X)$ denote by $M(g)$ a permutation matrix of order $n = |X|$,

$$M(g) = (m_{ij})_{1 \leq i,j \leq n}$$

where $m_{ij} = \begin{cases} 1 & j = i^g \\ 0 & \text{otherwise} \end{cases}$.
Clearly, $M(g^{-1}) = (M(g))^{-1} = (M(g))^t$, where M^t is the matrix transposed to M.

- Let K be a ring or a field. Let $M_n(K)$ be the ring (algebra) of all square matrices of order n over K. Define for a permutation group (G, X),

$$V_K(G, X) = \{A \in M_n(K) | \forall g \in G \ AM(g) = M(g)A\}$$

- $V_K(G, X)$ is a ring (algebra) which is called the *centralizer ring* (algebra) of (G, X). (Most important cases for us are $K = \mathbb{Z}$, $K = \mathbb{C}$. Usually sign K is omitted.)

- **Theorem** Let (G, X) be a permutation group of degree n and rank r. Then

  a) $V_K(G, X)$ as a vector space over K has dimension r;

  b) $V_K(G, X)$ has a special basis which consists of $(0, 1)$-matrices;

  c) The members of this *standard basis* are in bijective correspondence with the set $2-Orb(G, X)$.

- To prove this theorem we have to detect that if

$$2 - Orb(G, X) = (R_1, \ldots, R_r)$$

  then $V(G, X) =< A_1, \ldots, A_r >$ where $A_i = A(\Gamma_i)$, $\Gamma_i = (X, R_i)$, $1 \leq i \leq r$.

Example 1. *(Revisited)*

$$V(D_6, [0,5]) = < A_0, A_1, A_2, A_3 >$$

Multiplication table: $R_i \cdot R_j = \sum_{k=0}^{3} p_{ij}^k R_k$

$\backslash R_j$ $R_i \backslash$	R_0	R_1	R_2	R_3	control sum
R_0	1	0	0	0	1
	0	1	0	0	1
	0	0	1	0	1
	0	0	0	1	1
R_1	0	2	0	0	2
	1	0	1	0	2
	0	1	0	1	2
	0	0	2	0	2
R_2	0	0	2	0	2
	0	1	0	1	2
	1	0	1	0	2
	0	2	0	0	2
R_3	0	0	0	1	1
	0	0	1	0	1
	0	1	0	0	1
	1	0	0	0	1

Example 2. *Here $V(G,X)$ has rank 18, thus it has 18 basis matrices. All these matrices are presented in a compact form as a colour matrix*

$$A(V(G,X)) = \sum_{i=0}^{17} iA_i$$

where $A_0, A_1, \ldots, A_{17}$ is the standard basis of $V(G,X)$.

$$\begin{pmatrix} 0 & 3 & 4 & 5 & 5 & 4 & 3 & 6 \\ 7 & 1 & 8 & 9 & 10 & 11 & 12 & 13 \\ 13 & 8 & 1 & 9 & 10 & 12 & 11 & 7 \\ 14 & 15 & 15 & 2 & 16 & 17 & 17 & 14 \\ 14 & 17 & 17 & 16 & 2 & 15 & 15 & 14 \\ 13 & 11 & 12 & 10 & 9 & 1 & 8 & 7 \\ 7 & 12 & 11 & 10 & 9 & 8 & 1 & 13 \\ 6 & 4 & 3 & 5 & 5 & 3 & 4 & 0 \end{pmatrix}$$

3. Coherent configurations

- We are now giving an axiomatic analogue of $2 - Orb(G, X)$ and $V(G, X)$ for an arbitrary permutation group.

- Let $X = [1, n]$, $\mathfrak{R} = \{R_1, \ldots, R_r\}$ a collection of binary relations on X such that:

CC1 $R_i \cap R_j = \emptyset$ for $1 \leq i \neq j \leq r$;

CC2 $\bigcup_{i=1}^{r} R_i = X^2$;

CC3 $\forall i \in [1, r] \exists i' \in [1, r] R_i^t = R_{i'}$, where $R_i^t = \{(y, x) | (x, y) \in R_i\}$;

CC4 $\exists I' \subseteq [1, r] \bigcup_{i \in I'} R_i = \Delta$, where $\Delta = \{(x, x) | x \in X\}$;

CC5 $\forall i, j, k \in [1, r] \forall (x, y) \in R_k | \{z \in X | (x, z) \in R_i \wedge (z, y) \in R_j\}| = p_{ij}^k$

$\mathfrak{M} = (X, R)$ is called a *coherent configuration* (D. G. Higman, 1970).

- The numbers p_{ij}^k are *intersection numbers*.

- Switching to matrix language: Let $W \subseteq M_n(\mathbb{C})$ be a matrix algebra over $\mathbb{C}$ such that

CA1 W as a linear space over $\mathbb{C}$ has some basis, $A_1, A_2, \ldots, A_r$, consisting of $(0,1)$-matrices;

CA2 $\sum_{i=1}^{r} A_i = J_n$, where J_n is the matrix of order n all entries of which are equal to 1;

CA3 $\forall i \in [1,r] \exists i' \in [1,r] A_i^t = A_{i'}$;

Then W is called a *coherent algebra* of rank r and order n with the *standard basis* $A_1, A_2, \ldots, A_r$. We write $W = < A_1, \cdots, A_r >$.
Remark: similar notion of a *cellular algebra* (B. Weisfeiler, A. A. Leman, 1968).

- To each coherent configuration $\mathfrak{M} = (X, \mathfrak{R})$ we associate its *adjacency algebra* $W = < A_i = A(X, R_i) | R_i \in \mathfrak{R} >$ which is a coherent algebra.

- **Proposition**: For each permutation group (G, X) its centralizer algebra $V(G, X)$ is a coherent algebra.

- Remark: Converse proposition is not true: there are many examples of coherent algebras which are not centralizer algebras of a suitable permutation group.

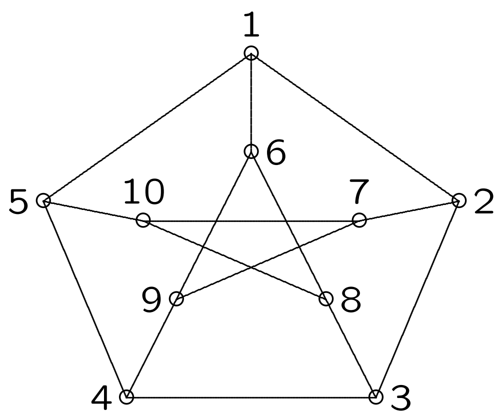
- **Theorem** A matrix algebra W is a coherent algebra $\iff$ W is closed with respect to Schur-Hadamard (entry-wise) multiplication, transposition, and contains matrices I, J, where I is the identity matrix.

- <u>Remark</u> An equivalent formulation uses closure with respect to the Hermitean adjoint instead of transposition.

- **Corollary** If W_1 and W_2 are two coherent algebras of the same order n, then $W = W_1 \cap W_2$ is also a coherent algebra of order n.

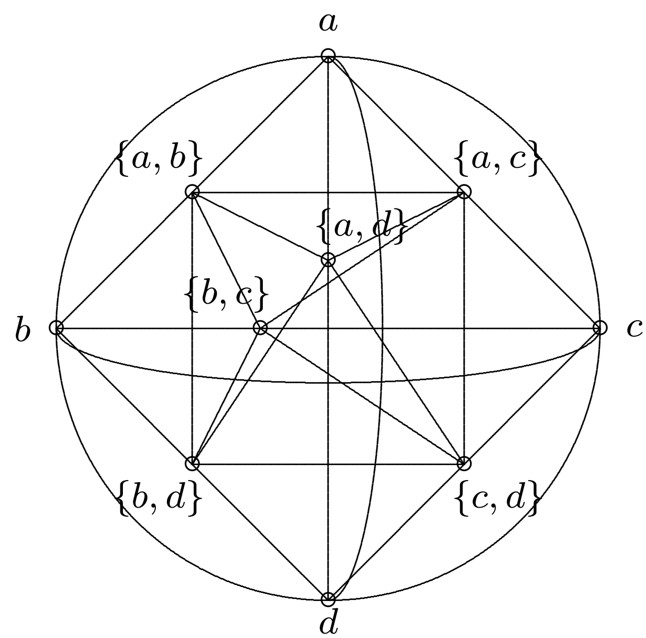
(More details in part II.)

Example 3. Part a)



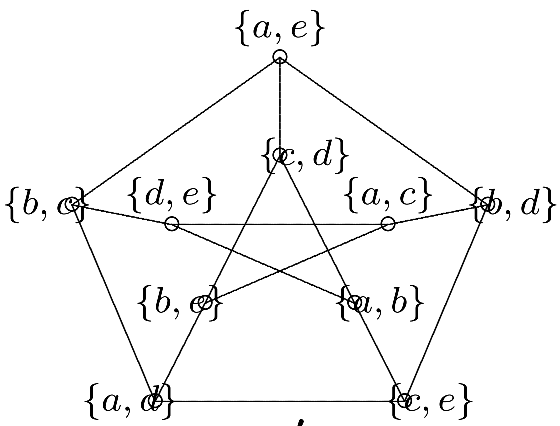
Γ'

$D_5 \leq Aut(\Gamma')$

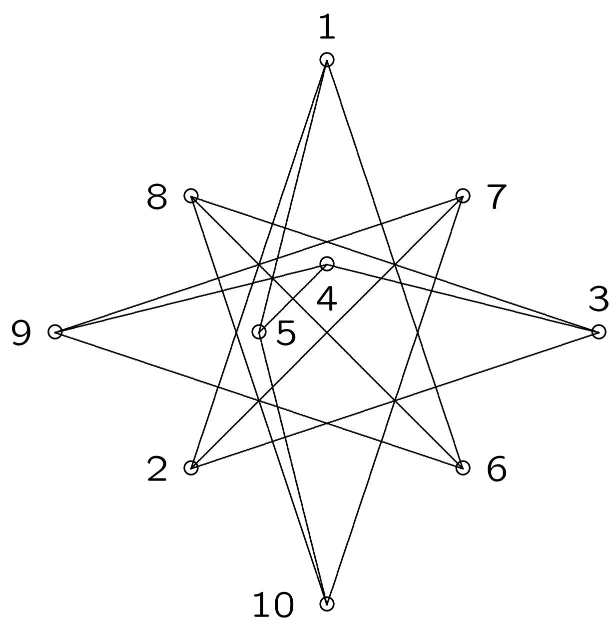


$\overline{\Gamma''}$

$S_4 \leq Aut(\overline{\Gamma''}) = Aut(\Gamma'')$



Γ'



Γ''

$\Gamma' = \Gamma'' = \Gamma =$ *Petersen graph.* $|Aut(\Gamma)| \geq$ gcd$(10, 24) = 120$. *In fact,* $Aut(\Gamma) \simeq S_5$.

Part b)

$$\begin{pmatrix} 0 & 2 & 3 & 3 & 2 & 4 & 5 & 6 & 6 & 5 \\ 2 & 0 & 2 & 3 & 3 & 5 & 4 & 5 & 6 & 6 \\ 3 & 2 & 0 & 2 & 3 & 6 & 5 & 4 & 5 & 6 \\ 3 & 3 & 2 & 0 & 2 & 6 & 6 & 5 & 4 & 5 \\ 2 & 3 & 3 & 2 & 0 & 5 & 6 & 6 & 5 & 4 \\ 7 & 8 & 9 & 9 & 8 & 1 & 10 & 11 & 11 & 10 \\ 8 & 7 & 8 & 9 & 9 & 10 & 1 & 10 & 11 & 11 \\ 9 & 8 & 7 & 8 & 9 & 11 & 10 & 1 & 10 & 11 \\ 9 & 9 & 8 & 7 & 8 & 11 & 11 & 10 & 1 & 10 \\ 8 & 9 & 9 & 8 & 7 & 10 & 11 & 11 & 10 & 1 \end{pmatrix}$$

$W' = V(D_5, [1..10])$, $rank(W') = \frac{1}{10}(10^2 + 5 \cdot 2^2) = 12$

$$\begin{pmatrix} 0 & 2 & 3 & 4 & 2 & 2 & 4 & 4 & 3 & 3 \\ 5 & 1 & 5 & 6 & 6 & 6 & 7 & 6 & 8 & 8 \\ 3 & 2 & 0 & 2 & 4 & 4 & 4 & 2 & 3 & 3 \\ 8 & 6 & 5 & 1 & 7 & 6 & 6 & 6 & 5 & 8 \\ 5 & 6 & 8 & 7 & 1 & 6 & 6 & 6 & 8 & 5 \\ 5 & 6 & 8 & 6 & 6 & 1 & 6 & 7 & 5 & 8 \\ 8 & 7 & 8 & 6 & 6 & 6 & 1 & 6 & 5 & 5 \\ 8 & 6 & 5 & 6 & 6 & 7 & 6 & 1 & 8 & 5 \\ 3 & 4 & 3 & 2 & 4 & 2 & 2 & 4 & 0 & 3 \\ 3 & 4 & 3 & 4 & 2 & 4 & 2 & 2 & 3 & 0 \end{pmatrix}$$

$W'' = V(S_4, [1, 10])$, $rank(W'') = \frac{1}{24}(10^2 + 6 \cdot 4^2 + 3 \cdot 2^2 + 8 \cdot 1^2) = 9$

$$\begin{pmatrix} 0 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 2 & 2 \\ 1 & 0 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 \\ 2 & 1 & 0 & 1 & 2 & 2 & 2 & 1 & 2 & 2 \\ 2 & 2 & 1 & 0 & 1 & 2 & 2 & 2 & 1 & 2 \\ 1 & 2 & 2 & 1 & 0 & 2 & 2 & 2 & 2 & 1 \\ 1 & 2 & 2 & 2 & 2 & 0 & 2 & 1 & 1 & 2 \\ 2 & 1 & 2 & 2 & 2 & 2 & 0 & 2 & 1 & 1 \\ 2 & 2 & 1 & 2 & 2 & 1 & 2 & 0 & 2 & 1 \\ 2 & 2 & 2 & 1 & 2 & 1 & 1 & 2 & 0 & 2 \\ 2 & 2 & 2 & 2 & 1 & 2 & 1 & 1 & 2 & 0 \end{pmatrix}$$

$W = W' \cap W'' = V(Aut(\Gamma)) = V(S_5, [1, 10])$ $rank(W) = 3$.

There exists a nice simple purely combinatorial algorithm which allows to construct W from W' and W''.

- If $W' \subseteq W$ are coherent algebras of the same order then W' is called a *coherent subalgebra* of W.

- If $\mathfrak{M}'$ is a coherent configuration with adjacency algebra W' and $\mathfrak{M}$ corresponds to W then in this case $\mathfrak{M}'$ is called *fusion* coherent configuration of $\mathfrak{M}$ or *merging* of relations (classes) of $\mathfrak{M}$.

- Enumeration of coherent subalgebras of a given coherent algebra is an important task (see further talks).

- *Diagonal relation* $\Delta = \{(x,x)|x \in X\}$ is splitted in a coherent configuration $\mathfrak{M}$ into (one or more) basis relations. Each such basis relation uniquely defines a *fiber* of $\mathfrak{M}$. (See more precise definition in abstract).

- Fiber is a combinatorial analogue of an orbit of a permutation group. Coherent configurations with one fiber are analogues of transitive permutation groups.

- Coherent configurations with one fiber is a *homogeneous coherent configuration* or an *association scheme* (not obligatory commutative).

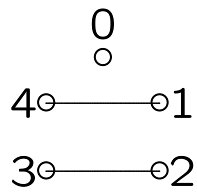
4. Association schemes

- Again: homogeneous coherent configuration = association scheme = coherent configuration with one fiber = diagonal relation Δ is one of basis relations = all basis graphs are regular.

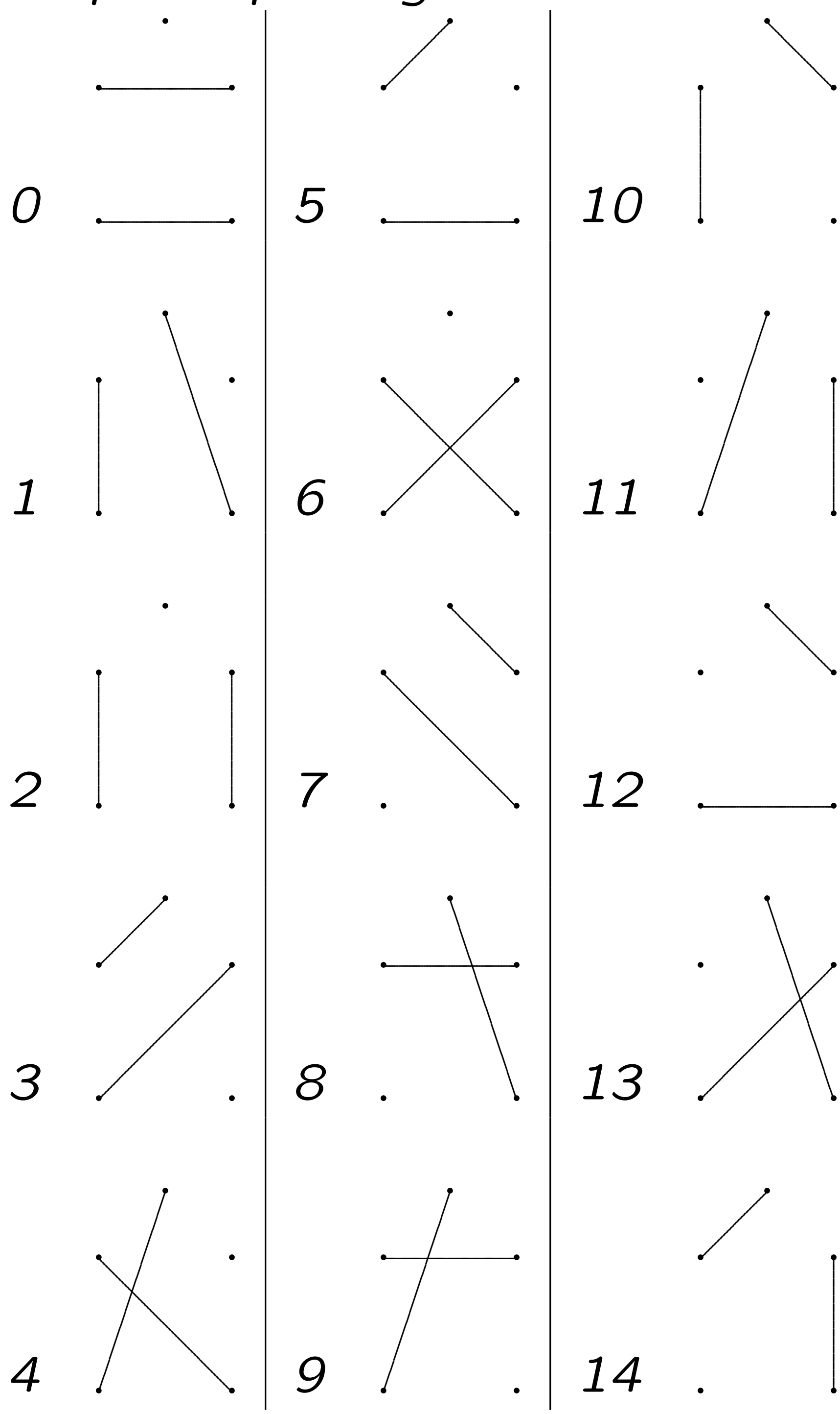
- In this context association scheme is a very natural combinatorial analogue of a transitive permutation group.

- *Schurian* association scheme $\mathfrak{M}$ corresponds to pair $(X, 2 - Orb(G,X))$ for a suitable transitive permutation group (G,X); otherwise, $\mathfrak{M}$ is *non-Schurian*.

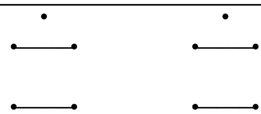
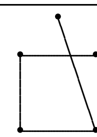
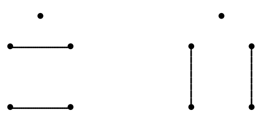
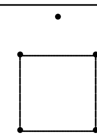
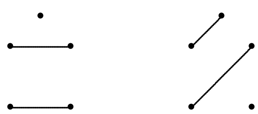
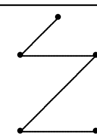
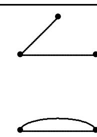
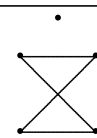
Example 4. *Let us start with* $(A_5, [0,4])$, *the* alternating group *of degree* 5 *and order* 60.
Let X *be set of all labeled graphs of form*



$|X| = 15$. A_5 *acts transitively on* X. *Labeling produced by computer package COCO:*

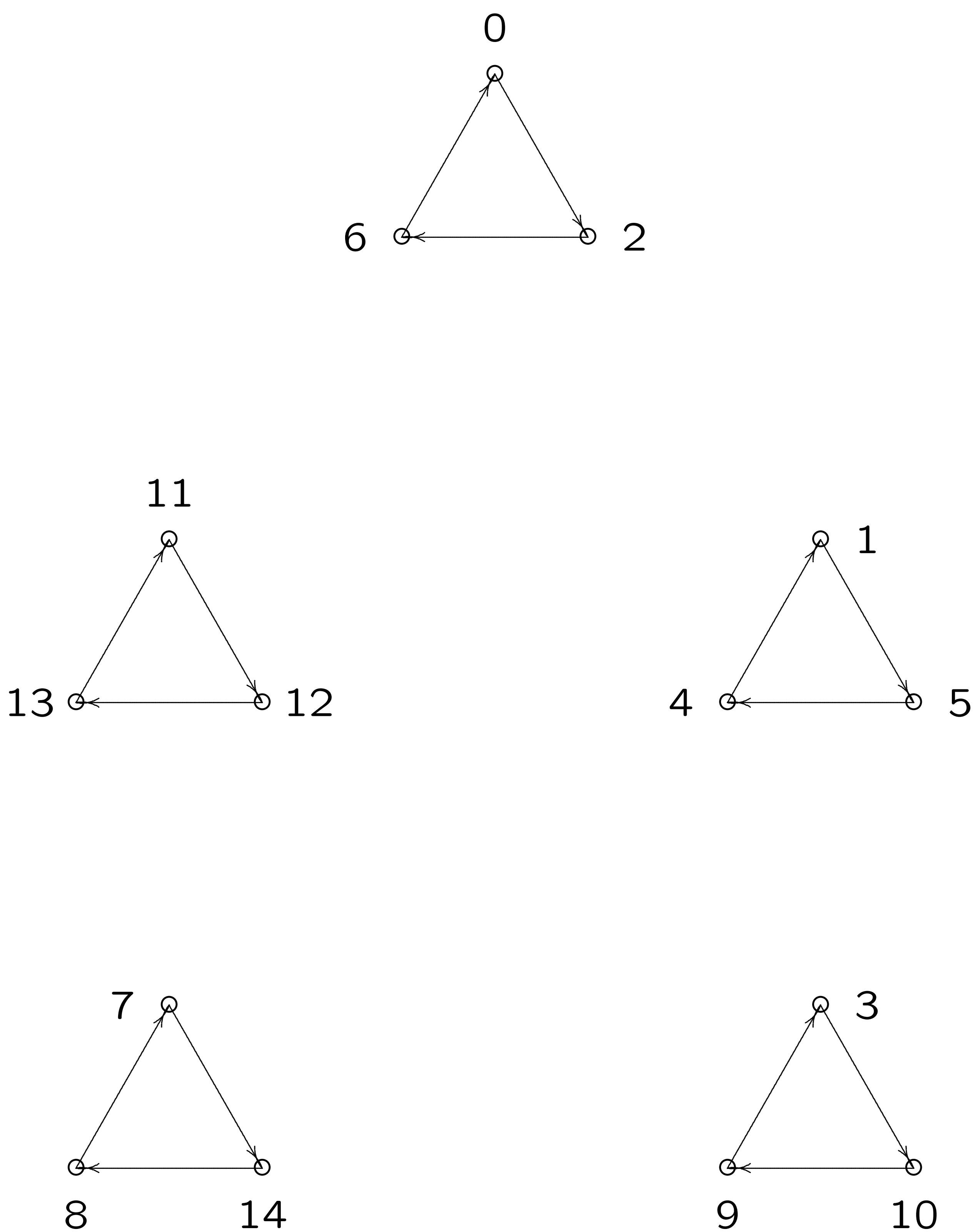


"Typical" representatives of 2-orbits:

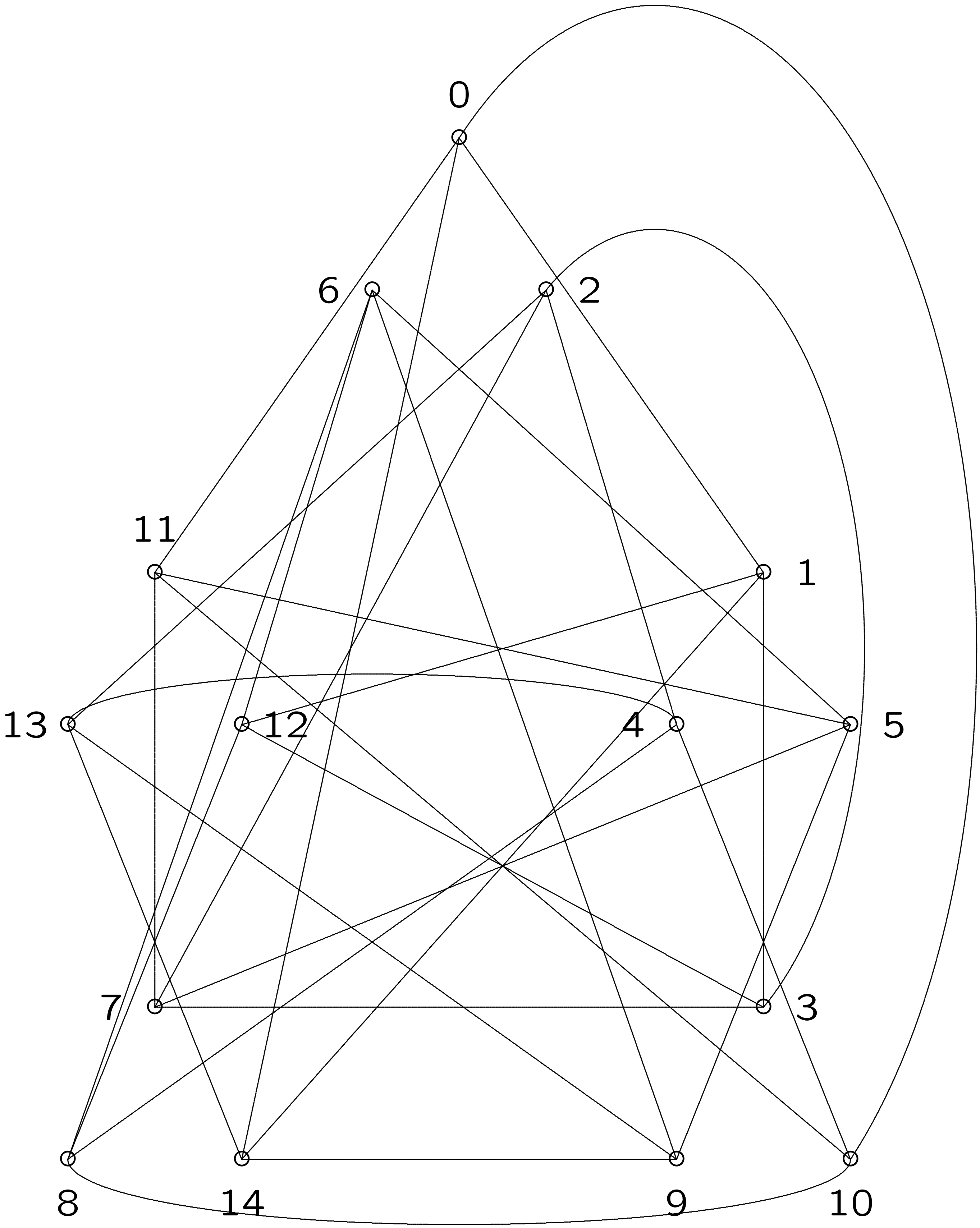
R_i	*Pair*	*Picture*	*Valency*	*Comments*
R_0	$(0,0)$		1	*diagonal*
R_1	$(0,1)$		4	
R_2	$(0,2)$		1	
R_3	$(0,3)$		4	
R_4	$(0,5)$		4	
R_5	$(0,6)$		1	
		Total:	*15*	

Remark: distinction between R_1 and R_3, R_2 and R_5 requires "chirality vision".

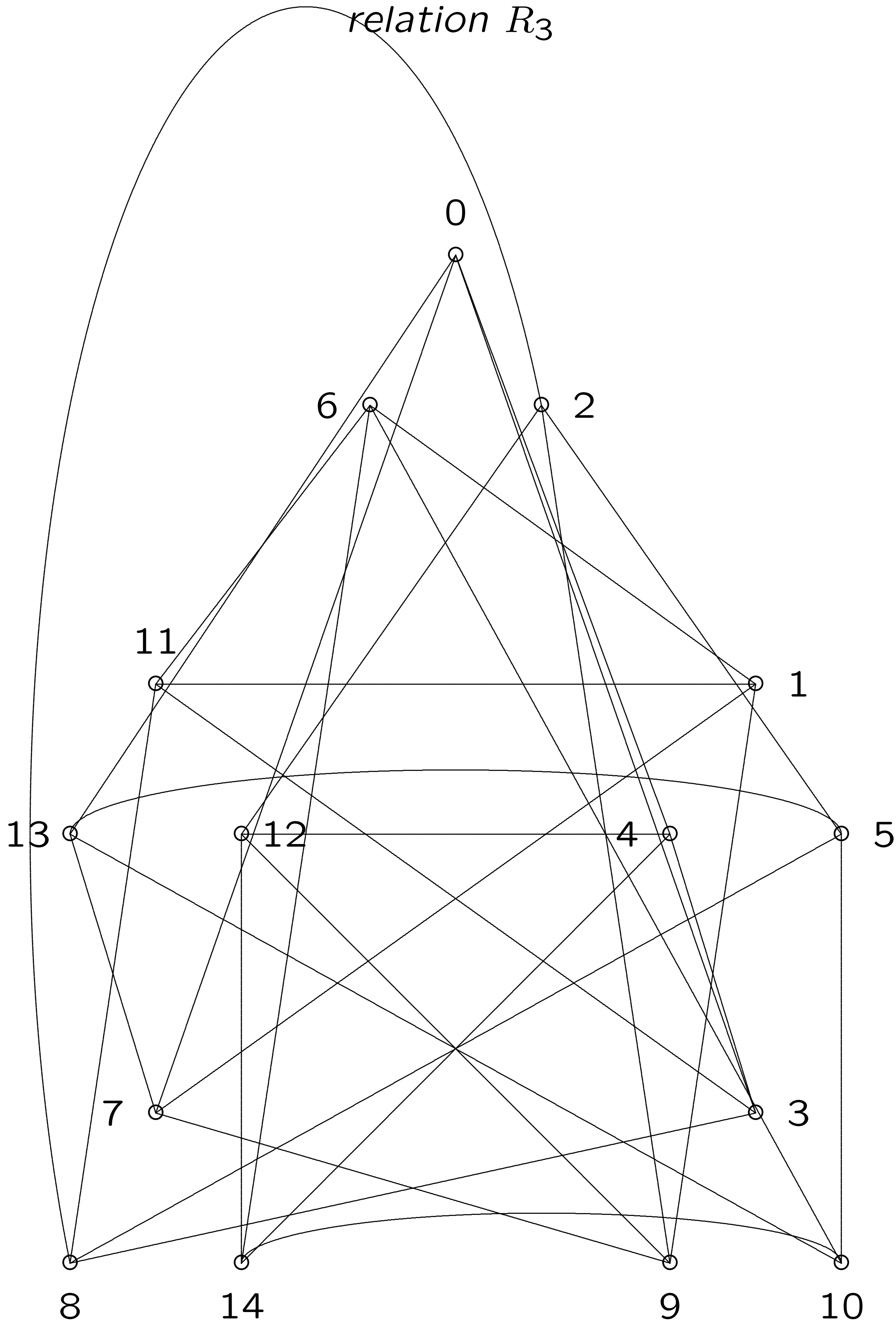
relation R_2 ($R_5 = R_2^t$)



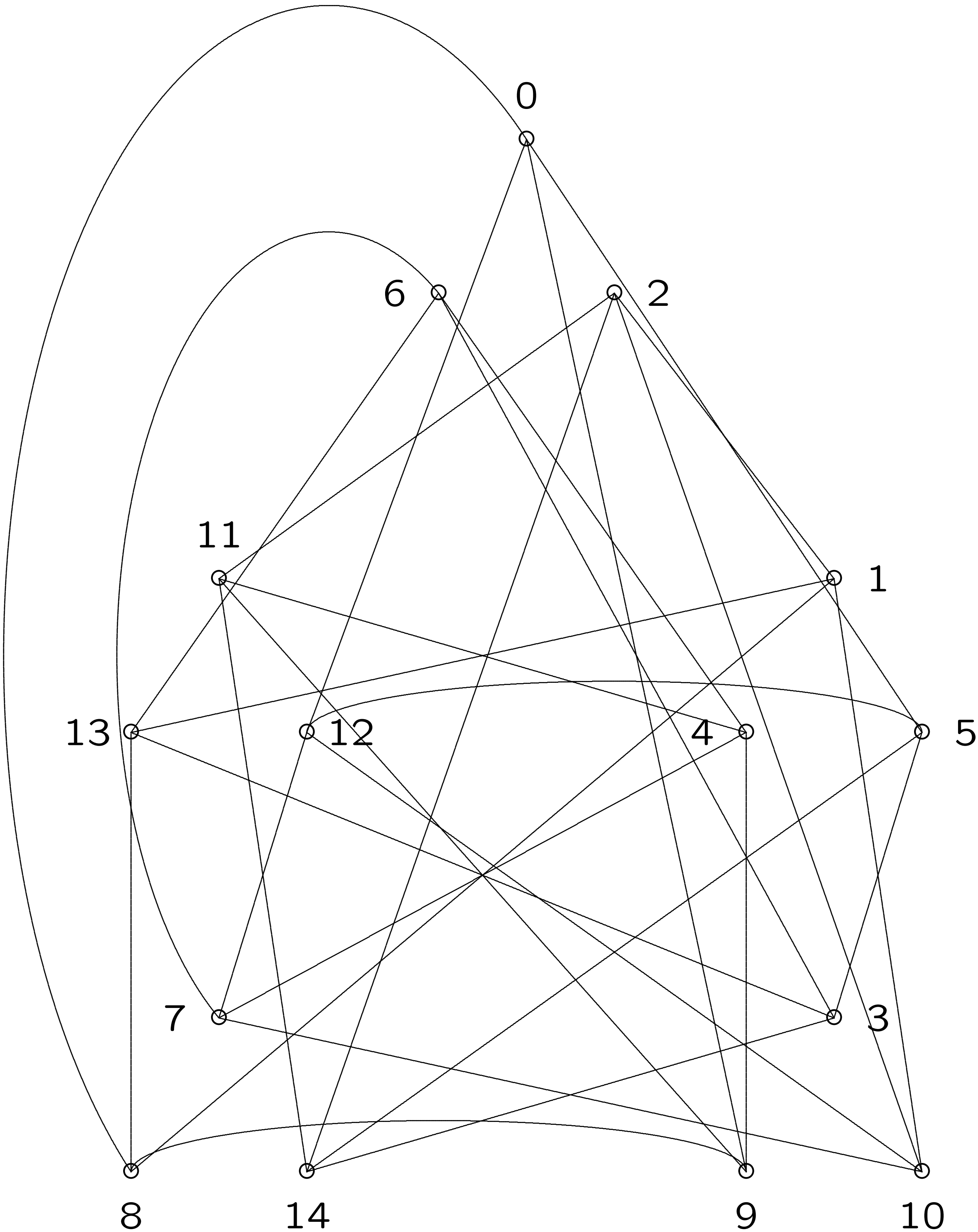
relation R_1



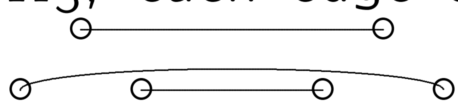
relation R_3



relation R_4



Comments

1. We have 3 times the same situation: 3-fold covering of K_5 (triangles in R_2 are vertices of K_5; each edge of K_5 is "blown-up" to 1-factor ).

2. The scheme is not commutative.

3. This is the first member of an exceptional infinite series of so-called *Siamese* association schemes on $(q+1)(q^2+1)$ vertices, q is a prime power (here, $q = 2$).

5. Adjacency and intersection algebras

- Recall that structure constants form a 3-dimensional tensor.

  Let $\mathfrak{M} = (X, \{R_0, R_1, \ldots, R_d\})$ be an association scheme with d *classes* (diagonal relation R_0 does not have status of a class). Let us arrange all structure constants P_{ij}^k into $d+1$ matrices $B_0, B_1, \ldots, B_d$

$$B_i = \begin{pmatrix} p_{i0}^0 & p_{i0}^1 & \cdots & p_{i0}^d \\ p_{i1}^0 & p_{i1}^1 & \cdots & p_{i1}^d \\ \cdots & \cdots & \cdots & \cdots \\ p_{id}^0 & p_{id}^1 & \cdots & p_{id}^d \end{pmatrix} \quad 0 \le i \le d$$

  Let $\mathfrak{B} = \ll B_0, B_1, \ldots, B_d \gg$ be the subalgebra of $M_{d+1}(\mathbb{C})$ generated by $B_0, B_1, \ldots, B_d$.

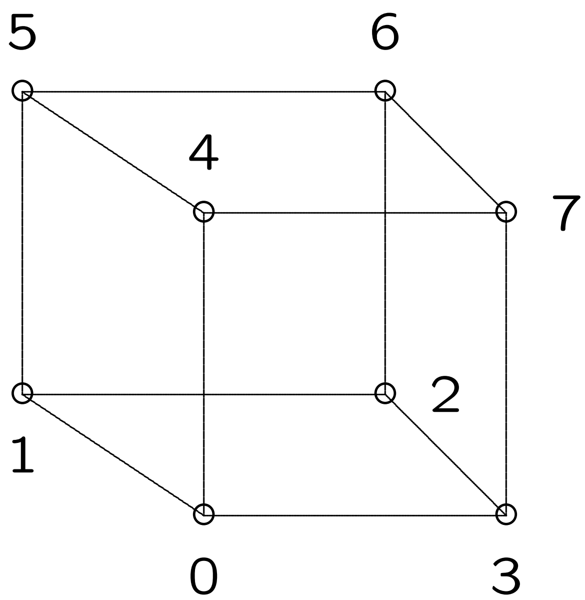
- **Theorem** The adjacency algebra of $\mathfrak{M}$ and algebra $\mathfrak{B}$ are anti-isomorphic; the isomorphism is established by correspondence $A_i \to B_i$. As a result, in particular, matrices A_i and B_i have the same minimal polynomial.

- **Corollary** In commutative case, both algebras are isomorphic.

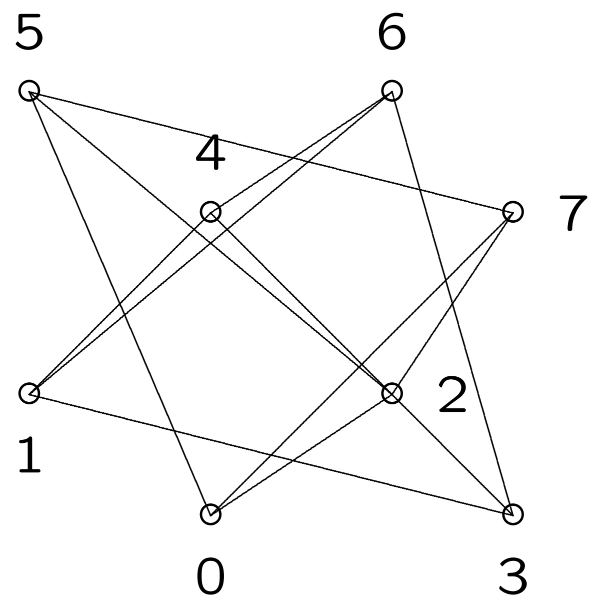
- The algebra $\mathfrak{B}$ is called *the intersection algebra* of association scheme $\mathfrak{M}$.

- Remark Similar theorem is valid for coherent configurations.

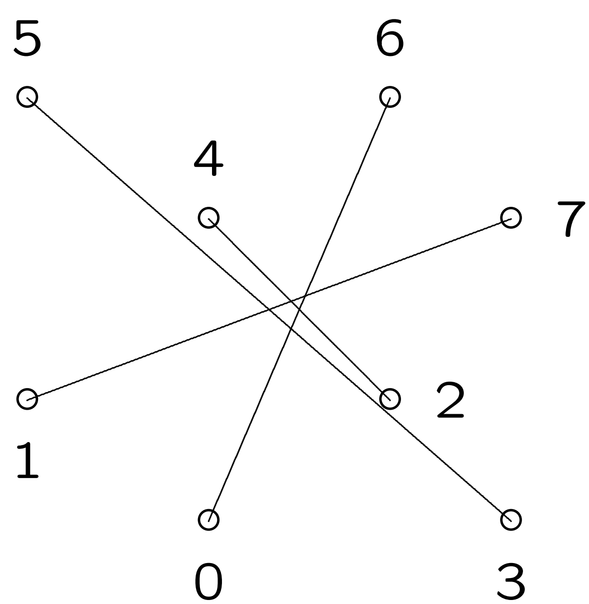
Example 5. *Metric association scheme with 3 classes, generated by 3-dimensional cube* Q_3.



R_1



R_2



R_3

Table of multiplication

$R_i \backslash R_j$	R_0	R_1	R_2	R_3
R_0	1	0	0	0
	0	1	0	0
	0	0	1	0
	0	0	0	1
R_1	0	3	0	0
	1	0	2	0
	0	2	0	1
	0	0	3	0
R_2	0	0	3	0
	0	2	0	1
	1	0	2	0
	0	3	0	0
R_3	0	0	0	1
	0	0	1	0
	0	1	0	0
	1	0	0	0

$$B_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 2 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 3 \\ 3 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad B_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

For example, $A_1^2 = 3A_0 + 2A_2$ (easy combinatorial inspection)

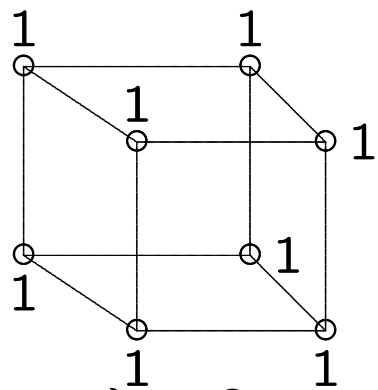
$$B_1^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 2 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 3 & 0 & 2 & 0 \\ 0 & 7 & 0 & 6 \\ 6 & 0 & 7 & 0 \\ 0 & 2 & 0 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 4 & 0 & 6 \\ 6 & 0 & 4 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix} = 3B_0 + 2B_2$$

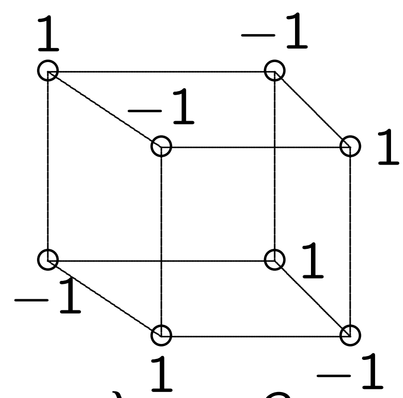
Why this knowledge is helpful?

We want to get $Spec(Q_3)$.

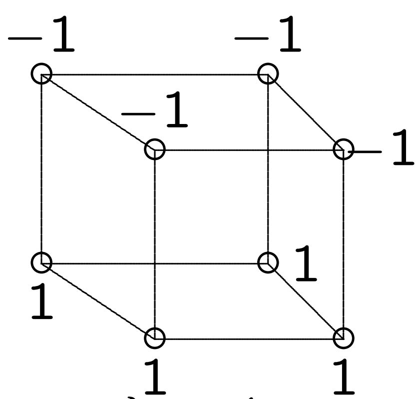
Way 1 Tricks a la C. Godsil.



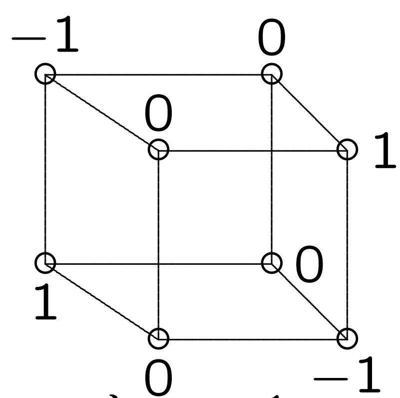
$\lambda = 3$



$\lambda = -3$



$\lambda = 1$



$\lambda = -1$

Way 2

$$\begin{vmatrix} \lambda & -1 & 0 & 0 \\ -3 & \lambda & -2 & 0 \\ 0 & -2 & \lambda & -3 \\ 0 & 0 & -1 & \lambda \end{vmatrix} = \lambda \begin{vmatrix} \lambda & -2 & 0 \\ -2 & \lambda & -3 \\ 0 & -1 & \lambda \end{vmatrix} + \lambda \begin{vmatrix} -3 & -2 & 0 \\ 0 & \lambda & -3 \\ 0 & -1 & \lambda \end{vmatrix} =$$

$$= \lambda(\lambda(\lambda^2 - 3) + 2(-2\lambda)) - 3(\lambda^2 + 3) = \lambda^4 - 10\lambda^2 - 9 =$$

$$= (\lambda^2 - 9)(\lambda^2 - 1)$$

Eigenvalues are $-1, 1, 3, -3$.

(More about spectrum in part II).

6. Strongly regular graphs

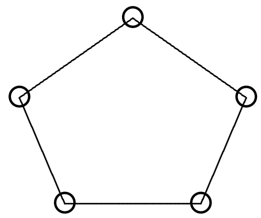
- A graph $\Gamma = (V, E)$ is *strongly regular* if it is regular of valency k and each pair of adjacent (unadjacent) vertices has exactly λ (μ) common neighbours.
  In this case we write $SRG(v, k, \lambda, \mu)$, where $v = |V|$.

- Γ is an srg $\iff$

$$\begin{cases} A^2 = kI + \lambda A + \mu(J - A - I) \\ AJ = JA = kJ \end{cases}$$

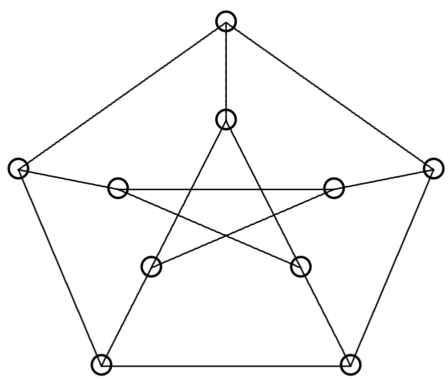
- Γ is an srg $\iff$ $\mathfrak{M}(\Gamma) = (V, < \Delta, E, \overline{E} >)$ is a symmetric association scheme with 2 classes, here $\overline{\Gamma} = (V, \overline{E})$ is graph *complementary* to Γ.

Example 6. *a) Pentagon*



$SRG(5, 2, 0, 1)$

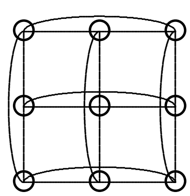
b) Petersen graph



$SRG(10, 3, 0, 1)$

c) $l \circ K_m$*: disjoint union of* l *copies of complete graph with* m *vertices.* $SRG(lm, m-1, m-2, 0)$.

d) Lattice square graph $L_2(3)$.



$SRG(9, 4, 1, 2)$

- SRG Γ is called *primitive* if both Γ and $\overline{\Gamma}$ are connected graphs.

- Using association scheme $\mathfrak{M}(\Gamma)$ with 2 classes (that is of dimension 3) it is easy to prove that an undirected regular Γ is an SRG $\iff$ Γ has exactly 3 distinct eigenvalues.

- There are various combinatorial and spectral equations and restrictions for sequence (v, k, λ, μ) of parameters of an SRG $\Rightarrow$
  feasible set of parameters.

- a central question: existence of an SRG with a given feasible set of parameters.

- Rank 3 graph $\Gamma \iff Aut(\Gamma)$ is acting transitively on vertices, ordered pairs of adjacent and non-adjacent vertices.

- Rank 3 graph $\Gamma \iff$ for $G = Aut(\Gamma)$, $2-Orb(G,V) = \{\Delta, E, \overline{E}\}$.

- Rank 3 graph $\Gamma \Rightarrow \Gamma$ is an SRG.

- All rank 3 graphs are described via the use of classification of finite simple groups.

Two most known infinite series of SRG's:

- Triangular graph $T(m)$ = line graph of K_m is $SRG(\frac{m(m-1)}{2}, 2(m-2), 1, 4)$. For $m \neq 8$ is uniquely determined by parameters.

- Lattice square graph $L_2(m)$ = line graph of the complete bipartite graph $K_{m,m}$ is

$$SRG(m^2, 2(m-1), 0, 2)$$

For $m \neq 4$ is uniquely determined by parameters.

- Many other infinite series and sporadic examples are known.

- Origins in design of statistical experiments and partial geometries (R. C. Bose, 1963).

- Important and beautiful links with coding theory and finite geometries. In particular, poster by Axel Kohnert.

- Diverse behaviour:

  - for some parameter sets there exists ”prolific” constructions;

  - for some parameter sets uniqueness is proved.

Example 7. *A non-rank-3 SRG on 280 vertices = non-Schurian association scheme with 2 classes (A.A. Ivanov, Klin, Faradžev, 1984), (Mathon-Rosa, 1985).*

- *Consider the symmetric group* S_9 *acting on the partitions of 9-set into 3 subsets of size 3;*

- *Degree is* $\frac{9!}{(3!)^4} = 280$*;*

- *Rank is 5 with subdegrees 1, 27, 36, 54, 162 (any two partitions have 3, 5, 9, 6, 7 non-empty intersections).*

- *The relation defined by 7 intersections gives*

$$\Gamma = SRG(280, 162, 96, 90)$$

$$\left(\overline{\Gamma} = SRG(280, 117, 44, 52)\right)$$

Problem Classify all SRG's Γ such that $Aut(\Gamma)$ is a primitive permutation group representation of the symmetric group S_n.
Known examples:

- K_n (trivial);
- $T(m)$ $(m \neq 4)$;
- a few sporadic examples, including the above one are known (Klin et al).

(Certain progress was achieved by Muzychuk and his followers.)

7. Distance regular graphs and metric schemes

- $\Gamma = (V, E)$ is (undirected) connected regular graph of valency k;
  for $x, y \in V$, $d(x, y)$ is a *distance* between vertices x and y, that is, the length of shortest path in Γ with the ends x, y;
  $d = d(\Gamma)$ is *diameter* of Γ.

- for $0 \leq i \leq d$ the *distance graph* $\Gamma_i = (V, R_i)$ has the same vertex set V; x, y are adjacent in $\Gamma_i \iff d(x, y) = i$;
  $\Gamma_0 = \Delta$, $\Gamma_1 = \Gamma$.

- Consider colour graph $(V, \{R_0, R_1, \ldots, R_d\})$: in general it does not form an association scheme.

- Graph Γ is called a *distance regular graph* (DRG) if $\mathfrak{M}(\Gamma) = (V, \{R_0, R_1, \ldots, R_d\})$ is an association scheme with d classes. In this case, $\mathfrak{M}(\Gamma)$ is called *metric* association scheme.

- Let W be an adjacency algebra of a metric association scheme. In this case W is generated by one matrix $A = A(\Gamma)$, moreover the generating process is in a sense canonical (cf talk by S. Reichard).

- We may associate to DRG Γ certain sequence of parameters (a part of intersection numbers of $\mathfrak{M}(\Gamma)$), which is called *intersection array*.

- A distance regular graph of diameter 2 = strongly regular graph.

- DRG Γ is called *primitive* if all distance graphs Γ_i, $1 \leq i \leq d$ are connected, otherwise Γ is *imprimitive*.

- DRG Γ is called *antipodal* if $\Gamma_d = m \circ K_l$ (a disjoint union of complete graphs, $v = |V| = ml$).

- **Theorem** Each imprimitive DRG is bipartite or antipodal.

- Graph Γ is called *distance transitive* (DTG) if for $G = Aut(\Gamma)$ the centralizer algebra $V(Aut(\Gamma))$ has rank $d + 1$ (in other words, G acts transitively on ordered pairs of equidistant vertices).

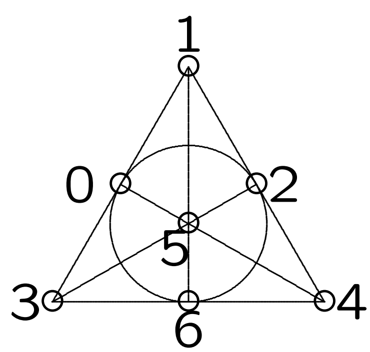
- Each DTG is DRG. There are many examples of DRG's which are not DTG's.

- Classification of DTG is one of the most monumental projects in algebraic graph theory (using CFSG): many infinite series and a lot of striking sporadic examples. This project is close to completion.

- Simplest non-trivial examples of imprimitive DRG's are coming from 3-dimensional euclidean geometry:
  - cube Q_3 ($k = 3$ $d = 3$, antipodal and bipartite),
  - icosahedron ($k = 5$, $d = 3$, antipodal),
  - dodecahedron ($k = 3$, $d = 6$, antipodal).

Example 8. *(Primitive graph on 21 vertices)*

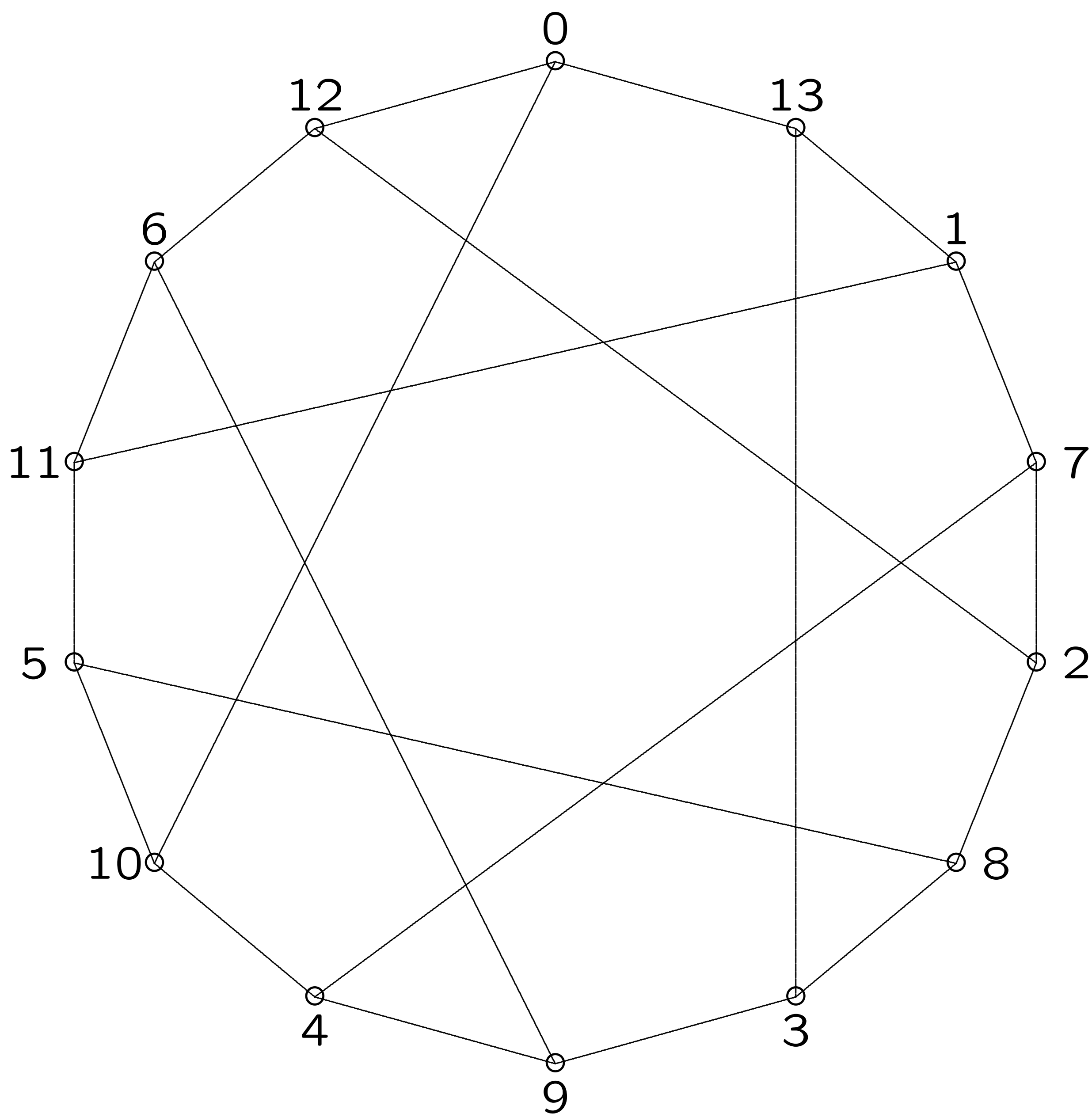
a) begin with Fano plane = $PG(2, 2)$*:*



$\{1, 2, 4\} = 7$
$\{2, 3, 5\} = 8$
$\{3, 4, 6\} = 9$
$\{4, 5, 0\} = 10$
$\{5, 6, 1\} = 11$
$\{6, 0, 2\} = 12$
$\{0, 1, 3\} = 13$

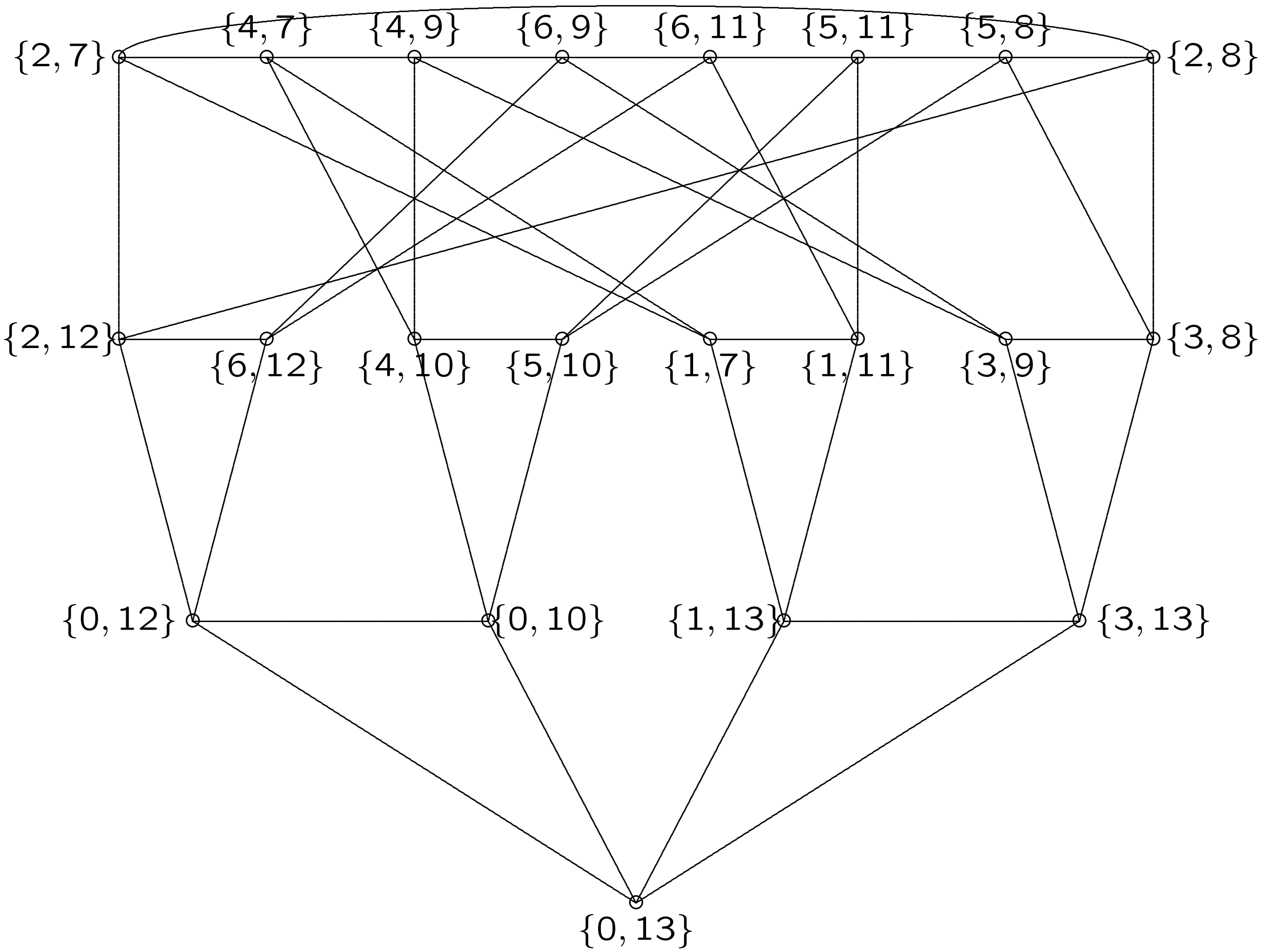
b) Construct its Levi graph: vertices are points and lines, adjacency=incidence.

Get Heawood graph H on 14 vertices:

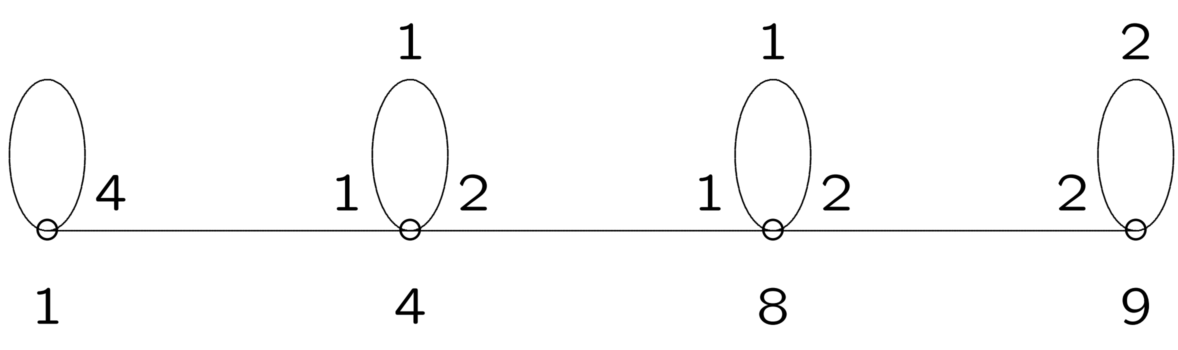


H *itself is antipodal DRG* ($k = 3$, $d = 3$).

c) *Construct the line graph $L(H)$ - vertices are edges of H, adjacency = common ends of edges:*



Intersection Diagram:



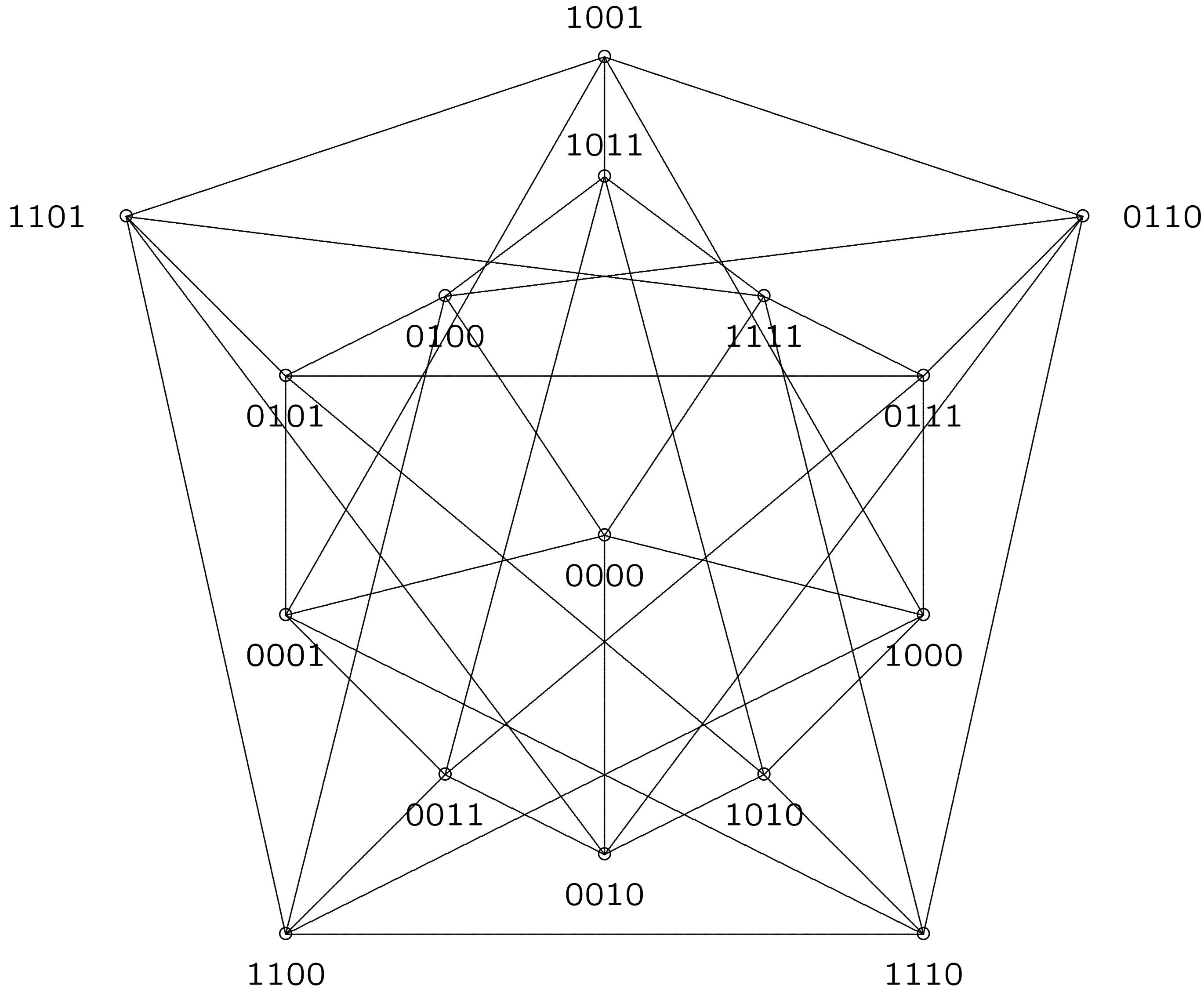
does not depend on a selection of vertex.

- A very famous family of DRG's is *n-dimensional cubes* Q_n with 2^n vertices which define *Hamming* (metric) association scheme $H(n,2)$ with n classes.

- merging of classes in various metric schemes will be a subject of PART II.

- Now we consider just one example of merging in $H(4,2)$.

Example 9. *Clebsch graph $\square_5$ is defined as merging of classes Γ_1 and Γ_4 in Q_4:*



Alternative way:

$$\square_5 = CAY(E_{2^4}, \{0001, 0010, 0100, 1000, 1111\})$$

is an $SRG(16, 5, 0, 1)$.

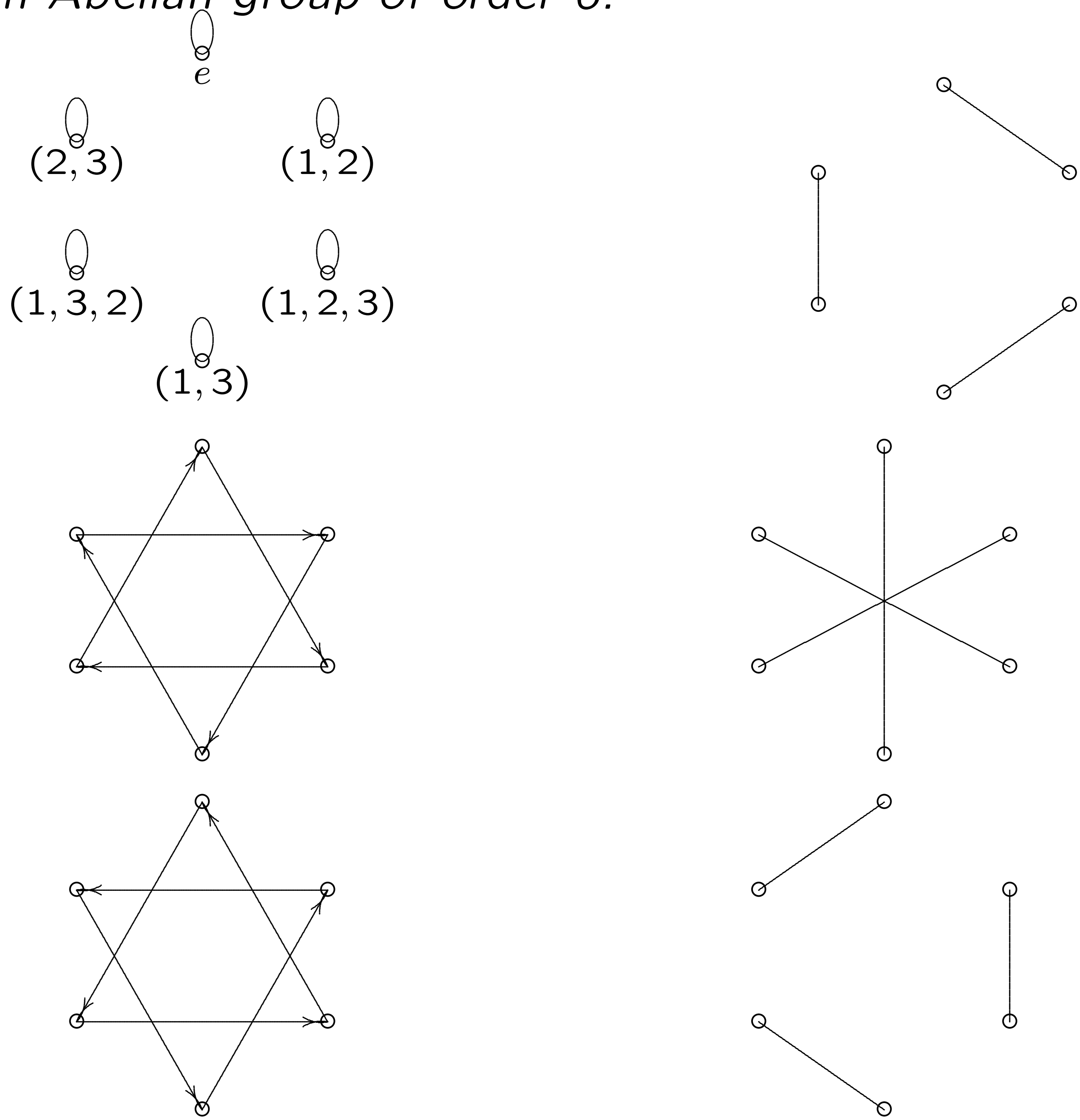
8. Dihedral association schemes

- Metric association schemes have in a sense one natural generator - a corresponding DRG.

- We now introduce a family of association schemes with two generators (*dihedral* schemes according to Zieschang).

- The name goes back to a simple example of a so-called *thin* scheme (all valencies of basis graphs are equal to 1).

Example 10. *Regular action of group $S_3 = D_3$, a non-Abelian group of order 6:*



Cayley graphs with respect to elements of S_3. Any two involutions generate the scheme.

Example 11.

- *Consider action of S_n on ordered pairs of distinct elements from* $[1, n]$, $n \geq 4$.

- *Get permutation group* (S_n, X) *of degree* $n(n-1)$.

- *Describe* $2-Orb(S_n, X)$ *via typical pairs of points:*
  R_0: $((a, b), (a, b))$
  R_1: $((a, b), (a, c))$
  R_2: $((a, b), (c, b))$
  R_3: $((a, b), (b, c))$
  R_4: $((a, b), (c, a))$
  R_5: $((a, b), (c, d))$
  R_6: $((a, b), (b, a))$
  Here R_1 *and* R_2 *generate association scheme* $(X, 2-Orb(S_n, X))$.

- *More examples in part 3.*
  (challenge for GB methods!)

9. Amorphic association schemes

- Association scheme is called *amorphic* if any partition of its classes leads to a fusion scheme.

- A necessary condition: each basic graph is an SRG.

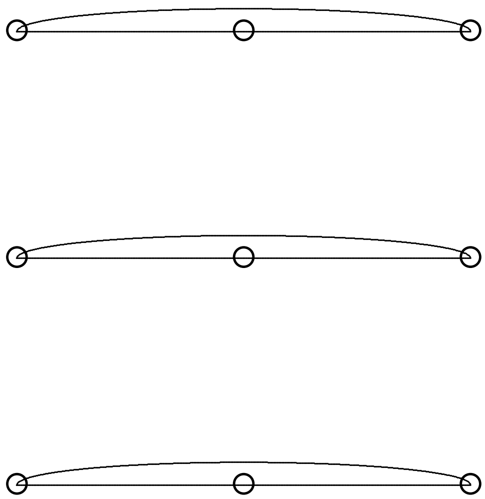
- **Theorem** (D.G. Higman, A.V. Ivanov)

  a) If number of classes is at least 3, then number of points is n^2, $n \in \mathbb{N}$;

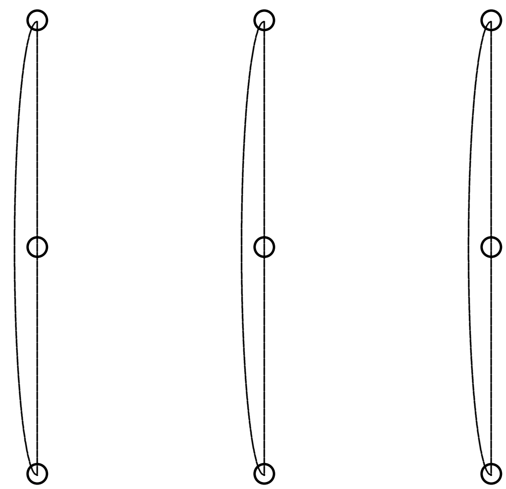
  b) Each basis graph is of a positive (negative) Latin square type graph.

- "Generic" examples: complete affine scheme of order n^2 with $n+1$ classes.

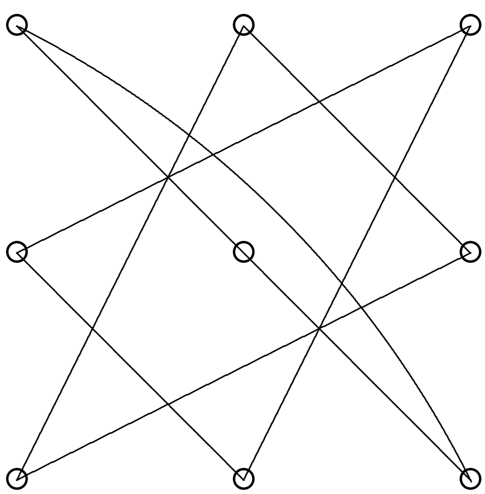
Example 12. $n = 3$



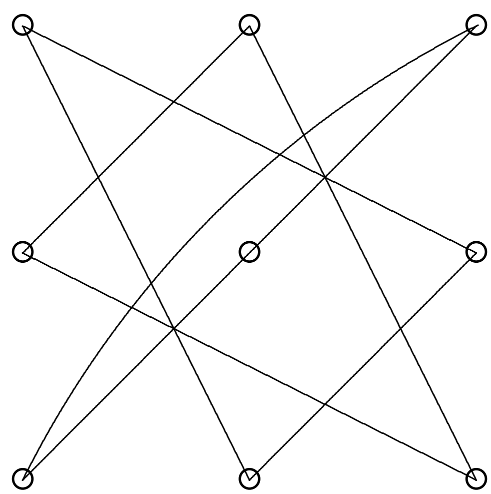
Γ_1



Γ_2



Γ_3



Γ_4

Here scheme is generated by arbitrary 3 basis relations (In general we need n generators).

Acknowledgments

- Boris Weisfeiler is indeed alive in our memory.

- Fruitful long-standing collaboration with:
  - late Ja. Ju. Golfand;
  - I. A. Faradžev, A. A. Ivanov, A. V. Ivanov, M. Muzychuk and other colleagues from former USSR;

- New helpful input of young generation: F. Fiedler, A. Heinze, Ch. Pech, S.Reichard;

- Permanent support of A. Woldar;

- Technical assistance in this lecture of Matan Ziv-Av.

References

1. E. Bannai, T. Ito. *Algebraic Combinatorics I.* Benjamin/Cummings, Menlo Park, 1984.
2. R. C. Bose. *Strongly regular graphs, partial geometries and partially balanced designs.* Pacific J. Math. 13(1963), 389–419.
3. R.C. Bose, D. M. Mesner. *On linear associative algebras corresponding to association schemes of partially balanced designs.* Ann. Math. Statist. 30(1959), 21-38.
4. A. E. Brouwer, A. M. Cohen, A. Neumaier. *Distance regular graphs.* Springer, Berlin, 1989.
5. Ph. Delsarte. *An algebraic approach to the association schemes of coding theory.* Philips Research reports supplements. 10, Eindhoven 1973.
6. I. A. Faradžev, A. A. Ivanov, M. M. Klin, A. Woldar (eds). *Investigations in algebraic theory of combinatorial objects.* Kluwer Acad. Publ. Dordrecht, 1994.
7. I. A. Faradžev, M. M. Klin. *Computer package for computations with coherent configurations.* Proc. ISSAC-91, Bonn, ACM Press, 1991, 219-223.
8. C. D. Godsil. *Algebraic combinatorics.* Chapman & Hall. N.Y., 1993.
9. D. G. Higman. *Coherent configurations.* I. Rend. Sem. Mat. Univ. Padova, **44** (1970), 1–25.
10. D. G. Higman. *Coherent algebras.* Linear Algebra Appl., **93** (1987), 209–239.
11. M. Klin. Ch. Rücker, G. Rücker, G. Tinhofer. *Algebraic combinatorics in mathematical chemistry. I. Permutation groups and coherent (cellular) algebras.* MATCH, 40(1999), 7-138.
12. M. Klin, A. Munemasa, M. Muzychuk, P. M. Zieschang. *Directed strongly regular graphs obtained from coherent algebras.* Linear Alg. and Appl. 377(2004), 83-109.
13. M. Klin, S. Reichard, A. Woldar, *Siamese objects and their relation to color graphs, association schemes and Steiner designs.* Bull. Belg. Math. Soc. Simon Stevin 12(2006), 845-857.
14. B. Ju. Weisfeiler, A. A. Leman. *A reduction of a graph to a canonical form and an algebra arising during this reduction.* Nauchno - Technicheskaja Informatsia, Seria 2, **9** (1968), 12–16 (Russian).
15. B. Weisfeiler (ed.). *On construction and identification of graphs.* Lecture Notes in Math., 558, Springer, Berlin, 1976.
16. H. Wielandt. *Finite permutation groups.* Acad. Press, 1964.