

Two applications of the footprint bound

Estimation of generalized Hamming weights

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Generator matrix case: a continuation of yesterdays talk

Parity check matrix case: another point of view

The footprint bound

Definition 1 Let \prec be a monomial ordering on $\mathcal{M}(X_1, \dots, X_m)$ and k a field. Given an ideal $I \subseteq k[X_1, \dots, X_m]$ the set

$$\Delta_{\prec}(I) = \{M \in \mathcal{M}(X_1, \dots, X_m) \mid \text{there does not exist any } F \in I \text{ with } \text{Im}(F) = M\}$$

Theorem 2 If $\Delta_{\prec}(I)$ is finite then $\#\mathbb{V}_{\bar{k}}(I) \leq \#\Delta_{\prec}(I)$ holds.

Generalized Hamming Weights

$$D = \{(1, 1, 0, 0, 1, 0) \\ (1, 0, 1, 0, 0, 0) \\ (0, 0, 1, 0, 1, 0)\}$$

$$\text{Supp}(D) = \{1, 2, 3, 5\}$$

$d_t(C) := \min\{\#\text{Supp}(D) \mid D \text{ is a subcode of } C \\ \text{of dimension } t\}$

$$d_1(C) = d(C)$$

Reed-Muller codes over \mathbb{F}_4

$$X^4 - X, Y^4 - Y$$

$$\begin{array}{cccc} \underline{Y^3} & \underline{XY^3} & \underline{X^2Y^3} & \underline{X^3Y^3} \\ \underline{Y^2} & \underline{XY^2} & \underline{X^2Y^2} & \underline{X^3Y^2} \\ \underline{Y} & \underline{XY} & \underline{X^2Y} & \underline{X^3Y} \\ \underline{1} & \underline{X} & \underline{X^2} & \underline{X^3} \end{array} \quad C = \text{RM}_q(3, 2)$$

$[16, 10, 4]$

A basis for a subspace $D \subseteq C$ can be assumed to “consist” of polynomials with different leading monomials.

d_2 -calculations: We look for worst case of

$$\#\Delta_{\prec}(\langle X^{i_1} Y^{j_1}, X^{i_2} Y^{j_2}, X^4, Y^4 \rangle)$$

An incident is

$$\#\Delta_{\prec}(\langle X^3, X^2 Y, X^4, Y^4 \rangle) = 9$$

So $d_2 \geq 16 - 9 = 7$.

d_3 -calculations: We look for worst case of

$$\#\Delta_{\prec}(\langle X^{i_1} Y^{j_1}, X^{i_2} Y^{j_2}, X^{i_3} Y^{j_3}, X^4, Y^4 \rangle)$$

An incident is

$$\#\Delta_{\prec}(\langle X^3, X^2 Y, X_2, X^4, Y^4 \rangle) = 8$$

So $d_3 \geq 16 - 8 = 8$.

Definition 7 $w(X_1), \dots, w(X_m) \in \mathbb{R}_+$, $X_1^{i_1} \cdots X_m^{i_m} \prec_w X_1^{j_1} \cdots X_m^{j_m}$
if (1) or (2) holds

- (1) $w(X_1^{i_1} \cdots X_m^{i_m}) < w(X_1^{j_1} \cdots X_m^{j_m})$
- (2) $w(X_1^{i_1} \cdots X_m^{i_m}) = w(X_1^{j_1} \cdots X_m^{j_m})$
and $X_1^{i_1} \cdots X_m^{i_m} \prec_{lex} X_1^{j_1} \cdots X_m^{j_m}$

Hermitian codes over \mathbb{F}_{16}

$\mathbb{V}_{\mathbb{F}_{16}}(\langle X^5 + Y^4 + Y \rangle) = 64$, $\text{ev} : \mathbb{F}_{16}[X, Y] \rightarrow \mathbb{F}_{16}^{64}$.

Let $w(X) = 5$, $w(Y) = 4$ and lexicographic ordering be $X \prec_{\text{lex}} Y$.

$\text{ev}(\Delta_{\prec_w}(\langle X^5 + Y^4 + Y, X^{16} + X, Y^{16} + Y \rangle))$ basis for \mathbb{F}_{16}^{64} .

Y^3	XY^3	X^2Y^3	X^3Y^3	...	$X^{11}Y^3$	$X^{12}Y^3$	$X^{13}Y^3$	$X^{14}Y^3$	$X^{15}Y^3$
Y^2	XY^2	X^2Y^2	X^3Y^2	...	$X^{11}Y^2$	$X^{12}Y^2$	$X^{13}Y^2$	$X^{14}Y^2$	$X^{15}Y^2$
Y	XY	X^2Y	X^3Y	...	$X^{11}Y$	$X^{12}Y$	$X^{13}Y$	$X^{14}Y$	$X^{15}Y$
1	X	X^2	X^3	...	X^{11}	X^{12}	X^{13}	X^{14}	X^{15}

15	19	23	27	31	35	39	43	47	51	55	59	63	67	71
10	14	18	22	26	30	34	38	42	46	50	54	58	62	66
5	9	13	17	21	25	29	33	37	41	45	49	53	57	61
0	4	8	12	16	20	24	28	32	36	40	44	48	52	56

For all $X^i Y^j$ "in basis" we list the values of

$$\# (\Delta_{\prec_w}(\langle X^i Y^j, X^5 + Y^4 \rangle) \cap \Delta_{\prec_w}(\langle X^5 + Y^4 + Y, X^{16} + X, Y^{16} + Y \rangle)).$$

15	19	23	27	31	35	39	43	47	51	55	59	60	61	62
10	14	18	22	26	30	34	38	42	46	50	54	56	58	60
5	9	13	17	21	25	29	33	37	41	45	49	52	55	58
0	4	8	12	16	20	24	28	32	36	40	44	48	52	56

Traditional codes corresponds to linear span of all $\text{ev}(X^i Y^j)$ with $w(X^i Y^j) \leq s$.

Improved codes corresponds to linear span of all $\text{ev}(X^i Y^j)$ with Δ -size at most some chosen number.

d_2 -calculations: Consider all pairs $X^{i_1} Y^{j_1}, X^{i_2} Y^{j_2}$ (from code basis). Calculate

$$n - \#(\Delta_{\prec_w}(\langle X^{i_1} Y^{j_1}, X^{i_2} Y^{j_2}, X^5 + Y^4 \rangle) \\ \cap \Delta_{\prec_w}(\langle X^5 + Y^4 + Y, X^{16} + X, Y^{16} + Y \rangle))$$

= "what can be hit by either $w(X^{i_1} Y^{j_1})$ or/and $w(X^{i_2} Y^{j_2})$."

Choose worstcase. Repeat process for d_3, \dots

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= "what can be hit by either $w(X^{i_1} Y^{j_1})$ or/and $w(X^{i_2} Y^{j_2})$."

Choose worstcase. Repeat process for d_3, \dots

	k	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9
Improved	55	6	8	9	11	12	14	15	16	18
Traditional	55	4	8	9	12	13	14	16	17	18
	k	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	
Improved	51	9	12	14	15	17	18	19	21	
Traditional	51	8	12	13	16	17	18	20	21	
Traditional	50	9	13	14	17	18	19	21	22	

Certainly, minimum distances are improved, but higher weights need NOT be.

Some notation

$$V = \{P_1, \dots, P_n\} = \mathbb{V}_{\mathbb{F}_q}(\langle G_1, \dots, G_g \rangle),$$
$$\text{ev}(F) = (F(P_1), \dots, F(P_n)).$$

$$A = \begin{bmatrix} \text{ev}(F_1) \\ \vdots \\ \text{ev}(F_a) \end{bmatrix}$$

$$[F_i] = \{F_i + \sum_{j=1}^{i-1} \alpha_j F_j \mid \alpha_j \in \mathbb{F}_q\}$$

$$D_{\{[F_{i_1}], \dots, [F_{i_s}]\}} = \max\{\#\{P_j \in V \mid F'_{i_1}(P_j) = \dots = F'_{i_s}(P_j) = 0\} \mid$$
$$F'_{i_t} \in [F_{i_t}], t = 1, \dots, s\}$$

$$D_s = \max\{D_{\{[F_{i_1}], \dots, [F_{i_s}]\}} \mid 1 \leq i_1 < \dots < i_s \leq r\}.$$

Theorem 4 Let C be a code with parity check matrix A (not necessarily of full rank) then for $d^* \leq a + t$, $t \leq k$, $d \leq n$ we have

$$d_t \geq d^* \Leftrightarrow D_{a-d^*+t+1} \leq d^* - 2$$
$$d_t \leq d^* \Leftrightarrow D_{a-d^*+t} \geq d^*$$

Improved Hermitian codes - parity check matrix

Let V be the 64 points on the Hermitian curve $X^5 + Y^4 + Y$ over \mathbb{F}_{16} . Let parity check matrix be

$$\begin{bmatrix} \text{ev}(1) \\ \text{ev}(X) \\ \text{ev}(Y) \\ \text{ev}(X^2) \\ \text{ev}(XY) \\ \text{ev}(Y^2) \\ \text{ev}(X^3) \\ \text{ev}(Y^3 + X^4) \end{bmatrix}$$

$$D_{\{[XY],[Y^3+X^4]\}} \leq 7 \text{ chose } w(X) = 3, w(Y) = 4$$

$$D_{\{[Y^2],[Y^3+X^4]\}} \leq 8 \text{ choose } w(X) = 1 \text{ and } w(Y) = 1.1$$

$$D_{\{[X^2],[X^3],[Y^3+X^4]\}} \leq 6 \text{ choose } w(X) = 1 \text{ and } w(Y) = 1.4$$

ABOVE WEIGHTS KEEP THE ORDERING OF ROWS.

Going through all combinations gives $D_1 \leq 16$, $D_2 \leq 8$, $D_3 \leq 6$ and $D_4 \leq 4$.

This implies $d_1 \geq 6$, $d_2 \geq 8$, $d_i \geq i + 7$ for $i = 3, \dots, 9$ and $d_i = i + 8$ for $i = 10, \dots, 56$. Not only minimum distance is improved.