

A Module View of Integral Closures

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The Problem

Let

$$P := \mathbf{F}[x_n, \dots, x_1]$$

and

$$Q := P[y_{m-1}, \dots, y_1]/I$$

be an *integrally closed* extension of P (with $Q = P$ corresponding to the case $m = 1$).

$$f(T) \in Q[T]$$

a monic, irreducible polynomial of some degree d , defining a *simple integral extension*

$$S := Q[z]/\langle f(z) \rangle,$$

which may or may not be integrally closed (in its field of fractions).

Find a *canonical form* for the *integral closure* $ic(S)$.

Module Bases

Suppose that $\Delta \in P$ satisfies

$$S \subseteq ic(S) \subseteq \frac{1}{\Delta} S.$$

With $y_0 := 1$,

$$S = P \langle z^i y_j : 0 \leq j < m, 0 \leq i < d \rangle$$

$$\frac{1}{\Delta} S = P \left\langle \frac{z^i y_j}{\Delta} : 0 \leq j < m, 0 \leq i < d \right\rangle.$$

So it is *obvious* to conjecture that

$$ic(S) = P \left\langle \frac{f_{i,j}}{\Delta} : 0 \leq i < d, 0 \leq j < m \right\rangle$$

with $LM(f_{i,j}) = \underline{x}^{\alpha_{i,j}} z^i y_j$ for some $\underline{x}^{\alpha_{i,j}} \in P$.

Example

$$P := \mathbf{F}_2[x_2, x_1]$$

$$f_1(T) := T^3 + T^1 x_2 x_1 + T^0 (x_2^3 x_1^2 + x_2^2 x_1^3)$$

$$Q := P[y] / \langle f_1(y) \rangle.$$

$$\Delta_1 := x_2^3 x_1^2 + x_2^2 x_1^3$$

$$Q = P\langle 1, y, y^2 \rangle \subseteq ic(Q) \subseteq P \left\langle \frac{1}{\Delta_1}, \frac{y}{\Delta_1}, \frac{y^2}{\Delta_1} \right\rangle = \frac{1}{\Delta_1} Q$$

$$\begin{aligned} ic(Q) &= P \left\langle 1, y, \frac{y^2}{x_2 x_1} \right\rangle \\ &= P[y_2, y_1] / I_1 \end{aligned}$$

$$I_1 := \langle y_1^2 + y_2 x_2 x_1, y_2 y_1 + y_1 + (x_2^2 x_1 + x_2 x_1^2), y_2^2 + y_2 + y_1(x_2 + x_1) \rangle$$

$$f_2(T) := T^3 + T^1 (y_2 + y_1)(x_2 + 1)x_1 + T^0 (y_2 + y_1)^2 x_2^2 x_1$$

$$S := ic(Q)[z] / \langle f_2(z) \rangle$$

$$\Delta_2 := x_2^8 x_1^5 + x_2^6 x_1^7 + x_2^8 x_1^3 + x_2^4 x_1^7 + x_2^6 x_1^3 + x_2^4 x_1^5$$

$$S = P\langle z^0 y_0, z^0 y_1, z^0 y_2, z^1 y_0, z^1 y_1, z^1 y_2, z^2 y_0, z^2 y_1, z^2 y_2 \rangle$$

$$P\langle z^i y_j : 0 \leq i, j < 3 \rangle \subseteq ic(S) \subseteq \frac{1}{\Delta_2} P\langle z^i y_j : 0 \leq i, j < 3 \rangle$$

$$ic(S) = \frac{1}{\delta_2} P\langle f_{i,j} : 0 \leq i, j < 3 \rangle, \quad LM(f_{i,j}) := \underline{x}^{\alpha_{i,j}} z^i y_j$$

$$ic(S) = P[z_8, \dots, z_1; x_2, x_1] / I_2, \quad z_{3i+j} := f_{i,j}$$

$$I_2 = \langle z_i z_j - NF(z_i z_j) \rangle$$

Truth

A P -module basis for S is actually

$$\begin{aligned}
 z_0 &:= y_0; \quad z_1 := y_1; \quad z_2 := y_2; \quad z_3 := zy_0; \quad z_4 := zy_1; \\
 z_5 &:= \frac{zy_2 + zy_1x_2 + zy_0}{x_1}; \quad z_6 := z^2y_0; \quad z_7 := \frac{z^2y_1}{x_1}; \\
 z_8 &:= \frac{z^2y_2 + z^2y_1(x_2x_1 + x_2 + x_1) + z^2y_0(x_2^3x_1^2 + x_2^3x_1 + x_2x_1^2 + 1)}{x_2(x_2 + 1)x_1^2(x_1 + 1)}; \\
 z_9 &:= (z^2y_2(x_1 + 1) + z^2y_1(x_2^2x_1 + x_2x_1^2 + x_2x_1 + x_1^2 + x_2 + x_1) \\
 &\quad + z^2y_0(x_2^3x_1 + x_2^2x_1^2 + x_1 + 1))/x_2(x_2 + 1)(x_2 + x_1)x_1^2
 \end{aligned}$$

The integral closure is then of the form

$$ic(S) = P[z_9, z_8, z_7, z_6, z_5, z_4, z_3, z_2, z_1; x_2, x_1]/\bar{I}$$

with \bar{I} the ideal of induced relations most of the form $z_i z_j - NF(z_i z_j)$ except for one of the form $SP(z_9, z_8) - NF(SP(z_9, z_8))$.

Weighted Truth

A P -module basis for S with the induced weight function

$$W := \begin{pmatrix} 19 & 12 & 15 & 9 & 9 \\ 12 & 9 & 9 & 9 & 0 \end{pmatrix}$$

is however

$$w_0 := y_0; \quad w_1 := y_1; \quad w_2 := y_2; \quad w_3 := zy_0; \quad w_4 := zy_1;$$

$$w_5 := \frac{zy_2 + zy_1x_2 + zy_0}{x_1};$$

$$w_6 := zy_2;$$

$$\begin{aligned} \delta w_7 := & z^2 y_0 (x_2^3 x_1^2 + x_2^2 x_1^3 + x_2^3 x_1 + x_2 x_1^3 + x_2 x_1 + 1) \\ & + z^2 y_2 (x_2 x_1 + 1) + z^2 y_1 (x_2 + x_1); \end{aligned}$$

$$\delta w_8 := z^2 y_1 (x_2^2 x_1 + x_2 x_1^2 + x_2 x_1 + x_1^2 + x_2 + x_1)$$

$$+ z^2 y_2 (x_2^2 x_1 + x_2 x_1 + x_1 + 1) + z^2 y_0 (x_2^3 x_1 + x_2^2 x_1^2 + x_2^2 x_1 + x_2 x_1 + x_1 + 1);$$

$$\delta w_9 := z^2 y_2 (x_2 x_1^2 + x_1) + z^2 y_1 (x_2 x_1 + x_1^2)$$

$$+ z^2 y_0 (x_2^2 x_1^2 + x_2 x_1^3 + x_2 x_1^2 + x_1)$$

for $\delta := x_1^2(x_1 + 1)x_2(x_2 + 1)(x_2 + x_1)$.

The integral closure is then of the form

$$ic(S) = P[w_9, w_8, w_7, w_6, w_5, w_4, w_3, w_2, w_1; x_2, x_1] / \bar{I}$$

with \bar{I} the ideal of induced relations most of the form $w_i w_j - NF(w_i w_j)$ except for $w_4 x_2 + w_5 x_1 + w_6 + w_3$, of the form $SP(w_4, w_5) - NF(SP(w_4, w_5))$.

Black Box Algorithm Ingredients

Any *good* algorithm for computing the integral closure $ic(S)$ as described above should produce, as output,

1. a $\delta \in P$ *minimal* such that $ic(S)\delta \subseteq S$;
2. a ring $R := \mathbf{F}[z_{s-1}, \dots, z_1; x_n, \dots, x_1]$ with a monomial order that *properly differentiates between dependent and independent variables*;
3. an ideal \bar{I} of induced relations *of predictable form*, so that $ic(S) = R/\bar{I}$;
4. an embedding $\phi : S \rightarrow ic(S)$;
5. an embedding $\psi : ic(S)\delta \rightarrow S$.

(The implementation *IntegralClosure* in MAGMA works only for $n = 1$ independent variable, and produces a P -module basis as a subset of S/δ , meaning δ is implicit, the R , \bar{I} , and ϕ are not given, and ψ is unnecessary. And the implementation *normal* in SINGULAR does not necessarily produce δ restricted to P , does give R , \bar{I} (called *norid*), ϕ (called *normap*), but no ψ , making it extremely difficult to figure out what the new variables are in terms of S , especially since the ordering used on R makes the new variables less important than even the independent variables of S .)

Module Basis for the Weighted Example

$$M_1 := \delta \cdot (1, 0, 0, 0, 0, 0, 0, 0, 0)$$

$$M_2 := \delta \cdot (0, 1, 0, 0, 0, 0, 0, 0, 0)$$

$$M_3 := \delta \cdot (0, 0, 1, 0, 0, 0, 0, 0, 0)$$

$$M_4 := \delta \cdot (0, 0, 0, 1, 0, 0, 0, 0, 0)$$

$$M_5 := \delta \cdot (0, 0, 0, 0, 1, 0, 0, 0, 0)$$

$$M_6 := \delta/x_1 \cdot (0, 0, 0, 1, x_2, 1, 0, 0, 0)$$

$$M_7 := \delta \cdot (0, 0, 0, 0, 0, 1, 0, 0, 0)$$

$$M_8 := (0, 0, 0, 0, 0, 0, x_2^3 x_1^2 + x_2^2 x_1^3 + x_2^3 x_1 + x_2 x_1^3 + x_2 x_1 + 1, x_2 + x_1, x_2 x_1 + 1)$$

$$M_9 := (0, 0, 0, 0, 0, 0,$$

$$x_2^3 x_1 + x_2^2 x_1^2 + x_2^2 x_1 + x_2 x_1 + x_1 + 1, x_2^2 x_1 + x_2 x_1^2 + x_2 x_1 + x_1^2 + x_2 + x_1, x_2^2 x_1 + x_2 x_1 + x_1 + 1)$$

$$M_{10} := x_1(0, 0, 0, 0, 0, 0, x_2^2 x_1 + x_2 x_1^2 + x_2 x_1 + 1, x_2 + x_1, x_2 x_1 + 1)$$

$$\delta = x_2^3 x_1^3 + x_2^2 x_1^4 + x_2^3 x_1^2 + x_2 x_1^4 + x_2^2 x_1^2 + x_2 x_1^3$$

Minimization

It may be necessary to switch to a new set in *independent* variables in order to *minimize* the description of the integral closure.

$$Q := P := \mathbf{F}_2[x_2, x_1], \text{WEIGHT}(Q) := \left(\begin{array}{c|cc} & 1 & 1 \\ \hline & 1 & 0 \end{array} \right)$$

$$f(T) := T^4 + T^2 x_2 x_1 + T^0 x_2^3 x_1^2$$

$$S := Q[z]/\langle f(z) \rangle, \text{WEIGHT}(S) = \left(\begin{array}{c|cc} 5 & 4 & 4 \\ \hline 3 & 4 & 0 \end{array} \right)$$

$$\Delta := x_2^3 x_1^2$$

$$ic(S) = \mathbf{F}_2[z_3, z_2, z_1; x_2, x_1]/I, \text{WEIGHT}(ic(S)) = \left(\begin{array}{ccc|cc} 3 & 2 & 5 & 4 & 4 \\ 1 & 2 & 3 & 4 & 0 \end{array} \right)$$

$$I = \langle z_1^2 + z_2 x_2 x_1, z_2 z_1 + z_3 x_2 + z_1, z_2^2 + z_2 + x_2, z_3 z_1 + x_2 x_1, z_3 z_2 + z_1, z_3^2 + z_2 x_1 + x_1 \rangle$$

$$\delta := x_2^2 x_1, z_3 \delta := z^3 + z x_2 x_1, z_2 \delta := z^2 x_2, z_1 \delta := z \delta$$

$$\text{weight}(z_2) = (2, 2) < \text{weight}(x_2) = (4, 4), x_2 = z_2^2 + z_2, z_1 = z_3 z_2$$

$$ic(S) = \mathbf{F}_2[\bar{z}_1; \bar{x}_2, \bar{x}_1]/\bar{I}, \text{WEIGHT}(ic(S)) = \left(\begin{array}{c|cc} 3 & 2 & 4 \\ \hline 1 & 2 & 0 \end{array} \right)$$

$$\bar{I} = \langle \bar{z}_1^2 + \bar{x}_2 \bar{x}_1 + \bar{x}_1 \rangle$$

$$\delta := x_2^2 x_1, \bar{z}_1 \delta := z^3 + z x_2 x_1, \bar{x}_2 \delta := z^2 x_2, \bar{x}_1 := x_1 \delta$$

$$\psi((\bar{z}_1, \bar{x}_2, \bar{x}_1) \delta) = (z^3 + z x_2 x_1, z^2 x_2, x_1 \delta)$$

$$\phi(z, x_2, x_1) = (\bar{z}_1 \bar{x}_2, \bar{x}_2^2 + \bar{x}_2, \bar{x}_1)$$

MAGMA isn't set up to do examples with two or more *independent* variables, but SINGULAR is. Compare the ideal returned by SINGULAR (edited to fit the page) immediately below to the suggested *predictable* (in the sense described above) form following it. The latter has $y_1 = m_2$, $y_2 = m_3$, $y_3 = z$ and the reverse map given above by the module basis page.) Note that the grevlex $T(1) = z \succ T(2) = y_2 \succ T(3) = y_1 \succ T(4) = x_2 \succ T(5) = x_1 \succ T(6) \succ T(7) \succ T(8) \succ T(9)$ order employed by SINGULAR's normal function completely ignores any original ordering, puts the *independent* variables in the middle, gives poor information about what the new variables are, and gives basis functions that, while correct, have no *meaning*.

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A Computer Algebra System for Polynomial Computations / version 2-0-4
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    by: G.-M. Greuel, G. Pfister, H. Schoenemann      \ April 2003
FB Mathematik der Universitaet, D-67653 Kaiserslautern \
> LIB "normal.lib";
> ring r1=2,(z,y2,y1,x2,x1),(dp(3),dp(2));
> poly f1=y1^2+y2*x2*x1;
> poly f2=(y2+1)*y1+x2*x1*(x2+x1);
> poly f3=(y2+1)*y2+y1*(x2+x1);
> poly f4=z^3+z*(y2+y1)*(x2+1)*x1+(y2+y1)^2*x2^2*x1;
> ideal i1=f1,f2,f3,f4;
> list nor=normal(i1);
> def r2=nor[1];
> setring r2;
> ideal i2=interred(norid);i2;
i2[1]=T(5)*T(6)+T(3)*T(7)

i2[2]=T(2)^2+T(3)*T(4)+T(3)*T(5)+T(2)

i2[3]=T(1)^2+T(2)*T(7)+T(3)*T(7)

i2[4]=T(4)^2*T(7)+T(4)*T(5)*T(7)+T(2)*T(6)+T(6)

i2[5]=T(2)*T(4)*T(7)+T(3)*T(6)

i2[6]=T(2)*T(3)*T(7)+T(3)^2*T(7)+T(3)*T(5)*T(7)+T(3)*T(6)*T(7)+T(4)*T(6)*T(7)
+T(1)*T(3)*T(8)+T(1)*T(4)*T(8)+T(1)*T(5)*T(8)+T(1)*T(2)*T(9)+T(1)*T(3)*T(9)
+T(1)*T(5)*T(9)+T(2)*T(5)*T(9)+T(2)*T(6)*T(9)+T(4)*T(6)*T(9)+T(2)*T(7)*T(9)
+T(3)*T(7)*T(9)+T(2)*T(9)^2+T(3)*T(9)^2+T(2)*T(6)+T(3)*T(6)+T(1)*T(8)
+T(1)*T(9)+T(3)*T(9)+T(6)*T(9)+T(6)

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$$\begin{aligned}
i2[7] = & T(1)*T(2)*T(7)+T(1)*T(3)*T(7)+T(3)^2*T(7)+T(3)*T(4)*T(7)+T(1)*T(5)*T(7) \\
& +T(2)*T(5)*T(7)+T(3)*T(5)*T(7)+T(4)*T(5)*T(7)+T(1)*T(6)*T(7)+T(2)*T(6)*T(7) \\
& +T(3)*T(6)*T(7)+T(1)*T(7)^2+T(2)*T(7)^2+T(3)*T(7)^2+T(1)*T(2)*T(8) \\
& +T(1)*T(4)*T(8)+T(4)*T(5)*T(8)+T(5)*T(7)*T(8)+T(1)*T(2)*T(9)+T(1)*T(3)*T(9) \\
& +T(3)*T(4)*T(9)+T(1)*T(5)*T(9)+T(2)*T(5)*T(9)+T(2)*T(6)*T(9)+T(3)*T(6)*T(9) \\
& +T(4)*T(6)*T(9)+T(2)*T(9)^2+T(3)*T(9)^2+T(1)*T(2)+T(2)*T(3)+T(3)^2+T(2)*T(5) \\
& +T(3)*T(5)+T(2)*T(6)+T(3)*T(6)+T(4)*T(6)+T(1)*T(7)+T(2)*T(7)+T(5)*T(7) \\
& +T(6)*T(7)+T(7)^2+T(3)*T(8)+T(7)*T(8)+T(2)*T(9)+T(3)*T(9)+T(6)*T(9)+T(7)*T(9) \\
& +T(1)
\end{aligned}$$

$$\begin{aligned}
i2[8] = & T(3)*T(6)^2+T(2)*T(6)*T(7)+T(4)*T(6)*T(7)+T(6)^2*T(7)+T(6)*T(7)^2 \\
& +T(1)*T(7)*T(8)+T(6)*T(7)
\end{aligned}$$

$$i2[9] = T(2)*T(4)*T(6)+T(2)*T(6)+T(4)*T(6)+T(6)^2+T(6)*T(7)+T(1)*T(8)+T(6)$$

$$\begin{aligned}
i2[10] = & T(1)*T(2)*T(6)+T(2)*T(6)^2+T(3)*T(4)*T(7)+T(3)*T(5)*T(7)+T(4)*T(6)*T(7) \\
& +T(3)*T(7)^2+T(2)*T(3)*T(8)+T(1)*T(4)*T(8)+T(3)*T(4)*T(8)+T(2)*T(5)*T(8) \\
& +T(3)*T(5)*T(8)+T(4)*T(5)*T(8)+T(5)^2*T(8)+T(2)*T(7)*T(8)+T(4)*T(7)*T(8) \\
& +T(5)*T(7)*T(8)+T(1)*T(2)*T(9)+T(1)*T(3)*T(9)+T(1)*T(4)*T(9)+T(1)*T(5)*T(9) \\
& +T(2)*T(5)*T(9)+T(1)*T(6)*T(9)+T(2)*T(6)*T(9)+T(3)*T(6)*T(9)+T(4)*T(6)*T(9) \\
& +T(1)*T(7)*T(9)+T(3)*T(7)*T(9)+T(4)*T(7)*T(9)+T(5)*T(7)*T(9)+T(5)*T(8)*T(9) \\
& +T(2)*T(9)^2+T(3)*T(9)^2+T(1)*T(2)+T(1)*T(3)+T(1)*T(5)+T(2)*T(6)+T(3)*T(6) \\
& +T(6)^2+T(1)*T(7)+T(3)*T(7)+T(1)*T(8)+T(3)*T(8)+T(5)*T(8)+T(3)*T(9)+T(6)*T(9) \\
& +T(1)+T(6)
\end{aligned}$$

$$i2[11] = T(4)^2*T(5)+T(4)*T(5)^2+T(2)*T(3)+T(3)$$

$$\begin{aligned}
i2[12] = & T(3)*T(4)*T(5)+T(4)*T(5)^2+T(3)^2+T(2)*T(5)+T(3)*T(5)+T(4)*T(5)+T(5)^2 \\
& +T(2)*T(7)+T(3)*T(7)+T(5)*T(7)+T(5)*T(9)+T(5)+T(7)
\end{aligned}$$

$$i2[13] = T(2)*T(4)*T(5)+T(3)^2$$

$$\begin{aligned}
i2[14] = & T(3)^2*T(5)+T(5)^3+T(1)*T(6)^2+T(1)*T(3)*T(7)+T(1)*T(4)*T(7) \\
& +T(3)*T(5)*T(7)+T(4)*T(5)*T(7)+T(1)*T(6)*T(7)+T(2)*T(6)*T(7)+T(3)*T(6)*T(7) \\
& +T(4)*T(6)*T(7)+T(4)*T(7)^2+T(5)*T(7)^2+T(1)*T(3)*T(8)+T(3)^2*T(8) \\
& +T(1)*T(4)*T(8)+T(3)*T(4)*T(8)+T(1)*T(5)*T(8)+T(3)*T(5)*T(8)+T(5)^2*T(8) \\
& +T(3)*T(7)*T(8)+T(4)*T(7)*T(8)+T(5)*T(7)*T(8)+T(1)*T(2)*T(9)+T(1)*T(3)*T(9) \\
& +T(2)*T(3)*T(9)+T(1)*T(4)*T(9)+T(1)*T(5)*T(9)+T(3)*T(5)*T(9)+T(4)*T(5)*T(9) \\
& +T(5)^2*T(9)+T(1)*T(6)*T(9)+T(3)*T(6)*T(9)+T(4)*T(6)*T(9)+T(1)*T(7)*T(9) \\
& +T(2)*T(7)*T(9)+T(4)*T(7)*T(9)+T(5)*T(7)*T(9)+T(5)*T(8)*T(9)+T(2)*T(9)^2
\end{aligned}$$

$$\begin{aligned}
&+T(1)*T(3)+T(3)^2+T(1)*T(5)+T(3)*T(5)+T(1)*T(6)+T(2)*T(6)+T(4)*T(6)+T(6)^2 \\
&+T(1)*T(7)+T(3)*T(7)+T(4)*T(7)+T(5)*T(7)+T(6)*T(7)+T(7)^2+T(3)*T(8)+T(5)*T(8) \\
&+T(7)*T(8)+T(1)*T(9)+T(2)*T(9)+T(5)*T(9)+T(3)+T(5)+T(7)
\end{aligned}$$

$$\begin{aligned}
i2[15]=&T(3)^2*T(4)+T(3)*T(4)^2+T(2)*T(5)+T(3)*T(6)+T(4)*T(6)+T(2)*T(7) \\
&+T(2)*T(9)+T(3)*T(9)+T(3)+T(6)
\end{aligned}$$

$$\begin{aligned}
i2[16]=&T(2)*T(3)*T(4)+T(3)*T(4)^2+T(4)*T(5)^2+T(2)*T(3)+T(3)*T(4)+T(4)*T(5) \\
&+T(5)^2+T(2)*T(6)+T(4)*T(6)+T(5)*T(7)+T(2)*T(9)+T(5)*T(9)+T(5)+T(7)
\end{aligned}$$

$$\begin{aligned}
i2[17]=&T(1)*T(3)*T(4)+T(1)*T(4)*T(5)+T(4)*T(6)^2+T(2)*T(6)*T(7)+T(3)*T(6)*T(7) \\
&+T(4)*T(6)*T(7)+T(6)^2*T(7)+T(3)*T(7)^2+T(6)*T(7)^2+T(1)*T(2)*T(8) \\
&+T(1)*T(3)*T(8)+T(1)*T(5)*T(8)+T(1)*T(7)*T(8)+T(3)*T(4)*T(9)+T(2)*T(6)*T(9) \\
&+T(1)*T(3)+T(2)*T(3)+T(3)^2+T(1)*T(5)+T(2)*T(5)+T(3)*T(5)+T(1)*T(6)+T(3)*T(6) \\
&+T(4)*T(6)+T(6)^2+T(1)*T(7)+T(2)*T(7)+T(1)*T(8)+T(2)*T(9)+T(6)
\end{aligned}$$

$$\begin{aligned}
i2[18]=&T(1)*T(2)*T(4)+T(1)*T(4)*T(6)+T(1)*T(4)*T(7)+T(3)*T(4)*T(7) \\
&+T(4)*T(5)*T(7)+T(4)*T(5)*T(8)+T(1)*T(4)+T(3)*T(7)+T(5)*T(7)+T(6)*T(7) \\
&+T(7)^2+T(3)*T(8)+T(7)*T(9)
\end{aligned}$$

$$\begin{aligned}
i2[19]=&T(3)*T(4)*T(7)*T(9)+T(2)*T(5)^2+T(3)*T(5)^2+T(4)*T(5)^2+T(1)*T(3)*T(6) \\
&+T(1)*T(4)*T(6)+T(1)*T(6)^2+T(2)*T(6)^2+T(4)*T(6)^2+T(1)*T(3)*T(7) \\
&+T(1)*T(4)*T(7)+T(3)*T(4)*T(7)+T(2)*T(5)*T(7)+T(3)*T(5)*T(7)+T(4)*T(5)*T(7) \\
&+T(5)^2*T(7)+T(1)*T(6)*T(7)+T(4)*T(6)*T(7)+T(6)^2*T(7)+T(4)*T(7)^2+T(5)*T(7)^2 \\
&+T(6)*T(7)^2+T(1)*T(3)*T(8)+T(2)*T(3)*T(8)+T(3)^2*T(8)+T(1)*T(4)*T(8) \\
&+T(1)*T(5)*T(8)+T(2)*T(5)*T(8)+T(4)*T(5)*T(8)+T(2)*T(6)*T(8)+T(3)*T(6)*T(8) \\
&+T(2)*T(7)*T(8)+T(1)*T(3)*T(9)+T(2)*T(3)*T(9)+T(3)*T(4)*T(9)+T(3)*T(6)*T(9) \\
&+T(4)*T(6)*T(9)+T(1)*T(7)*T(9)+T(2)*T(7)*T(9)+T(3)*T(7)*T(9)+T(4)*T(7)*T(9) \\
&+T(5)*T(7)*T(9)+T(6)*T(7)*T(9)+T(3)*T(8)*T(9)+T(2)*T(9)^2+T(3)*T(9)^2 \\
&+T(1)*T(2)+T(1)*T(3)+T(2)*T(3)+T(3)^2+T(1)*T(5)+T(4)*T(5)+T(5)^2+T(1)*T(6) \\
&+T(2)*T(6)+T(3)*T(6)+T(4)*T(6)+T(6)^2+T(1)*T(7)+T(5)*T(7)+T(6)*T(8)+T(7)*T(8) \\
&+T(2)*T(9)+T(3)*T(9)+T(5)*T(9)+T(6)*T(9)+T(7)*T(9)+T(1)+T(5)+T(7)
\end{aligned}$$

$$\begin{aligned}
i2[20]=&T(2)*T(6)*T(7)*T(8)+T(1)*T(2)*T(8)^2+T(1)*T(3)*T(8)^2+T(3)*T(4)^2*T(9) \\
&+T(1)*T(6)^2*T(9)+T(2)*T(6)^2*T(9)+T(1)*T(3)*T(7)*T(9)+T(3)^2*T(7)*T(9) \\
&+T(1)*T(5)*T(7)*T(9)+T(2)*T(5)*T(7)*T(9)+T(3)*T(5)*T(7)*T(9) \\
&+T(4)*T(5)*T(7)*T(9)+T(5)^2*T(7)*T(9)+T(2)*T(6)*T(7)*T(9)+T(6)^2*T(7)*T(9) \\
&+T(1)*T(7)^2*T(9)+T(4)*T(7)^2*T(9)+T(5)*T(7)^2*T(9)+T(6)*T(7)^2*T(9) \\
&+T(1)*T(3)*T(8)*T(9)+T(3)^2*T(8)*T(9)+T(3)*T(4)*T(8)*T(9)+T(2)*T(5)*T(8)*T(9) \\
&+T(4)*T(5)*T(8)*T(9)+T(5)^2*T(8)*T(9)+T(2)*T(6)*T(8)*T(9)+T(1)*T(7)*T(8)*T(9) \\
&+T(3)*T(7)*T(8)*T(9)+T(5)*T(7)*T(8)*T(9)+T(3)*T(6)*T(9)^2+T(1)*T(7)*T(9)^2
\end{aligned}$$

$$\begin{aligned}
&+T(3)*T(7)*T(9)^2+T(3)*T(8)*T(9)^2+T(5)*T(8)*T(9)^2+T(2)*T(3)*T(5)+T(3)*T(5)^2 \\
&+T(4)*T(5)^2+T(1)*T(3)*T(6)+T(4)^2*T(6)+T(1)*T(6)^2+T(2)*T(6)^2+T(4)*T(6)^2 \\
&+T(3)^2*T(7)+T(1)*T(4)*T(7)+T(3)*T(4)*T(7)+T(3)*T(5)*T(7)+T(4)*T(5)*T(7) \\
&+T(5)^2*T(7)+T(1)*T(6)*T(7)+T(2)*T(6)*T(7)+T(3)*T(6)*T(7)+T(4)*T(7)^2 \\
&+T(5)*T(7)^2+T(1)*T(2)*T(8)+T(1)*T(3)*T(8)+T(2)*T(3)*T(8)+T(3)^2*T(8) \\
&+T(3)*T(5)*T(8)+T(6)^2*T(8)+T(2)*T(7)*T(8)+T(3)*T(7)*T(8)+T(3)^2*T(9) \\
&+T(2)*T(4)*T(9)+T(3)*T(4)*T(9)+T(1)*T(5)*T(9)+T(2)*T(5)*T(9)+T(3)*T(5)*T(9) \\
&+T(2)*T(6)*T(9)+T(4)*T(6)*T(9)+T(6)^2*T(9)+T(3)*T(7)*T(9)+T(6)*T(7)*T(9) \\
&+T(5)*T(8)*T(9)+T(6)*T(8)*T(9)+T(7)*T(8)*T(9)+T(2)*T(9)^2+T(3)*T(9)^2 \\
&+T(7)*T(9)^2+T(3)*T(4)+T(4)*T(5)+T(5)^2+T(3)*T(6)+T(6)^2+T(5)*T(7)+T(6)*T(7) \\
&+T(3)*T(8)+T(1)*T(9)+T(2)*T(9)+T(5)*T(9)+T(6)*T(9)+T(7)*T(9)+T(3)+T(5)+T(6) \\
&+T(7)
\end{aligned}$$

$$\begin{aligned}
i2[21]=&T(2)*T(4)^2*T(8)+T(3)*T(4)^2*T(8)+T(3)^2*T(7)*T(9)+T(3)*T(5)*T(7)*T(9) \\
&+T(5)^2*T(7)*T(9)+T(2)*T(6)*T(7)*T(9)+T(4)*T(6)*T(7)*T(9)+T(3)*T(7)^2*T(9) \\
&+T(4)*T(7)^2*T(9)+T(5)*T(7)^2*T(9)+T(1)*T(3)*T(8)*T(9)+T(1)*T(4)*T(8)*T(9) \\
&+T(1)*T(2)*T(9)^2+T(1)*T(3)*T(9)^2+T(3)*T(4)*T(9)^2+T(1)*T(5)*T(9)^2 \\
&+T(2)*T(5)*T(9)^2+T(2)*T(6)*T(9)^2+T(4)*T(6)*T(9)^2+T(3)*T(7)*T(9)^2 \\
&+T(4)*T(7)*T(9)^2+T(5)*T(7)*T(9)^2+T(2)*T(9)^3+T(3)*T(9)^3+T(1)*T(6)^2 \\
&+T(3)^2*T(7)+T(1)*T(4)*T(7)+T(3)*T(4)*T(7)+T(1)*T(5)*T(7)+T(2)*T(5)*T(7) \\
&+T(3)*T(5)*T(7)+T(5)^2*T(7)+T(3)*T(6)*T(7)+T(4)*T(6)*T(7)+T(6)^2*T(7) \\
&+T(1)*T(7)^2+T(2)*T(7)^2+T(4)*T(7)^2+T(5)*T(7)^2+T(6)*T(7)^2+T(1)*T(2)*T(8) \\
&+T(1)*T(3)*T(8)+T(3)^2*T(8)+T(4)^2*T(8)+T(3)*T(5)*T(8)+T(5)^2*T(8) \\
&+T(5)*T(7)*T(8)+T(1)*T(3)*T(9)+T(2)*T(3)*T(9)+T(3)^2*T(9)+T(2)*T(4)*T(9) \\
&+T(1)*T(5)*T(9)+T(2)*T(5)*T(9)+T(3)*T(5)*T(9)+T(4)*T(5)*T(9)+T(1)*T(6)*T(9) \\
&+T(2)*T(6)*T(9)+T(3)*T(6)*T(9)+T(1)*T(7)*T(9)+T(2)*T(7)*T(9)+T(3)*T(7)*T(9) \\
&+T(4)*T(7)*T(9)+T(6)*T(7)*T(9)+T(7)^2*T(9)+T(5)*T(8)*T(9)+T(1)*T(9)^2 \\
&+T(2)*T(9)^2+T(3)*T(9)^2+T(6)*T(9)^2+T(7)*T(9)^2+T(1)*T(2)+T(2)*T(3)+T(3)^2 \\
&+T(2)*T(5)+T(3)*T(5)+T(4)*T(6)+T(2)*T(7)+T(3)*T(7)+T(5)*T(7)+T(7)^2+T(1)*T(8) \\
&+T(3)*T(8)+T(5)*T(8)+T(1)*T(9)+T(3)*T(9)+T(4)*T(9)+T(5)*T(9)+T(6)*T(9) \\
&+T(7)*T(9)+T(9)^2+T(1)+T(6)+T(9)
\end{aligned}$$

$$\begin{aligned}
i2[22]=&T(4)*T(5)^3+T(2)*T(3)*T(5)+T(1)*T(4)*T(5)+T(4)*T(5)^2+T(3)*T(4)*T(7) \\
&+T(4)*T(5)*T(7)+T(4)*T(5)*T(9)+T(2)*T(3)+T(1)*T(5)+T(2)*T(5)+T(5)^2+T(3)*T(6) \\
&+T(1)*T(7)+T(2)*T(7)+T(3)*T(7)+T(4)*T(7)+T(5)*T(7)+T(5)*T(9)+T(3)+T(5)+T(7)
\end{aligned}$$

$$\begin{aligned}
i2[23]=&T(1)*T(5)^3+T(1)*T(6)*T(7)^2+T(1)*T(7)^3+T(1)*T(2)*T(5)*T(8) \\
&+T(2)*T(5)^2*T(8)+T(5)^3*T(8)+T(1)*T(3)*T(6)*T(8)+T(1)*T(6)^2*T(8) \\
&+T(3)*T(4)*T(7)*T(8)+T(1)*T(5)*T(7)*T(8)+T(2)*T(5)*T(7)*T(8) \\
&+T(3)*T(5)*T(7)*T(8)+T(5)^2*T(7)*T(8)+T(4)*T(6)*T(7)*T(8)+T(1)*T(7)^2*T(8) \\
&+T(2)*T(7)^2*T(8)+T(3)*T(7)^2*T(8)+T(4)*T(7)^2*T(8)+T(1)*T(2)*T(8)^2
\end{aligned}$$

$$\begin{aligned}
&+T(1)*T(3)*T(8)^2+T(3)^2*T(8)^2+T(3)*T(4)*T(8)^2+T(1)*T(5)*T(8)^2 \\
&+T(3)*T(5)*T(8)^2+T(5)^2*T(8)^2+T(3)*T(7)*T(8)^2+T(4)*T(7)*T(8)^2 \\
&+T(3)*T(4)^2*T(9)+T(1)*T(2)*T(5)*T(9)+T(1)*T(3)*T(5)*T(9)+T(2)*T(5)^2*T(9) \\
&+T(3)*T(5)^2*T(9)+T(1)*T(3)*T(6)*T(9)+T(1)*T(4)*T(6)*T(9)+T(1)*T(3)*T(7)*T(9) \\
&+T(3)^2*T(7)*T(9)+T(2)*T(5)*T(7)*T(9)+T(3)*T(5)*T(7)*T(9)+T(1)*T(6)*T(7)*T(9) \\
&+T(3)*T(6)*T(7)*T(9)+T(4)*T(6)*T(7)*T(9)+T(6)^2*T(7)*T(9)+T(1)*T(7)^2*T(9) \\
&+T(6)*T(7)^2*T(9)+T(1)*T(3)*T(8)*T(9)+T(1)*T(4)*T(8)*T(9)+T(2)*T(5)*T(8)*T(9) \\
&+T(4)*T(5)*T(8)*T(9)+T(1)*T(6)*T(8)*T(9)+T(2)*T(6)*T(8)*T(9) \\
&+T(5)*T(7)*T(8)*T(9)+T(5)*T(8)^2*T(9)+T(1)*T(3)*T(9)^2+T(2)*T(3)*T(9)^2 \\
&+T(1)*T(4)*T(9)^2+T(3)*T(4)*T(9)^2+T(2)*T(5)*T(9)^2+T(1)*T(6)*T(9)^2 \\
&+T(3)*T(6)*T(9)^2+T(1)*T(7)*T(9)^2+T(2)*T(7)*T(9)^2+T(4)*T(7)*T(9)^2 \\
&+T(5)*T(7)*T(9)^2+T(3)*T(8)*T(9)^2+T(5)*T(8)*T(9)^2+T(1)*T(2)*T(3) \\
&+T(1)*T(2)*T(5)+T(1)*T(3)*T(5)+T(2)*T(3)*T(5)+T(1)*T(5)^2+T(4)*T(5)^2+T(5)^3 \\
&+T(1)*T(6)^2+T(2)*T(6)^2+T(4)*T(6)^2+T(6)^3+T(3)^2*T(7)+T(1)*T(4)*T(7) \\
&+T(3)*T(4)*T(7)+T(1)*T(5)*T(7)+T(2)*T(5)*T(7)+T(5)^2*T(7)+T(1)*T(6)*T(7) \\
&+T(3)*T(6)*T(7)+T(4)*T(6)*T(7)+T(2)*T(7)^2+T(3)*T(7)^2+T(6)*T(7)^2 \\
&+T(1)*T(2)*T(8)+T(2)*T(3)*T(8)+T(3)^2*T(8)+T(1)*T(4)*T(8)+T(2)*T(5)*T(8) \\
&+T(3)*T(5)*T(8)+T(1)*T(6)*T(8)+T(3)*T(6)*T(8)+T(6)^2*T(8)+T(1)*T(7)*T(8) \\
&+T(2)*T(7)*T(8)+T(3)*T(7)*T(8)+T(4)*T(7)*T(8)+T(5)*T(7)*T(8)+T(6)*T(7)*T(8) \\
&+T(3)*T(8)^2+T(5)*T(8)^2+T(2)*T(3)*T(9)+T(3)*T(4)*T(9)+T(2)*T(5)*T(9) \\
&+T(4)*T(5)*T(9)+T(5)^2*T(9)+T(2)*T(6)*T(9)+T(1)*T(7)*T(9)+T(2)*T(7)*T(9) \\
&+T(3)*T(7)*T(9)+T(5)*T(7)*T(9)+T(6)*T(7)*T(9)+T(6)*T(8)*T(9)+T(7)*T(8)*T(9) \\
&+T(1)*T(3)+T(2)*T(3)+T(1)*T(5)+T(3)*T(5)+T(4)*T(5)+T(5)^2+T(6)^2+T(3)*T(7) \\
&+T(4)*T(7)+T(6)*T(7)+T(7)^2+T(1)*T(8)+T(3)*T(8)+T(5)*T(8)+T(7)*T(8)+T(6)*T(9) \\
&+T(3)
\end{aligned}$$

$$\begin{aligned}
i_2[24]=&T(4)^4+T(4)*T(5)*T(8)^2+T(4)^3*T(9)+T(3)^2*T(7)*T(9) \\
&+T(3)*T(5)*T(7)*T(9)+T(5)^2*T(7)*T(9)+T(3)*T(6)*T(7)*T(9)+T(4)*T(6)*T(7)*T(9) \\
&+T(6)^2*T(7)*T(9)+T(4)*T(7)^2*T(9)+T(5)*T(7)^2*T(9)+T(6)*T(7)^2*T(9) \\
&+T(1)*T(2)*T(8)*T(9)+T(1)*T(4)*T(8)*T(9)+T(1)*T(5)*T(8)*T(9) \\
&+T(1)*T(7)*T(8)*T(9)+T(1)*T(2)*T(9)^2+T(1)*T(3)*T(9)^2+T(1)*T(5)*T(9)^2 \\
&+T(2)*T(5)*T(9)^2+T(4)*T(6)*T(9)^2+T(3)*T(7)*T(9)^2+T(4)*T(7)*T(9)^2 \\
&+T(5)*T(7)*T(9)^2+T(2)*T(9)^3+T(3)*T(9)^3+T(1)*T(4)^2+T(2)*T(4)^2+T(3)*T(5)^2 \\
&+T(4)*T(5)^2+T(5)^3+T(1)*T(3)*T(6)+T(1)*T(6)^2+T(4)*T(6)^2+T(6)^3 \\
&+T(1)*T(5)*T(7)+T(2)*T(6)*T(7)+T(3)*T(6)*T(7)+T(1)*T(7)^2+T(4)*T(7)^2 \\
&+T(5)*T(7)^2+T(6)*T(7)^2+T(1)*T(3)*T(8)+T(2)*T(3)*T(8)+T(1)*T(4)*T(8) \\
&+T(2)*T(4)*T(8)+T(1)*T(5)*T(8)+T(2)*T(5)*T(8)+T(3)*T(5)*T(8)+T(4)*T(5)*T(8) \\
&+T(2)*T(6)*T(8)+T(4)*T(6)*T(8)+T(1)*T(7)*T(8)+T(2)*T(7)*T(8)+T(4)*T(7)*T(8) \\
&+T(2)*T(8)^2+T(5)*T(8)^2+T(1)*T(3)*T(9)+T(2)*T(3)*T(9)+T(2)*T(4)*T(9) \\
&+T(4)^2*T(9)+T(1)*T(5)*T(9)+T(3)*T(5)*T(9)+T(2)*T(6)*T(9)+T(3)*T(6)*T(9) \\
&+T(1)*T(7)*T(9)+T(2)*T(7)*T(9)+T(3)*T(7)*T(9)+T(7)^2*T(9)+T(1)*T(8)*T(9)
\end{aligned}$$

$$\begin{aligned}
&+T(3)*T(8)*T(9)+T(7)*T(8)*T(9)+T(5)*T(9)^2+T(6)*T(9)^2+T(7)*T(9)^2+T(1)*T(2) \\
&+T(3)*T(4)+T(4)^2+T(1)*T(5)+T(2)*T(5)+T(3)*T(5)+T(4)*T(5)+T(5)^2+T(3)*T(6) \\
&+T(6)^2+T(1)*T(7)+T(2)*T(7)+T(3)*T(7)+T(6)*T(7)+T(7)^2+T(3)*T(8)+T(5)*T(8) \\
&+T(6)*T(8)+T(8)^2+T(2)*T(9)+T(5)*T(9)+T(7)*T(9)+T(1)+T(3)+T(6)
\end{aligned}$$

$$\begin{aligned}
i2[25]=&T(3)*T(4)^3+T(3)*T(4)^2+T(1)*T(5)^2+T(1)*T(4)*T(6)+T(4)*T(6)^2 \\
&+T(1)*T(5)*T(7)+T(2)*T(5)*T(7)+T(4)*T(5)*T(7)+T(5)^2*T(7)+T(3)*T(6)*T(7) \\
&+T(6)^2*T(7)+T(2)*T(7)^2+T(4)*T(7)^2+T(5)*T(7)^2+T(6)*T(7)^2+T(2)*T(3)*T(8) \\
&+T(3)*T(4)*T(8)+T(2)*T(5)*T(8)+T(3)*T(5)*T(8)+T(5)^2*T(8)+T(1)*T(7)*T(8) \\
&+T(2)*T(7)*T(8)+T(4)*T(7)*T(8)+T(5)*T(7)*T(8)+T(1)*T(2)*T(9)+T(1)*T(4)*T(9) \\
&+T(1)*T(5)*T(9)+T(1)*T(6)*T(9)+T(2)*T(6)*T(9)+T(3)*T(6)*T(9)+T(1)*T(7)*T(9) \\
&+T(3)*T(7)*T(9)+T(4)*T(7)*T(9)+T(5)*T(7)*T(9)+T(5)*T(8)*T(9)+T(1)*T(2) \\
&+T(1)*T(3)+T(1)*T(6)+T(3)*T(6)+T(6)^2+T(7)*T(8)+T(1)*T(9)+T(7)*T(9)+T(1)
\end{aligned}$$

$$\begin{aligned}
i2[26]=&T(2)*T(4)^3+T(3)*T(4)^2+T(4)*T(6)^2+T(3)^2*T(7)+T(3)*T(4)*T(7) \\
&+T(3)*T(5)*T(7)+T(5)^2*T(7)+T(3)*T(6)*T(7)+T(6)^2*T(7)+T(4)*T(7)^2+T(5)*T(7)^2 \\
&+T(6)*T(7)^2+T(1)*T(2)*T(8)+T(1)*T(4)*T(8)+T(1)*T(5)*T(8)+T(4)*T(5)*T(8) \\
&+T(1)*T(7)*T(8)+T(1)*T(2)*T(9)+T(1)*T(3)*T(9)+T(1)*T(4)*T(9)+T(1)*T(5)*T(9) \\
&+T(2)*T(5)*T(9)+T(4)*T(6)*T(9)+T(3)*T(7)*T(9)+T(4)*T(7)*T(9)+T(5)*T(7)*T(9) \\
&+T(2)*T(9)^2+T(3)*T(9)^2+T(1)*T(2)+T(1)*T(4)+T(1)*T(5)+T(1)*T(6)+T(2)*T(6) \\
&+T(3)*T(6)+T(6)^2+T(1)*T(7)+T(5)*T(7)+T(7)^2+T(1)*T(8)+T(2)*T(8)+T(5)*T(8) \\
&+T(3)*T(9)+T(6)*T(9)+T(1)+T(6)
\end{aligned}$$

$$\begin{aligned}
i2[27]=&T(1)*T(4)^3+T(4)*T(6)^2+T(3)^2*T(7)+T(3)*T(5)*T(7)+T(5)^2*T(7) \\
&+T(3)*T(6)*T(7)+T(6)^2*T(7)+T(4)*T(7)^2+T(5)*T(7)^2+T(6)*T(7)^2+T(1)*T(2)*T(8) \\
&+T(1)*T(4)*T(8)+T(1)*T(5)*T(8)+T(1)*T(7)*T(8)+T(1)*T(2)*T(9)+T(1)*T(3)*T(9) \\
&+T(1)*T(5)*T(9)+T(2)*T(5)*T(9)+T(4)*T(6)*T(9)+T(3)*T(7)*T(9)+T(4)*T(7)*T(9) \\
&+T(5)*T(7)*T(9)+T(2)*T(9)^2+T(3)*T(9)^2+T(2)*T(6)+T(3)*T(6)+T(6)^2+T(3)*T(7) \\
&+T(5)*T(7)+T(6)*T(7)+T(7)^2+T(1)*T(8)+T(1)*T(9)+T(3)*T(9)+T(6)*T(9)+T(6)
\end{aligned}$$

$$\begin{aligned}
i2[28]=&T(4)*T(5)*T(8)^2*T(9)+T(4)^3*T(9)^2+T(2)*T(6)*T(7)*T(9)^2 \\
&+T(3)*T(6)*T(7)*T(9)^2+T(6)^2*T(7)*T(9)^2+T(3)*T(7)^2*T(9)^2 \\
&+T(6)*T(7)^2*T(9)^2+T(1)*T(2)*T(8)*T(9)^2+T(1)*T(3)*T(8)*T(9)^2 \\
&+T(1)*T(5)*T(8)*T(9)^2+T(1)*T(7)*T(8)*T(9)^2+T(2)*T(6)*T(9)^3 \\
&+T(3)*T(6)*T(7)*T(8)+T(6)^2*T(7)*T(8)+T(4)*T(7)^2*T(8)+T(6)*T(7)^2*T(8) \\
&+T(1)*T(2)*T(8)^2+T(2)*T(4)*T(8)^2+T(3)*T(4)*T(8)^2+T(1)*T(7)*T(8)^2 \\
&+T(4)*T(7)*T(8)^2+T(3)*T(5)^2*T(9)+T(5)^3*T(9)+T(1)*T(3)*T(7)*T(9) \\
&+T(1)*T(4)*T(7)*T(9)+T(3)*T(5)*T(7)*T(9)+T(5)^2*T(7)*T(9)+T(3)*T(6)*T(7)*T(9) \\
&+T(4)*T(6)*T(7)*T(9)+T(3)*T(7)^2*T(9)+T(5)*T(7)^2*T(9)+T(2)*T(3)*T(8)*T(9) \\
&+T(1)*T(4)*T(8)*T(9)+T(2)*T(4)*T(8)*T(9)+T(4)^2*T(8)*T(9)+T(4)*T(5)*T(8)*T(9) \\
&+T(1)*T(6)*T(8)*T(9)+T(3)*T(6)*T(8)*T(9)+T(1)*T(7)*T(8)*T(9)
\end{aligned}$$

$$\begin{aligned}
&+T(2)*T(7)*T(8)*T(9)+T(4)*T(7)*T(8)*T(9)+T(2)*T(8)^2*T(9)+T(1)*T(2)*T(9)^2 \\
&+T(1)*T(3)*T(9)^2+T(2)*T(3)*T(9)^2+T(2)*T(4)*T(9)^2+T(3)*T(4)*T(9)^2 \\
&+T(4)^2*T(9)^2+T(3)*T(5)*T(9)^2+T(1)*T(6)*T(9)^2+T(3)*T(6)*T(9)^2 \\
&+T(4)*T(6)*T(9)^2+T(6)^2*T(9)^2+T(3)*T(7)*T(9)^2+T(4)*T(7)*T(9)^2 \\
&+T(6)*T(7)*T(9)^2+T(1)*T(8)*T(9)^2+T(2)*T(8)*T(9)^2+T(3)*T(8)*T(9)^2 \\
&+T(5)*T(8)*T(9)^2+T(7)*T(8)*T(9)^2+T(3)*T(9)^3+T(5)*T(9)^3+T(2)*T(4)^2+T(4)^3 \\
&+T(3)*T(5)^2+T(4)*T(5)^2+T(5)^3+T(1)*T(3)*T(6)+T(4)*T(6)^2+T(1)*T(3)*T(7) \\
&+T(1)*T(4)*T(7)+T(3)*T(4)*T(7)+T(1)*T(5)*T(7)+T(4)*T(5)*T(7)+T(1)*T(6)*T(7) \\
&+T(3)*T(6)*T(7)+T(4)*T(6)*T(7)+T(6)^2*T(7)+T(1)*T(7)^2+T(3)*T(7)^2+T(4)*T(7)^2 \\
&+T(5)*T(7)^2+T(6)*T(7)^2+T(1)*T(2)*T(8)+T(1)*T(3)*T(8)+T(2)*T(3)*T(8) \\
&+T(1)*T(4)*T(8)+T(3)*T(4)*T(8)+T(5)^2*T(8)+T(3)*T(6)*T(8)+T(4)*T(6)*T(8) \\
&+T(6)^2*T(8)+T(1)*T(7)*T(8)+T(3)*T(7)*T(8)+T(7)^2*T(8)+T(1)*T(8)^2+T(2)*T(8)^2 \\
&+T(4)*T(8)^2+T(5)*T(8)^2+T(6)*T(8)^2+T(7)*T(8)^2+T(1)*T(2)*T(9)+T(2)*T(3)*T(9) \\
&+T(1)*T(5)*T(9)+T(4)*T(5)*T(9)+T(5)^2*T(9)+T(2)*T(6)*T(9)+T(4)*T(6)*T(9) \\
&+T(3)*T(7)*T(9)+T(1)*T(8)*T(9)+T(2)*T(8)*T(9)+T(3)*T(8)*T(9)+T(5)*T(8)*T(9) \\
&+T(6)*T(8)*T(9)+T(8)^2*T(9)+T(2)*T(9)^2+T(3)*T(9)^2+T(4)*T(9)^2+T(5)*T(9)^2 \\
&+T(6)*T(9)^2+T(9)^3+T(1)*T(3)+T(2)*T(3)+T(1)*T(4)+T(4)^2+T(2)*T(5)+T(3)*T(5) \\
&+T(4)*T(5)+T(2)*T(6)+T(2)*T(7)+T(4)*T(7)+T(5)*T(7)+T(6)*T(7)+T(7)^2+T(1)*T(8) \\
&+T(2)*T(8)+T(3)*T(8)+T(5)*T(8)+T(7)*T(8)+T(8)^2+T(7)*T(9)+T(8)*T(9)+T(9)^2 \\
&+T(3)+T(5)+T(6)+T(7)
\end{aligned}$$

(This was produced by the author's own program, written in MAGMA but generally untested, employing the author's q-th power algorithm, and may contain transcription errors.)

$$\begin{aligned}
& m2 * m2 \\
& \quad + m3 * (x_2 x_1); \\
& m3 * m2 \\
& \quad + m2 \\
& \quad + m1 * (x_2^2 x_1 + x_2 x_1^2); \\
& m3 * m3 \\
& \quad + m3 \\
& \quad + m2 * (x_2 + x_1); \\
& m4 * m2 \\
& \quad + m5; \\
& m4 * m3 \\
& \quad + m7; \\
& m4 * m4 \\
& \quad + m10 \\
& \quad + m8 * x_1; \\
& m5 * m2 \\
& \quad + m7 * (x_2 x_1); \\
& m5 * m3 \\
& \quad + m5 \\
& \quad + m4 * (x_2^2 x_1 + x_2 x_1^2); \\
& m5 * m4 \\
& \quad + m10 * (x_2^2 x_1 + x_2 x_1 + x_1) \\
& \quad + m9 * (x_2 x_1^2 + x_1) \\
& \quad + m8 * x_1; \\
& m5 * m5 \\
& \quad + m10 * (x_2^3 x_1^2 + x_2^2 x_1^3 + x_2^2 x_1^2 + x_2 x_1^3 + x_2 x_1) \\
& \quad + m9 * (x_2^2 x_1^2 + x_2 x_1^3) \\
& \quad + m8 * (x_2^2 x_1^2 + x_2 x_1^3 + x_2 x_1^2);
\end{aligned}$$

$$\begin{aligned}
& m6 * m2 \\
& \quad +m7 * x_2^2 \\
& \quad +m4 * (x_2^2 + x_2x_1); \\
& m6 * m3 \\
& \quad +m5 \\
& \quad +m4 * (x_2^3 + x_2^2x_1); \\
& m6 * m4 \\
& \quad +m10 * (x_2^3 + x_2x_1 + x_1) \\
& \quad +m9 * (x_2^2x_1 + x_1) \\
& \quad +m8 * x_1; \\
& m6 * m5 \\
& \quad +m10 * (x_2^4x_1 + x_2^3x_1^2 + x_2^3x_1 + x_2^2x_1^2 + x_2x_1) \\
& \quad +m9 * (x_2^3x_1 + x_2^2x_1^2) \\
& \quad +m8 * (x_2^3x_1 + x_2^2x_1^2 + x_2x_1^2); \\
& m6 * m6 \\
& \quad +m10 * (x_2^5 + x_2^4x_1 + x_2^4 + x_2^3x_1 + x_2^2) \\
& \quad +m9 * (x_2^4 + x_2^3x_1 + x_2^2 + x_2x_1) \\
& \quad +m8 * (x_2^4 + x_2^3x_1 + x_2^3); \\
& m7 * m2 \\
& \quad +m5 \\
& \quad +m4 * (x_2^2x_1 + x_2x_1^2); \\
& m7 * m3 \\
& \quad +m7 \\
& \quad +m5 * (x_2 + x_1); \\
& m7 * m4 \\
& \quad +m10 * (x_2^2x_1 + x_2x_1^2 + x_2x_1 + x_1^2 + 1) \\
& \quad +m9 * (x_2x_1 + x_1^2) \\
& \quad +m8 * (x_2x_1 + x_1^2 + x_1); \\
& m7 * m5 \\
& \quad +m10 * (x_2x_1^2 + x_2x_1 + x_1); \\
& \quad +m9 * (x_2x_1^2 + x_1) \\
& \quad +m8 * (x_2^2x_1^2 + x_2x_1^3 + x_1); \\
& m7 * m6 \\
& \quad +m10 * (x_2^3 + x_2x_1 + x_1) \\
& \quad +m9 * (x_2x_1^2 + x_1) \\
& \quad +m8 * (x_2^3x_1 + x_2^2x_1^2 + x_1); \\
& m7 * m7 \\
& \quad +m10 * (x_2^3x_1 + x_2^2x_1^2 + 1) \\
& \quad +m9 * (x_2^2x_1^2 + x_2x_1^3) \\
& \quad +m8 * x_1;
\end{aligned}$$

$$\begin{aligned}
& m8 * m2 \\
& \quad +m10 * (x_2^2 + x_2) \\
& \quad +m9 * (x_2x_1); \\
& m8 * m3 \\
& \quad +m10 * (x_2^2 + x_2x_1 + x_2 + x_1 + 1) \\
& \quad +m9 * (x_2 + x_1 + 1) \\
& \quad +m8 * (x_2 + x_1 + 1); \\
& m8 * m4 \\
& \quad +m7 * (x_2 + 1) \\
& \quad +m5 * (x_2 + 1) \\
& \quad +m4 * (x_2 + 1) \\
& \quad +m3 * (x_2^3x_1) \\
& \quad +m2 * (x_2^3 + x_2^2x_1 + x_2^2); \\
& m8 * m5 \\
& \quad +m7 * (x_2^2x_1 + x_2x_1) \\
& \quad +m4 * (x_2^3x_1 + x_2^3x_1^2 + x_2^2x_1 + x_2x_1^2) \\
& \quad +m3 * (x_2^4x_1 + x_2^3x_1^2 + x_2^3x_1) \\
& \quad +m2 * (x_2^3x_1) \\
& \quad +m1 * (x_2^5x_1^2 + x_2^4x_1^3); \\
& m8 * m6 \\
& \quad +m7 * (x_2^3 + x_2^2) \\
& \quad +m6 * (x_2 + 1) \\
& \quad +m5 * (x_2 + 1) \\
& \quad +m4 * (x_2^4 + x_2^3x_1 + x_2^2 + x_2x_1) \\
& \quad +m3 * (x_2^5 + x_2^4x_1 + x_2^4) \\
& \quad +m2 * (x_2^3x_1) \\
& \quad +m1 * (x_2^6x_1 + x_2^5x_1^2 + x_2^5 + x_2^3x_1^2 + x_2^4 + x_2^3x_1); \\
& m8 * m7 \\
& \quad +m5 * (x_2^2 + x_2x_1 + x_1 + 1) \\
& \quad +m4 * (x_2^3x_1 + x_2^2x_1^2 + x_2^2x_1 + x_2x_1^2) \\
& \quad +m3 * (x_2^3x_1) \\
& \quad +m2 * (x_2^4x_1 + x_2^3x_1^2 + x_2^3 + x_2^2x_1 + x_2^2) \\
& \quad +m1 * (x_2^5x_1 + x_2^3x_1^3 + x_2^4x_1 + x_2^3x_1^2); \\
& m8 * m8 \\
& \quad +m10 * (x_2^2x_1 + x_2 + x_1 + 1) \\
& \quad +m9 * (x_2^2x_1 + x_2^2 + x_1 + 1) \\
& \quad +m8 * (x_2^2 + x_2x_1 + x_2 + x_1) \\
& \quad +m7 * x_2^3 \\
& \quad +m6 * x_2^2 \\
& \quad +m5 * x_2^2;
\end{aligned}$$

$$\begin{aligned}
& m9 * m2 \\
& \quad +m10 * (x_2^2 + x_2x_1 + x_1) \\
& \quad +m9 * x_1 \\
& \quad +m8 * x_1; \\
& m9 * m3 \\
& \quad +m10 * (x_2 + 1) \\
& \quad +m9 * x_2 \\
& \quad +m8 * (x_2 + x_1); \\
& m9 * m4 \\
& \quad +m7 * (x_2 + 1) \\
& \quad +m4 * (x_2x_1 + x_2 + x_1 + 1) \\
& \quad +m3 * (x_2^2x_1) \\
& \quad +m2 * x_2^3 \\
& \quad +m1 * (x_2^4x_1 + x_2^3x_1^2); \\
& m9 * m5 \\
& \quad +m5 * (x_2x_1 + x_1) \\
& \quad +m4 * (x_2^3x_1 + x_2^2x_1^2 + x_2^2x_1 + x_2x_1^2) \\
& \quad +m3 * (x_2^4x_1) \\
& \quad +m2 * (x_2^4x_1 + x_2^3x_1^2 + x_2^2x_1) \\
& \quad +m1 * (x_2^4x_1^2 + x_2^3x_1^3); \\
& m9 * m6 \\
& \quad +m7 * (x_2 + 1) \\
& \quad +m6 * (x_2 + 1) \\
& \quad +m5 * (x_2^2 + 1) \\
& \quad +m4 * (x_2^3 + x_2^2x_1 + x_2 + 1) \\
& \quad +m3 * (x_2^5 + x_2^4 + x_2^3x_1) \\
& \quad +m2 * (x_2^5 + x_2^4x_1 + x_2^2x_1) \\
& \quad +m1 * (x_2^5x_1 + x_2^4x_1^2 + x_2^5 + x_2^4x_1 + x_2^4 + x_2^3x_1); \\
& m9 * m7 \\
& \quad +m7 * (x_2x_1 + x_1) \\
& \quad +m5 * (x_2^2 + x_2x_1 + x_2 + x_1) \\
& \quad +m3 * (x_2^4x_1 + x_2^3x_1^2 + x_2^2x_1) \\
& \quad +m2 * (x_2^3x_1 + x_2^2x_1^2 + x_2^3) \\
& \quad +m1 * (x_2^5x_1 + x_2^4x_1^2);
\end{aligned}$$

$$\begin{aligned}
& m9 * m8 \\
& \quad +m10 * (x_2^3 + x_2^2 x_1 + x_1 + 1) \\
& \quad +m9 * (x_2^2 + x_2 x_1 + x_1 + 1) \\
& \quad +m8 * (x_2^2 + x_2) \\
& \quad +m7 * x_2^2 \\
& \quad +m6 * x_2^2 \\
& \quad +m4 * (x_2^4 + x_2^3 x_1 + x_2^2); \\
& m9 * m9 \\
& \quad +m10 * (x_2^2 + 1) \\
& \quad +m9 * (x_2^2 + x_2 x_1 + x_1 + 1) \\
& \quad +m8 * (x_2^2 + x_2 x_1 + x_2 + x_1) \\
& \quad +m6 * x_2^2 \\
& \quad +m5 * x_2^2 \\
& \quad +m4 * (x_2^2 x_1); \\
& m10 * m2 \\
& \quad +m10 * x_1 \\
& \quad +m9 * x_1 \\
& \quad +m8 * x_1; \\
& m10 * m3 \\
& \quad +m10 * (x_1 + 1) \\
& \quad +m9 * x_1; \\
& m10 * m4 \\
& \quad +m4 * (x_2 x_1 + x_1) \\
& \quad +m3 * (x_2^2 x_1) \\
& \quad +m2 * (x_2^2 x_1); \\
& m10 * m5 \\
& \quad +m5 * (x_2 x_1 + x_1) \\
& \quad +m3 * (x_2^3 x_1^2) \\
& \quad +m2 * (x_2^2 x_1) \\
& \quad +m1 * (x_2^4 x_1^2 + x_2^3 x_1^3);
\end{aligned}$$

$$\begin{aligned}
m_{10} * m_6 & \quad +m_7 * (x_2 + 1) \\
& \quad +m_5 * (x_2^2 + x_2) \\
& \quad +m_4 * (x_2 + 1) \\
& \quad +m_3 * (x_2^4 x_1) \\
& \quad +m_2 * (x_2^2 x_1) \\
& \quad +m_1 * (x_2^5 x_1 + x_2^4 x_1^2 + x_2^4 x_1 + x_2^3 x_1^2); \\
m_{10} * m_7 & \quad +m_7 * (x_2 x_1 + x_1) \\
& \quad +m_3 * (x_2^2 x_1) \\
& \quad +m_2 * (x_2^3 x_1 + x_2^2 x_1^2 + x_2^2 x_1) \\
& \quad +m_1 * (x_2^4 x_1^2 + x_2^3 x_1^3); \\
m_{10} * m_8 & \quad +m_8 * (x_2 x_1 + x_1) \\
& \quad +m_7 * x_2^2 \\
& \quad +m_5 * x_2^2 \\
& \quad +m_4 * x_2^2; \\
m_{10} * m_9 & \quad +m_9 * (x_2 x_1 + x_1) \\
& \quad +m_7 * x_2^2 \\
& \quad +m_4 * (x_2^2 x_1 + x_2^2); \\
m_{10} * m_{10} & \quad +m_{10} * (x_2 x_1 + x_1) \\
& \quad +m_4 * (x_2^2 x_1); \\
m_6 * x_1 + m_5 * x_2 & \quad +m_7 + m_4;
\end{aligned}$$