ON ALGEBRAS ARISING FROM SEMIREGULAR GROUP ACTIONS

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1. SCHUR RINGS

For a permutation group (G, X), its **centralizer algebra** $V(G, X)$ is defined to be the algebra of all $n \times n$ matrices with entries in $\mathbb{Q}$ that commute with the permutation matrices corresponding to the elements of G.

In the case when G contains a regular subgroup H, $V(G, X)$ has a nice embedding into the group algebra $\mathbb{Q}H$. For a subset A of H, $\underline{A}$ denotes the element $\sum_{h \in A} h$ in $\mathbb{Q}H$, such elements are called simple quantities. Fix a point x in X. For a relation $R \subseteq X^2$ invariant under G, its Schur projection on H at X is defined as

$$\mathrm{s}pr_{H,x}(R) = \{h \in H \mid (x, x^h) \in R\}.$$

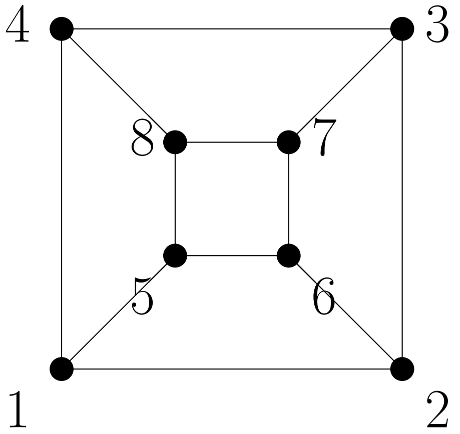
The simple quantities $\underline{\mathrm{s}pr_{H,x}(R)}$ that R are the 2-orbits of G span a subalgebra of $\mathbb{Q}H$ isomorphic to $V(G, X)$. It is called the **transitivity module** of G, and denoted by $V(H, G_x)$.

A transitivity module $\mathcal{A}$ satisfies the following porperties:

(S1) $\mathcal{A}$ has a basis of simple quantities $\underline{T_0}, ..., \underline{T_d}$, where $T_0 = \{e\}$ (e denotes the identity of H).
(S2) $T_i \cap T_j = \emptyset$ for all $i \neq j$, and $\cup_{i=0}^{d} T_i = H$.
(S3) For each $i \in \{0, ..., d\}$, there exists a $i' \in \{0, ..., d\}$ such that $T_{i'} = T_i^{-1} := \{t^{-1} \mid t \in T_i\}$.

The converse is not true in general, the class of subalgebras $\mathcal{A}$ of $\mathbb{Q}H$ satisfying (S1)-(S3) does not coincide with the class of transitivity modules over H. The members of this class are called **Schur rings** over H. The sets T_i are called the basic sets of $\mathcal{A}$.

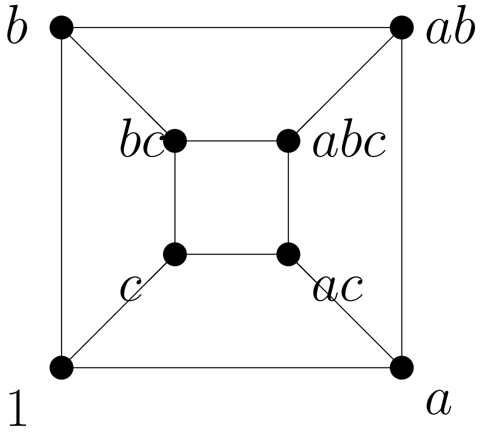
EXAMPLE 1. *Let* $Q_3 = (V, E)$ *denote the cubical graph:*



We determine $\mathcal{A} = V(H, G_x)$ *that* $(G, X) = (\mathrm{Aut}\,(Q_3), V)$, $x = 1$, *and* $H = \big\langle\, a, b, c \big\rangle$, *where*

$a = (12)(34)(56)(78)$, $b = (14)(23)(58)(67)$ *and* $c = (15)(26)(37)(48)$.

There is a simple way to determine the basic sets of $\mathcal{A}$. *Label the vertices of* V *with elements of* H *as follows: assign* $e \in H$ *to the vertex* $x = 1$, *and* $h \in H$ *to another vertex* $i \in V$ *if* $x^h = i$.



The basic sets of $\mathcal{A}$ *are obtained as the orbits of the stabilizer* G_1 *in* G*, that is*

$$\mathcal{A} = \left\langle\, \underline{e},\ \underline{a, b, c},\ \underline{ab, bc, ac},\ \underline{abc} \,\right\rangle.$$

$\square$

The definition of centralizer algebra is due to H. Wielandt [**12**], it goes back to the work I. Schur, who was the first asssociating (G, X), which contains a regular subgroup H, with its transitivity module $V(H, G_x)$ [**11**].

2. SCHUR RINGS OF LARGER DEGREES

For the rest we fix the following situation: (G, X) is a permutation group containing a semiregular subgroup H. Let $X_1, ..., X_n$ denote the orbits of H, and fix a point x_i in X_i for all i.

Let $\mathcal{M}_n(\mathbb{Q}H)$ denote the algebra of $n \times n$ matrices with entries in $\mathbb{Q}H$. An $n \times n$ matrix $\mathbf{S} = (S_{i,j})$, $S_{i,j}$ is a subset of H, not all are the emptyset, is called a **symbol** of H of order n, the corresponding element $\underline{\mathbf{S}} := (\underline{S_{i,j}})$ in $\mathcal{M}_n(\mathbb{Q}H)$ will be called a simple quantity. We define $\underline{\emptyset} := \mathbf{0}$, where $\mathbf{0}$ is the zero element of $\mathbb{Q}H$. Intersection and union of two symbols of the same order are defined entrywise. $\mathbf{H}$ will denote the symbol that has H in each of its entires.

DEFINITION 1. *For a relation $R \subseteq X^2$ invariant under G, the* **Schur projection** *of R on H at (x_i) is defined to be the symbol*

$$spr_{H,(x_i)}(R) = \begin{pmatrix} R_{1,1} & \cdots & R_{1,m} \\ \vdots & \cdots & \vdots \\ R_{m,1} & \cdots & R_{m,m} \end{pmatrix},$$

where $R_{i,j} = \{h \in H \mid (x_i, x_j^h) \in R\}$.

Definition 2. *The transitivity module of G relative to (x_i) is defined as the subalgebra of $\mathcal{M}_n(\mathbb{Q}H)$ spanned by the simple quantities $\underline{\mathrm{spr}_{H,(x_i)}(T)}$ that T are the 2-orbits of G. It is denoted by $V(H, G_{(x_i)})$.*

Definition 3. *A subalgebra $\mathcal{A}$ of $\mathcal{M}_n(\mathbb{Q}H)$ is called a* **Schur ring over H of degree n** *if the following axioms hold:*

(S1′) *$\mathcal{A}$ has a basis of simple quantities $\underline{A_0}, ..., \underline{A_d}$, where $A_0 = (T_{i,j})$ with $T_{i,i} = \{e\}$ and $T_{i,j} = \emptyset$ otherwise.*

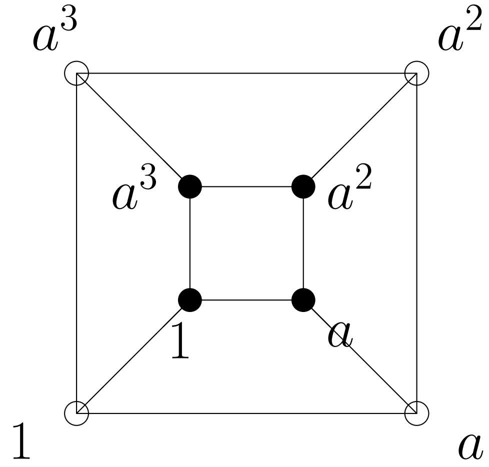
(S2′) *$A_i \cap A_j = \emptyset$ for all $i \neq j$, and $\cup_{i=0}^{d} A_i = \mathbf{H}$.*

(S3′) *For each $i \in \{0, ..., d\}$, there exists $i' \in \{0, ..., d\}$ such that $A_{i'} = A_i^*$, where for $A_i = (T_{k,l})$, A_i^* is defind to be $(T'_{k,l})$ such that $T'_{k,l} = T_{l,k}^{-1}$.*

The symbols $A_0, ..., A_d$ are called the **basic symbols** *of $\mathcal{A}$.*

Example 2. *Let $Q_3 = (V, E)$ be the cubical graph as in Example 1. We determine $\mathcal{A} = V(H, G_{(x_1,x_2)})$, where $G = \mathrm{Aut}(Q_3)$, $H = \langle a = (1234) \rangle$, and $x_1 = 1$, $x_2 = 5$. (X_1 and X_2 are the orbits of H with $1 \in X_1$ and $5 \in X_2$.)*

Since Q_3 is arc-transitive, E is a 2-orbit of G. In order to determine the basic symbol $\mathbf{E}$ *corresponding* *to E, we label the vertices in V with the elements of H as follows: assign $e \in H$ to the vertices x_i, and $h \in H$ to another vertex $j \in V$ if $x_i^h = j$.*



The basic symbol $\mathbf{E}$ can be red out of the picture to be

$$\mathbf{E} = \begin{pmatrix} \{a, a^3\} & \{e\} \\ \{e\} & \{a, a^3\} \end{pmatrix}.$$

Further calculation gives that

$$\mathcal{A} = \left\langle \begin{pmatrix} \underline{e} & \mathbf{0} \\ \mathbf{0} & \underline{e} \end{pmatrix}, \begin{pmatrix} \underline{a, a^3} & \underline{e} \\ \underline{e} & \underline{a, a^3} \end{pmatrix}, \begin{pmatrix} \underline{a^2} & \underline{a, a^3} \\ \underline{a, a^3} & \underline{a^2} \end{pmatrix}, \begin{pmatrix} \mathbf{0} & \underline{a^2} \\ \underline{a^2} & \mathbf{0} \end{pmatrix} \right\rangle.$$

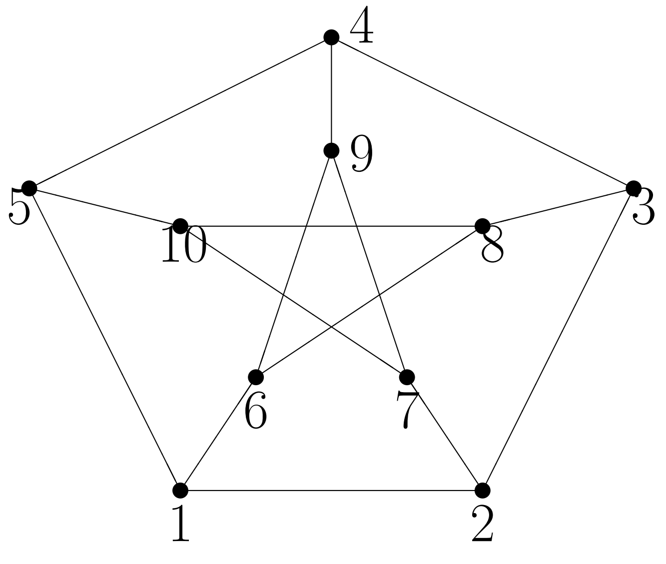
$\square$

3. Semi-Cayley configurations

A combinatorial generalization of the concept of Schur ring is the concept of coherent configuration due to D. G. Higman [**5**]. In this context, a Schur ring over H is the same object as a Cayley configuration over H. On the other hand, a simple graph is called semi-Cayley if it has a group of auotmorphisms having exactly two orbits of vertices. Suggested by these terminologies we call a Schur ring over H of degree 2 a **semi-Cayley configuration over H.**

Examples of such objects are provided by strongly regular semi-Cayley garphs.

Example 3. *Let* $\Gamma = (V, E)$ *denote the Petersen graph:*



Let $H = \langle a \rangle \leq \mathrm{Aut}(\Gamma)$*, where*

$$a = (1\,2\,3\,4\,5)(6\,7\,8\,9\,10).$$

H *has orbits* $X_1 = \{1, 2, 3, 4, 5\}$ *and* $X_2 = \{6, 7, 8, 9, 10\}$*, fix* $x_1 = 1$ *and* $x_2 = 6$*. The simple quantity corresponding to the Schur projection* $\mathrm{spr}_{H,(1,6)}(E)$ *generates the following semi-Cayley configuration over* H*:*

$$\left\langle \begin{pmatrix} \underline{e} & \mathbf{0} \\ \mathbf{0} & \underline{e} \end{pmatrix}, \begin{pmatrix} \underline{a, a^4} & \underline{e} \\ \underline{e} & \underline{a, a^4} \end{pmatrix}, \begin{pmatrix} \underline{a^2, a^3} & \underline{a, a^2, a^3, a^4} \\ \underline{a, a^2, a^3, a^4} & \underline{a, a^4} \end{pmatrix} \right\rangle.$$

$\square$

For a strongly regular semi-Cayley graph (X, E), if

$$\mathrm{spr}_{H,(x_1,x_2)}(E) = \begin{pmatrix} D & S \\ S^{-1} & D' \end{pmatrix},$$

then the triple (D, D', S) was called a **partial difference triple**, see [**4, 8, 10**].

The Schur rings described in Examples 2 and 3 share the property that $T_{i,j} = T_{i,j}^{-1}$ for all i, j and basic symbol $(T_{i,j})$.

DEFINITION 4. *A Schur ring* $\mathcal{A}$ *over* H *is* **reversible** *if* $T_{i,j}^{-1} = T_{i,j}$ *for all* i, j *and basic symbol* (T_{ij}) *of* $\mathcal{A}$*.*

We mention a nice property of reversible Schur rings of degree 2 proved in [**6**]. First, we recall a theorem of Schur.

Let A denote the group of permutations of H of the form $\pi_k\colon h \mapsto h^k$, $\gcd(h, |H|) = 1$. A has an action on $\mathbb{Q}H$ as for $\pi_k \in A$ and $\nu = \sum_{h\in H} a_h h$,

$$(\nu, \pi_k) \mapsto \nu^{(k)} := \sum_{h\in H} a_h h^k.$$

THEOREM 1. [**12**, Theorem 23.9.(a)]
If $\mathcal{A}$ is a Schur ring over an abelian group H, then as a subset of $\mathbb{Q}H$, $\mathcal{A}$ is A-invariant.

To reformulate our theorem we need the following definition.

DEFINITION 5. *Let $\mathcal{A}$ be Schur ring over H of degree n with basic symbols $A_0, .., A_d$. The* **trace module** $\widetilde{\mathcal{A}}$ *of $\mathcal{A}$ is defined as the submodule of $\mathbb{Q}H$ spanned by the elements $Tr(\underline{A_i}) := \underline{T_{1,1}} + \cdots + \underline{T_{n,n}}$ if $A_i = (T_{i,j})$.*

THEOREM 2. [**6**]
If $\mathcal{A}$ is a reversible Schur ring of degree 2 over an abelian group H, then its trace module $\widetilde{\mathcal{A}}$ is A-invariant.

4. n-B-GROUPS

First applications of Schur rings concerned B-groups. A group H is called a **B-group**, if any primitive permutation group which contains a regular subgroup isomorphic to H is 2-transitive. The problem to classify abelian B-groups (asked by W. Burnside) was recently settled by C. H. Li [**9**].

DEFINITION 6. *A group H is called an* **n-B-group** *if any permutation group which contains a semiregular subgroup isomorphic to H with n orbits is* 2-*transitive.*

In [**7**] we investigate the primitivity of (G, X) using the Schur projections of the G-invariant relations. For the rest H is assumed to be abelian. First, we recall some facts from the regular case, see [**2, 3**].

For the moment we assume that H is regular. If $\mathcal{B}$ is a G-invariant partition of X, then it is equal to the partition consisting of the orbits of the group $\mathcal{K} \cap H$, where $\mathcal{K}$ is the kernel of G acting on $\mathcal{B}$.

Let $\mathcal{A}$ denote the transitivity module $V(H, G_x)$. For an arbitrary $K \leq H$, the K-orbits form a G-invariant partition if and only if $\underline{K} \in \mathcal{A}$, such K is also called an $\mathcal{A}$-subgoup of H.

Let Λ_H denote the subgroup lattice of H, and given an arbitrary Schur ring $\mathcal{A}$ over H, let $\Lambda_{H,\mathcal{A}}$ denote its sublattice containing the $\mathcal{A}$-subroups of H.

Let H^* denote dual group of H consisting of its complex irreducible characters. $H \cong H^*$. For Schur ring $\mathcal{A}$ over H with basic sets $T_0, ..., T_d$, consider the classes C's of the equivalence relation $\sim$ on H^* defined as

$$\chi \sim \chi' \text{ if and only if } \chi(T_i) = \chi'(T_i) \text{ for all i,}$$

where $\chi(T_i) = \sum_{t \in T_i} \chi(t)$. The simple quantities $\underline{C}$'s span a Schur ring over H^*, called the **dual Schur ring** to $\mathcal{A}$, and denoted by $\mathcal{A}^*$.

The **Dirichlet correspondence** Φ is the antiisomorphism between Λ_H and Λ_{H^*} defined as

$$\begin{array}{rcl} K & \mapsto & K^\perp = \{\, \chi \in H^* \mid K \leq \ker \chi \,\},\ K \leq H \\ K^* & \mapsto & (K^*)^\perp = \cap_{\chi \in K^*} \ker \chi,\ K^* \leq H^*. \end{array}$$

The restriction of Φ to $\Lambda_{H,\mathcal{A}}$ induces an antiisomorphism between the sublattices $\Lambda_{H,\mathcal{A}}$ and $\Lambda_{H^*,\mathcal{A}^*}$. Therefore, imprimitivity systems of G can be studied by considering the Schur ring over H^* dual to $V(H, G_x)$. This observation, combined with certain properties of sums of complex roots of unity, was explored by W. Burnside to show that, a cyclic group of composite prime power order is a B-group [**1**].

Let us turn back to our general situation, that is, H is semiregular with n orbits. First, we describe a class of partitions of X.

DEFINITION 7. *Given an n-tuple (y_i) $(y_i \in X_i)$, a parititon Δ of $\{1, ..., n\}$ and a subgroup $K \leq H$, define the partition as*

$$\Pi((y_i), \Delta, K) := \Big\{ \bigcup_{i \in T} x_i^{hK} \mid T \in \Delta,\ h \in H \Big\}.$$

the (n)-tuple (y_i) is called the **base vector** of $\Pi((y_i), \Delta, K)$.

EXAMPLE 4. *Let $X = \{1, \ldots, 12\}$ and $H = \langle a \rangle$ where $a = (1\,2\,3\,4)(5\,6\,7\,8)(9\,10\,11\,12)$. The orbits of H are $X_1 = \{1, 2, 3, 4\}$, $X_2 = \{5, 6, 7, 8\}$ and $X_3 = \{9, 10, 11, 12\}$. Choose $\Delta = \big\{ \{1, 2\}, \{3\} \big\}$ and $K = \langle a^2 \rangle$. There are two different partitions of X of the form $\Pi((x_1, x_2), \Delta, K)$ which are shown below.*

base vector (x_i)	*elements of* $\Pi(\underline{x}, \Delta, K)$
$(1,5,9)$	$\{1,3,5,7\}$, $\{2,4,6,8\}$, $\{9,11\}$
	$\{10,12\}$
$(1,6,9)$	$\{1,3,6,8\}$, $\{2,4,5,7\}$, $\{9,11\}$
	$\{10,12\}$

It can be shown that if $\mathcal{B}$ is a G-invariant partition of X, then $\mathcal{B} = \Pi((y_i), \Delta, K)$, where $K = \mathcal{K} \cap H$, $\mathcal{K}$ is the kernel of the action of G on $\mathcal{B}$ and Δ is a uniform partition, that is, having classes of the same size. Note that, if H is regular then this is the partition consisting of K-orbits.

Characters are extended to symbols as for $\chi \in H^*$ and symbol $\mathbf{S} = (S_{i,j})$, let $\chi(\mathbf{S})$ be the $n \times n$ complex matrix defined as $\chi(\mathbf{S}) = (\chi(S_{i,j}))$, where $\chi(\emptyset) := 0$.

Definition 8. *With the above assumptions, for a relation* $R \subseteq X^2$ *invariant under* G*, and an eigenvalue* λ *of* (X, R) *(that is, of an adjcency matrix of* (X, R)*), define the subgroup of* H^* *as*

$$K_{R,\lambda} = \left\langle\, \chi \in H^* \mid \det(\chi(spr_{H,x}(R)) - \lambda I) = 0 \,\right\rangle.$$

We have the following theorem.

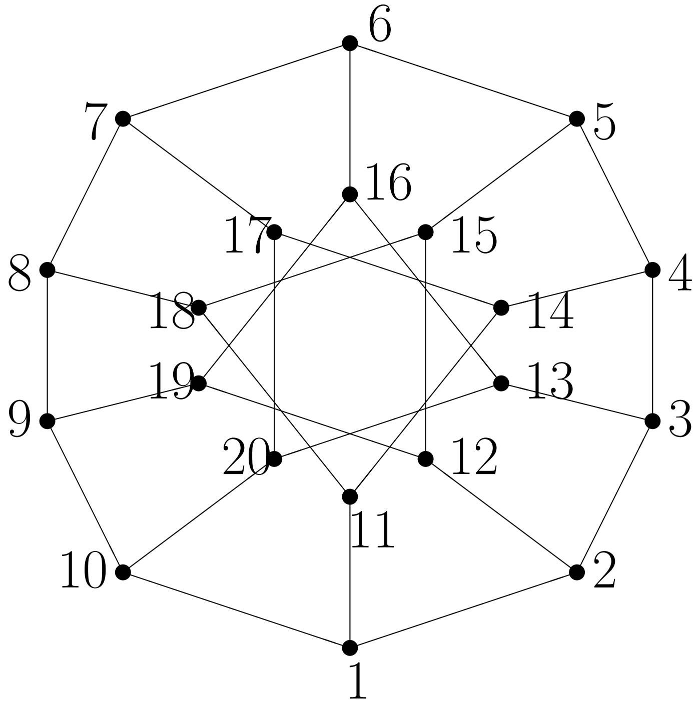
Theorem 3. [**7**]
With the above notations, if $R \subseteq X^2$ *is invariant under* G *and* λ *is an eigenvalue of* (X, R)*, then there exits a* G*-invariant partition of the form* $\Pi(\underline{y}, \Delta, K^{\perp}_{R,\lambda})$.

In certain cases we can also calculate Δ from $\mathrm{spr}_{H,\underline{x}}(R)$. In particular, we have the following theorem.

Theorem 4. [**7**]
With the notations of Theorem 3, if n *is a prime and* $\lambda \neq |R|/|X|$*, then* $\Pi(\underline{y}, \Delta, K^{\perp}_{R,\lambda})$ *is equal to the partition consisting of the* $K^{\perp}_{R,\lambda}$*-orbits.*

Our final example illustrates Theorem 4.

Example 5. *Let* $\Gamma = (V, E)$ *be the Desargues graph:*



Let $(G, X) = (\mathrm{Aut}(\Gamma), V)$, $H = \langle a \rangle$, *where*

$$a = (1\,2\,\cdots\,10)(11\,12\,\cdots\,20).$$

H *has orbits* $X_1 = \{1, ..., 10\}$ *and* $X_2 = \{11, ..., 20\}$. *Fix* $x_1 = 1$ *and* $x_2 = 11$. *Then*

$$spr_{H,(1,11)}(E) = \begin{pmatrix} \underline{a, a^{-1}} & \underline{1} \\ \underline{1} & \underline{a^3, a^{-3}} \end{pmatrix}.$$

It can be checked that -1 *is an eigenvalue of* Γ. *We obtain that for* $\chi \in H^*$, $\chi \in K_{E,-1}$ *if and only if* $\langle a^5 \rangle \leq \ker \chi$. *Therefore, by definition,*

$$K_{E,-1}^{\perp} = \langle a^5 \rangle.$$

In view of Theorem 4, conclude that the $\langle a^5 \rangle$*-orbits form an equitable partition* Δ *of* Γ. *One can see that the corresponding quotient graph* Γ/Δ *is equal to the Petersen graph. Moreover,* $\langle a^5 \rangle \lhd G$, *and hence* Γ *is obtained as a regular covering of the Petersen graph.*

□

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