

# Automated Synthesis of a Gröbner Bases Algorithm

Bruno Buchberger  
Research Institute for Symbolic Computation  
Johannes Kepler University, Linz, Austria

Talk at Workshop "Formal Gröbner Bases Theory"  
RICAM - RISC, Linz, Austria  
March 6, 2006

**Copyright Note:** Copyright Bruno Buchberger 2006

This file may be copied and stored in data bases under the following conditions:

- The file is kept unchanged (including this copyright note).
- A message is sent to [Bruno.Buchberger@jku.at](mailto:Bruno.Buchberger@jku.at).
- If you use material contained in this file, cite it appropriately referring to the above talk and workshop.

---

## Objectives

Demonstrate that

- automated synthesis of non-trivial algorithms is possible,
- main idea (S-polynomials) of algorithmic Gröbner bases theory can be "invented" automatically.

The algorithm synthesis method described here can be implemented in any reasoning system that provides:

- access to the proof objects even in the case of failing proofs,
- a "natural" proof style (temporary assumptions and temporary goals).

## **An "Algorithm" for Algorithm Synthesis**

### **Synthesis of a Gröbner Bases Algorithm**

#### **Conclusion**

## **An "Algorithm" for Algorithm Synthesis**

### **Synthesis of a Gröbner Bases Algorithm**

#### **Conclusion**

---

## **The Algorithm Invention ("Synthesis") Problem**

Given a problem specification  $P$  (in predicate logic), find an algorithm  $A$  such that

$$\forall_x P[x, A[x]] .$$

Examples of specifications P:

```
P[x, y] ⇔ is-greater[x, y]
P[x, y] ⇔ is-sorted-version[x, y]
P[x, y] ⇔ has-derivative[x, y]
P[x, y] ⇔ are-factors-of[x, y]
P[x, y] ⇔ is-Gröbner-basis[x, y]
....
```

A general algorithm S for "all" P cannot exist but ...

---

## Literature

There is a rich literature on algorithm synthesis methods, see survey

[Basin et al. 2004] D. Basin, Y. Deville, P. Flener, A. Hamfelt, J. F. Nilsson. Synthesis of Programs in Computational Logic. In: M. Bruynooghe, K. K. Lau (eds.), Program Development in Computational Logic, Lecture Notes in Computer Science, Vol. 3049, Springer, 2004, pp. 30-65.

Our method is in the class of "scheme-based" methods. Closest (but essentially different):

[Lau et al. 1999] K. K. Lau, M. Ornaghi, S. Tärnlund. Steadfast logic programs. Journal of Logic Programming, 38/3, 1999, pp. 259-294. (*Synthesis from successful proofs.*)

Work in the Group of A. Bundy, Critics. (*Synthesis of the appropriate induction scheme.*)

---

## The "Lazy Thinking" Method for Algorithm Synthesis (BB 2001): Intuition

"Lazy Thinking" Method for Algorithm Synthesis = My Advice to "Students" (= "Computers") How to Invent Algorithms

Given: A problem P.

Find: An algorithm A for P.

- ♣ Completely understand the problem. ("Specification" of the problem.)
- ♣ Learn how to prove.
- ♣ Collect (prove) "complete" knowledge on the auxiliary notion appearing in the problem.

- ♣ Consider known fundamental ideas of how to structure algorithms in terms of subalgorithms ("algorithm schemes").  
Try one scheme A after the other.
- ♣ Try to prove that A solves P: From the **failing proof** construct specifications of the subalgorithms that make the proof successful.

## The "Lazy Thinking" Method for Algorithm Synthesis: Sketch

Given a problem specification P

- consider various "algorithm schemes" for A, e.g.
 
$$A[\langle \rangle] = c$$

$$\forall_x A[\langle x \rangle] = s[\langle x \rangle]$$

$$\forall_{x, \bar{y}} (A[\langle x, y, \bar{z} \rangle] = i[x, y, A[\langle y, \bar{z} \rangle]])$$
- and try to **prove (automatically)**  $\forall_x P[x, A[x]]$ .
- This proof will normally **fail** because nothing is known on the unspecified sub-algorithms c, s, i, ... in the algorithm scheme.
- From the temporary assumptions and goals in the failing proof situation **(automatically) generate such specifications for the unspecified sub-algorithms** c, s, i, ... that make the proof successful.

Now, apply the method recursively to the auxiliary functions.

This synthesis method reduces the problem of finding an algorithm that satisfies the specification to finding algorithms for (automatically generated) other specifications (that are hopefully easier to satisfy or for which algorithms are already known).

## Example: Synthesis of Merge-Sort [BB et al. 2003]

**Problem:** Synthesize "sorted" such that

$$\forall_x \text{is-sorted-version}[x, \text{sorted}[x]].$$

("Correctness Theorem")

**Knowledge on Problem:**

$$\forall_{x,y} \left( \text{is-sorted-version}[x, y] \Leftrightarrow \begin{array}{l} \text{is-sorted}[y] \\ \text{is-permuted-version}[x, y] \end{array} \right)$$

`is-sorted[⟨⟩]`

$\forall_x \text{is-sorted}[\langle x \rangle]$

$$\forall_{x,y,\bar{z}} \left( \text{is-sorted}[\langle x, y, \bar{z} \rangle] \Leftrightarrow \begin{array}{l} x \geq y \\ \text{is-sorted}[\langle y, \bar{z} \rangle] \end{array} \right)$$

etc.

## An Algorithm Scheme: Divide and Conquer

$$\forall_x \left( A[x] = \begin{cases} s[x] & \Leftarrow \text{is-trivial-tuple}[x] \\ m[A[l[x]], A[r[x]]] & \Leftarrow \text{otherwise} \end{cases} \right)$$

$s, m, l, r$  are unknowns.

We Now Start Proving the Correctness Theorem and Analyze the Failing Proof: see notebooks with failing proofs.

## Automated Invention of Sufficient Specifications for the Subalgorithms

A simple (but amazingly powerful) **rule** ( $m$  ... an unknown subalgorithm ):

Collect temporary assumptions  $T[x_0, \dots, A[l], \dots]$

and temporary goals  $G[x_0, \dots, m[A[l], \dots]]$

and produces specification

$$\forall_{x, \dots, y, \dots} \left( T[x, \dots, Y, \dots] \Rightarrow G[y, \dots, m[Y]] \right).$$

Details: see papers [BB 2003] and example.

## The Result of Applying Lazy Thinking in the Sorting Example

Lazy Thinking, [automatically](#) (in approx. 2 minutes on a laptop using the *Theorema* system), finds the following specifications for the sub-algorithms that provenly guarantee the correctness of the above algorithm (scheme):

$$\forall_x (\text{is-trivial-tuple}[x] \Rightarrow \mathbf{s}[x] = x)$$

$$\forall_{y,z} \left( \begin{array}{l} \text{is-sorted}[y] \\ \text{is-sorted}[z] \end{array} \Rightarrow \begin{array}{l} \text{is-sorted}[\mathbf{m}[y, z]] \\ \mathbf{m}[y, z] \approx (y \prec z) \end{array} \right)$$

$$\forall_x (\mathbf{l}[x] \prec \mathbf{r}[x] \approx x)$$

Note: the specifications generated are not only sufficient but natural !

The four proof notebooks generated automatically by Theorema that develop these specifications successively, are given in the appendix.

## What Do We Have Now?

- **Case A:** We find algorithms **S, M, L, R** in our knowledge base for which the properties specified above for **s, m, l, r** are already contained in the knowledge base or can be derived (proved) from the knowledge base.

In this case, we are done, i.e. we have synthesized a sorting algorithm.

- **Case B:** We do not find such algorithms **S, M, L, R** in our knowledge base.

In this case, we apply Lazy Thinking again in order to synthesize appropriate **s, m, l, r**

until we arrive at sub-sub-...-algorithms in our knowledge base (e.g. the basic operations of tuple theory like append, prepend etc.)

Case B can be avoided, if we proceed systematically bottom-up ("complete theory exploration" in layers).

## Example: Synthesis of Insertion-Sort

Synthesize A such that

$$\forall_{\mathbf{x}} \text{is-sorted-version}[\mathbf{x}, \mathbf{A}[\mathbf{x}]].$$

Algorithm Scheme: "simple recursion"

$$\begin{aligned} \mathbf{A}[\langle \rangle] &= \mathbf{c} \\ \forall_{\mathbf{x}} \mathbf{A}[\langle \mathbf{x} \rangle] &= \mathbf{s}[\langle \mathbf{x} \rangle] \\ \forall_{\mathbf{x}, \bar{\mathbf{y}}} (\mathbf{A}[\langle \mathbf{x}, \bar{\mathbf{y}} \rangle] &= \mathbf{i}[\mathbf{x}, \mathbf{A}[\langle \bar{\mathbf{y}} \rangle]]) \end{aligned}$$

Lazy Thinking, [automatically](#) (in approx. 2 minutes on a laptop using the *Theorema* system), finds the following specifications for the auxiliary functions

$$\begin{aligned} \mathbf{c} &= \langle \rangle \\ \forall_{\mathbf{x}} (\mathbf{s}[\langle \mathbf{x} \rangle] &= \langle \mathbf{x} \rangle) \\ \forall_{\mathbf{x}, \bar{\mathbf{y}}} (\text{is-sorted}[\langle \bar{\mathbf{y}} \rangle] &\Rightarrow \mathbf{i}[\mathbf{x}, \langle \bar{\mathbf{y}} \rangle] \approx (\mathbf{x} \sim \langle \bar{\mathbf{y}} \rangle)) \end{aligned}$$

## How Far Can We Go With the Method ?

Can we automatically synthesize algorithms for [non-trivial problems](#)? What is "non-trivial"?

Example of a non-trivial problem: construction of Gröbner bases.

## The Problem of Constructing Gröbner Bases is Non-trivial

[Dozens of fundamental problems](#) in algebraic geometry, invariant theory, optimization, graph theory, coding theory, cryptography, statistics, symbolic summation, symbolic solution of differential equations, ... [can be reduced](#) to the construction of Gröbner bases. (Approx. 1000 papers on the application of the Gröbner bases method, see B. Buchberger, A. Zapletal, Papers Data Base on Gröbner Bases Theory, 2006, [www.ricam.oeaw.ac.at/Groebner-Bases-Bibliography/index.php](http://www.ricam.oeaw.ac.at/Groebner-Bases-Bibliography/index.php).)

Some of these problems were [open for decades](#).

[Main algorithmic idea](#) of Gröbner bases theory: The "S-polynomials" together with the S-polynomial theorem.

[ Buchberger 1965] and [ Buchberger 1970].

Hence, question: Can Lazy Thinking [automatically invent the notion of S-polynomial](#) and automatically deliver the [algorithm](#) based on it together with its correctness [proof](#)?

## The S-Polynomials

S-polynomial[ $xy - 2yz - z - 1, y^2 - x^2z + xz + 2$ ] =  
 $xz(xy - 2yz - z - 1) + y(y^2 - x^2z + xz + 2)$

$y(2 + y^2 + xz - x^2z) + xz(-1 + xy - z - 2yz)$

$xz(xy - 2yz - z - 1) + y(y^2 - x^2z + xz + 2)$  // Expand

$2y + y^3 - xz + xyz - xz^2 - 2xyz^2$

```

S-polynomial[g1, g2] =
  form the
least-common-multiple[leading-power-product[g1], leading-power-product[g2]]
  and then ...

```

## Algorithm-Supported Mathematical Theory Exploration

### An "Algorithm" for Algorithm Synthesis

### Synthesis of a Gröbner Bases Algorithm

### Conclusion

## The Problem of Constructing Gröbner Bases

Find algorithm **Gb** such that

$$\forall \text{ is-finite}[F] \left( \begin{array}{l} \text{is-finite}[ \text{Gb}[F] ] \\ \text{is-Gröbner-basis}[ \text{Gb}[F] ] \\ \text{ideal}[F] = \text{ideal}[ \text{Gb}[F] ]. \end{array} \right)$$

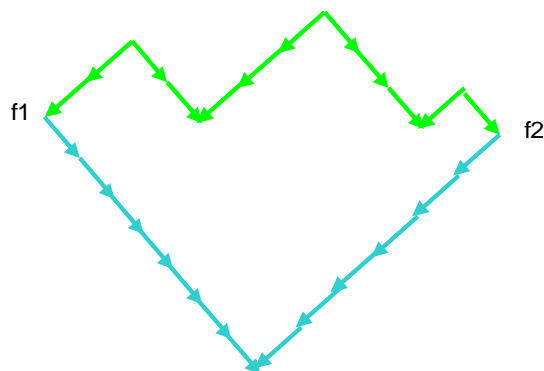
Definitions [BB 1965] :

$$\text{is-Gröbner-basis}[G] \Leftrightarrow \text{is-confluent}[ \rightarrow_G ].$$

$\rightarrow_G$  ... a division step.

## Confluence of Division $\rightarrow_G$

`is-confluent`[  $\rightarrow$  ] :  $\Leftrightarrow \forall_{f1, f2} (f1 \leftrightarrow^* f2 \Rightarrow f1 \downarrow^* f2)$



## Knowledge on the Concepts Involved

$h1 \rightarrow_G h2 \Rightarrow p . h1 \rightarrow_G p . h2$

etc.

## Algorithm Scheme "Critical Pair / Completion"

```

A[F] = A[F, pairs[F]]
A[F, ⟨⟩] = F

A[F, ⟨⟨g1, g2⟩, p̄⟩] =
  where [f = lc[g1, g2], h1 = trd[rd[f, g1], F], h2 = trd[rd[f, g2], F],
        { A[F, ⟨p̄⟩]                                     ← h1 = h2
          A[F - df[h1, h2], ⟨p̄⟩] ≃ ⟨⟨Fk, df[h1, h2]⟩k=1,...,|F|⟩ ] ← otherwise ]

```

This scheme can be tried in any domain, in which we have a reduction operation  $rd$  that depends on sets  $F$  of objects and a Noetherian relation  $>$  which interacts with  $rd$  in the following natural way:

$$\forall_{f, g} (f > rd[f, g]).$$

## The Essential Problem

The problem of synthesizing a Gröbner bases algorithm can now be also stated by asking whether, starting with the proof of

$$\forall_F \left( \begin{array}{l} \text{is-finite}[A[F]] \\ \text{is-Gröbner-basis}[A[F]] \\ \text{ideal}[F] = \text{ideal}[A[F]] \end{array} \right),$$

we can *automatically produce the idea* that

$$lc[g1, g2] = lcm[lp[g1], lp[g2]]$$

and

$$df[h1, h2] = h1 - h2$$

and prove that the idea is correct.

## Now Start the (Automated) Correctness Proof

With current theorem proving technology, in the *Theorema* system (and other provers), the proof attempt could be done automatically. ([Ongoing PhD thesis of A. Craciun.](#))

## Details

It should be clear that, if the algorithm terminates, the final result is a finite set (of polynomials)  $G$  that has the property

$$\forall_{g_1, g_2 \in G} \left( \text{where } [f = \text{lc}[g_1, g_2], h_1 = \text{trd}[\text{rd}[f, g_1], G], \right. \\ \left. h_2 = \text{trd}[\text{rd}[f, g_2], G], \bigvee \left\{ \begin{array}{l} h_1 = h_2 \\ \text{df}[h_1, h_2] \in G \end{array} \right\} \right) \cdot$$

We now try to prove that, if  $G$  has this property, then

```
is-finite[G],
ideal[F] = ideal[G],
is-Gröbner-basis[G],
  i.e. is-Church-Rosser[→G].
```

Here, we only deal with the third, most important, property.

## Using Available Knowledge

Using Newman's lemma and some elementary properties it can be shown that it is sufficient to prove

$$\text{is-Church-Rosser}[\rightarrow_G] \Leftrightarrow \forall_p \forall_{f_1, f_2} \left( \left( \left\{ \begin{array}{l} p \rightarrow_G f_1 \\ p \rightarrow_G f_2 \end{array} \right\} \right) \Rightarrow f_1 \downarrow_G^* f_2 \right).$$

## The (Automated) Proof Attempt

Let now the power product  $p$  and the polynomials  $f_1, f_2$  be arbitrary but fixed and assume

$$\begin{cases} p \rightarrow_G f_1 \\ p \rightarrow_G f_2. \end{cases}$$

We have to find a polynomial  $g$  such that

$$\begin{aligned} f_1 &\rightarrow_G^* g, \\ f_2 &\rightarrow_G^* g. \end{aligned}$$

From the assumption we know that there exist polynomials  $g_1$  and  $g_2$  in  $G$  such that

$$\begin{aligned} lp[g_1] &| p, \\ f_1 &= rd[p, g_1], \\ lp[g_2] &| p, \\ f_2 &= rd[p, g_2]. \end{aligned}$$

From the final situation in the algorithm scheme we know that for these  $g_1$  and  $g_2$

$$\bigvee \begin{cases} h_1 = h_2 \\ df[h_1, h_2] \in G, \end{cases}$$

where

$$\begin{aligned} h_1 &:= trd[f_1', G], f_1' := rd[lc[g_1, g_2], g_1], \\ h_2 &:= trd[f_2', G], f_2' := rd[lc[g_1, g_2], g_2]. \end{aligned}$$

## Case $h_1=h_2$

$$\begin{aligned} lc[g_1, g_2] \rightarrow_{g_1} rd[lc[g_1, g_2], g_1] \rightarrow_G^* trd[rd[lc[g_1, g_2], g_1], G] = \\ trd[rd[lc[g_1, g_2], g_2], G] \leftarrow_G^* rd[lc[g_1, g_2], g_2] \leftarrow_{g_2} lc[g_1, g_2]. \end{aligned}$$

(Note that here we used the requirements  $rd[lc[g_1, g_2], g_1] < lc[g_1, g_2]$  and  $rd[lc[g_1, g_2], g_2] < lc[g_1, g_2]$ .)

Hence, by elementary properties of polynomial reduction,

$$\bigvee_{a, q} ( a q lc[g_1, g_2] \rightarrow_{g_1} a q rd[lc[g_1, g_2], g_1] \rightarrow_G^* a q trd[rd[lc[g_1, g_2], g_1], G] = \\ a q trd[rd[lc[g_1, g_2], g_2], G] \leftarrow_G^* a q rd[lc[g_1, g_2], g_2] \leftarrow_{g_2} a q lc[g_1, g_2] ).$$

Now we are stuck in the proof.

## Now Use the Specification Generation Algorithm

Using the above specification generation rule, we see that we could proceed successfully with the proof if  $lc[g1,g2]$  satisfied the following requirement

$$\forall_{p,g1,g2} \left( \left( \left\{ \begin{array}{l} lp[g1] | p \\ lp[g2] | p \end{array} \right\} \right) \Rightarrow \left( \exists_{a,q} (p = a \ q \ lc[g1, g2]) \right) \right), \quad (lc \text{ requirement})$$

With such an  $lc$ , we then would have

$$p \rightarrow_{g1} rd[p, g1] = a \ q \ rd[lc[g1, g2], g1] \rightarrow_G^* a \ q \ trd[rd[lc[g1, g2], g1], G] = \\ a \ q \ trd[rd[lc[g1, g2], g2], G] \leftarrow_G^* a \ q \ rd[lc[g1, g2], g2] = rd[p, g2] \leftarrow_{g2} p$$

and, hence,

$$f1 \rightarrow_G^* a \ q \ trd[rd[lc[g1, g2], g1], G],$$

$$f2 \rightarrow_G^* a \ q \ trd[rd[lc[g1, g2], g1], G],$$

i.e. we would have found a suitable  $g$ .

## Summarize the (Automatically Generated) Specifications of the Subalgorithm $lc$

( $lc$  requirement), which also could be written in the form:

$$\forall_{p,g1,g2} \left( \left( \left\{ \begin{array}{l} lp[g1] | p \\ lp[g2] | p \end{array} \right\} \right) \Rightarrow (lc[g1, g2] | p) \right),$$

and

$$lp[g1] | lc[g1, g2], \\ lp[g2] | lc[g1, g2],$$

which is a consequence of

$$\begin{aligned} \text{lc}[g_1, g_2] &\rightarrow_{g_1} \text{rd}[\text{lc}[g_1, g_2], g_1], \\ \text{lc}[g_1, g_2] &\rightarrow_{g_2} \text{rd}[\text{lc}[g_1, g_2], g_2]. \end{aligned}$$

## Summarize Again

For synthesizing an algorithm for the Gröbner bases problem it suffices to find an lc satisfying

$$\forall_{p, g_1, g_2} \left( \left( \left\{ \begin{array}{l} \text{lp}[g_1] \mid p \\ \text{lp}[g_2] \mid p \end{array} \right\} \right) \Rightarrow (\text{lc}[g_1, g_2] \mid p) \right),$$

and

$$\begin{aligned} \text{lp}[g_1] \mid \text{lc}[g_1, g_2], \\ \text{lp}[g_2] \mid \text{lc}[g_1, g_2]. \end{aligned}$$

This problem can be solved by any high-school student (or university professor)! No knowledge on Gröbner bases theory necessary!

## A Suitable lc

$$\text{lc}[g_1, g_2] = \text{lcm}[\text{lp}[g_1], \text{lp}[g_2]]$$

is a suitable function that satisfies the above requirements.

**Eureka! The crucial function lc (the "critical pair" function) in the critical pair / completion algorithm scheme has been synthesized automatically!**

## Case $h_1 \neq h_2$

In this case,  $\text{df}[h_1, h_2] \in G$ :

In this part of the proof (which is much easier) we are basically stuck right at the beginning. By the requirement generation algorithm we obtain the following requirement for df:

$$\forall_{h1, h2} (h1 \downarrow_{\{df[h1, h2]\}} * h2) \quad (\text{df requirement}).$$

## Looking to the Knowledge Base for a Suitable df

(Looking to the knowledge base of elementary properties of polynomial reduction, it is now easy to find a function df that satisfies (df requirement), namely

$$df[h1, h2] = h1 - h2,$$

because, in fact,

$$\forall_{f, g} (f \downarrow_{\{f-g\}} * g).$$

Eureka! The function  $df$  (the "completion" function) in the critical pair / completion algorithm scheme has been "automatically" synthesized!

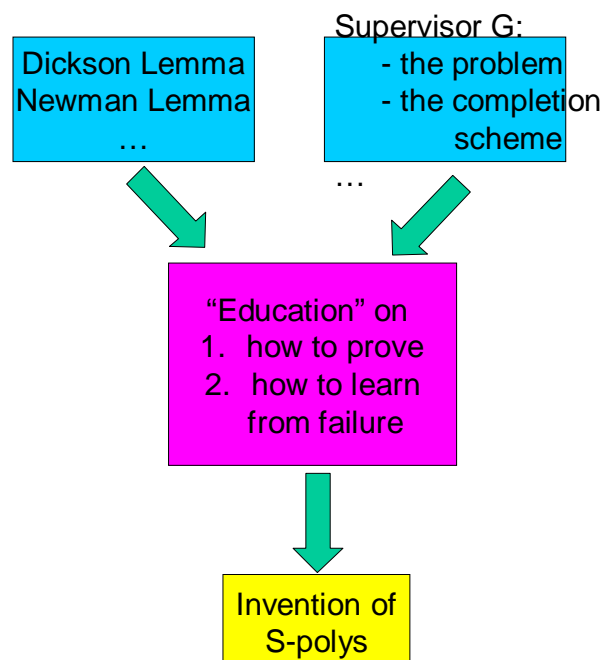
## An "Algorithm" for Algorithm Synthesis

## Synthesis of a Gröbner Bases Algorithm

## Conclusion

---

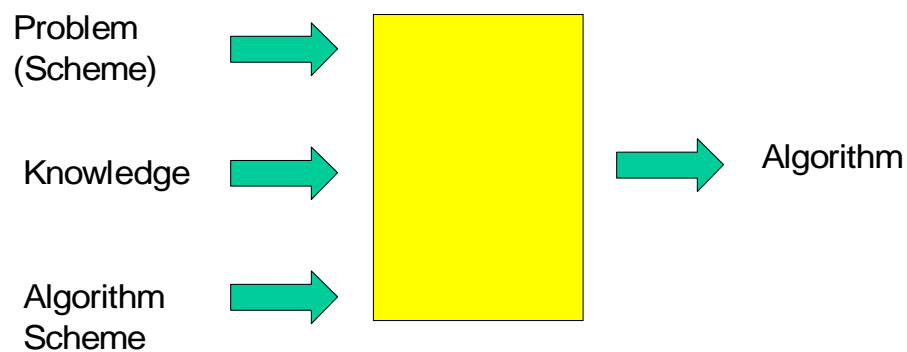
**A way of looking at it ("what would have happened if ..."):**



Pairs idea? (The CPC algorithm scheme did not really exist at that time.)

$$\begin{aligned}
 & A[\mathbb{F}, \langle \langle g1, g2 \rangle, \bar{p} \rangle] = \\
 & \text{where } [f = \text{lc}[g1, g2], h1 = \text{trd}[\text{rd}[f, g1], \mathbb{F}], h2 = \text{trd}[\text{rd}[f, g2], \mathbb{F}], \\
 & \left\{ \begin{array}{l} A[\mathbb{F}, \langle \bar{p} \rangle] \\ A[\mathbb{F} - \text{df}[h1, h2], \langle \bar{p} \rangle \approx \langle \langle \mathbb{F}_k, \text{df}[h1, h2] \rangle_{k=1, \dots, |\mathbb{F}|} \rangle] \end{array} \right\} \begin{array}{l} \leftarrow h1 = h2 \\ \leftarrow \text{otherwise} \end{array} ]
 \end{aligned}$$

## Research Topics



- **Libraries** of algorithm **schemes**.

More generally, libraries of formulae schemes for definitions, propositions, problems, and algorithms.

- **Case studies** of problem (schemes), knowledge, algorithm schemes and how they produce algorithms.
- How well are **current reasoning systems suited** for supporting this approach to algorithm (definition, theorem, problem, ...) synthesis?
- Improved **algorithms for generating problem specifications** from failing proofs.
- Automated synthesis of **refinements of the S-polynomials-based Gröbner bases algorithm**.
- Automated synthesis of **Gröbner bases algorithms not based on S-polynomials** (do such algorithms exist?)

---

## References

### ■ On Gröbner Bases

[Buchberger 1965]

B. Buchberger. Ein Algorithmus zum Auffinden der Basiselemente des Restklassenrings nach einem nulldimensionalen Polynomideal. PhD Thesis, Mathematical Institute, University of Innsbruck, December 1965. (English Translation by Michael P. Abramson: "An Algorithm for Finding the Basis Element of the Residue Class Ring of a Zero Dimensional Polynomial Ideal", Journal of Symbolic Computation, Vol. 41 (Numbers 3-4), March / April 2006, pp. 475 - 511).

[Buchberger 1970]

B. Buchberger. Ein algorithmisches Kriterium für die Lösbarkeit eines algebraischen Gleichungssystems (An Algorithmical Criterion for the Solvability of Algebraic Systems of Equations). Aequationes mathematicae 4/3, 1970, pp. 374-383. (English translation in: [Buchberger, Winkler 1998], pp. 535 -545.) Published version of the PhD Thesis of B. Buchberger, University of Innsbruck, Austria, 1965.

[Buchberger 1998]

B. Buchberger. Introduction to Gröbner Bases. In: [Buchberger, Winkler 1998], pp.3-31.

[Buchberger, Winkler, 1998]

B. Buchberger, F. Winkler (eds.). Gröbner Bases and Applications, Proceedings of the International Conference "33 Years of Gröbner Bases", 1998, RISC, Austria, London Mathematical Society Lecture Note Series, Vol. 251, Cambridge University Press, 1998.

[Becker, Weispfenning 1993]

T. Becker, V. Weispfenning. Gröbner Bases: A Computational Approach to Commutative Algebra, Springer, New York, 1993.

### ■ On Mathematical Knowledge Management

B. Buchberger, G. Gonnet, M. Hazewinkel (eds.)

Mathematical Knowledge Management.

Special Issue of Annals of Mathematics and Artificial Intelligence, Vol. 38, No. 1-3, May 2003, Kluwer Academic Publisher, 232 pages.

A.Asperti, B. Buchberger, J.H.Davenport (eds.)

Mathematical Knowledge Management.

Proceedings of the Second International Conference on Mathematical Knowledge Management (MKM 2003), Bertinoro, Italy, Feb.16-18, 2003, Lecture Notes in Computer Science, Vol. 2594, Springer, Berlin-Heidelberg-NewYork, 2003, 223 pages.

A.Asperti, G.Bancerek, A.Trybulec (eds.).

Proceedings of the Third International Conference on Mathematical Knowledge Management, MKM 2004, Bialowieza, Poland, September 19-21, 2004, Lecture Notes in Computer Science, Vol. 3119, Springer, Berlin-Heidelberg-NewYork, 2004

### ■ On Theorema

[Buchberger et al. 2000]

B. Buchberger, C. Dupre, T. Jebelean, F. Kriftner, K. Nakagawa, D. Vasaru, W. Windsteiger. The Theorema Project: A Progress Report. In: M. Kerber and M. Kohlhase (eds.), Symbolic Computation and Automated Reasoning (Proceedings of CALCULEMUS 2000, Symposium on the Integration of Symbolic Computation and Mechanized Reasoning, August 6-7, 2000, St. Andrews, Scotland), A.K. Peters, Natick, Massachusetts, ISBN 1-56881-145-4, pp. 98-113.

### ■ On Theory Exploration and Algorithm Synthesis

[Buchberger 2000]

B. Buchberger. Theory Exploration with *Theorema*.

Analele Universitatii Din Timisoara, Ser. Matematica-Informatica, Vol. XXXVIII, Fasc.2, 2000, (Proceedings of SYNASC 2000, 2nd International Workshop on Symbolic and Numeric Algorithms in Scientific Computing, Oct. 4-6, 2000, Timisoara, Rumania, T. Jebelean, V. Negru, A. Popovici eds.), ISSN 1124-970X, pp. 9-32.

[Buchberger 2003]

B. Buchberger. Algorithm Invention and Verification by Lazy Thinking.

In: D. Petcu, V. Negru, D. Zaharie, T. Jebelean (eds), Proceedings of SYNASC 2003 (Symbolic and Numeric Algorithms for Scientific Computing, Timisoara, Romania, October 1-4, 2003), Mirton Publishing, ISBN 973-661-104-3, pp. 2-26.

[Buchberger, Craciun 2003]

B. Buchberger, A. Craciun. Algorithm Synthesis by Lazy Thinking: Examples and Implementation in Theorema. in: Fairouz Kamareddine (ed.), Proc. of the Mathematical Knowledge Management Workshop, Edinburgh, Nov. 25, 2003, Electronic Notes on Theoretical Computer Science, volume dedicated to the MKM 03 Symposium, Elsevier, ISBN 044451290X, to appear.

[Buchberger 2005]

B. Buchberger.

Towards the Automated Synthesis of a Gröbner Bases Algorithm.

RACSAM (Review of the Royal Spanish Academy of Science), Vol. 98/1, pp. 65-75, 2005.

## Appendix: The Proof Notebooks Automatically Generated by Theorema for the Synthesis of the Merge-Sort Algorithm

### Notebook 1:

Prove:

(Theorem (correctness of sort))  $\forall_{\text{is-tuple}[\mathbf{x}]} \text{is-sorted-version}[\mathbf{x}, \text{sorted}[\mathbf{x}]],$

under the assumptions:

(Definition (is sorted): 1)  $\text{is-sorted}[\langle \rangle],$

(Definition (is sorted): 2)  $\forall_{\mathbf{x}} \text{is-sorted}[\langle \mathbf{x} \rangle],$

(Definition (is sorted): 3)  $\forall_{\mathbf{x}, \mathbf{y}, \bar{\mathbf{z}}} (\text{is-sorted}[\langle \mathbf{x}, \mathbf{y}, \bar{\mathbf{z}} \rangle] \Leftrightarrow \mathbf{x} \geq \mathbf{y} \wedge \text{is-sorted}[\langle \mathbf{y}, \bar{\mathbf{z}} \rangle]),$

(Definition (is permuted version): 1)  $\langle \rangle \approx \langle \rangle,$

(Definition (is permuted version): 2)  $\forall_{\mathbf{y}, \bar{\mathbf{y}}} (\langle \rangle \neq \langle \mathbf{y}, \bar{\mathbf{y}} \rangle),$

(Definition (is permuted version): 3)  $\forall_{\mathbf{x}, \bar{\mathbf{x}}, \bar{\mathbf{y}}} (\langle \bar{\mathbf{y}} \rangle \approx \langle \mathbf{x}, \bar{\mathbf{x}} \rangle \Leftrightarrow \mathbf{x} \in \langle \bar{\mathbf{y}} \rangle \wedge \text{dfo}[\mathbf{x}, \langle \bar{\mathbf{y}} \rangle] \approx \langle \bar{\mathbf{x}} \rangle),$

(Definition (is sorted version))

$\forall_{\substack{\mathbf{x}, \mathbf{y} \\ \text{is-tuple}[\mathbf{x}]}} (\text{is-sorted-version}[\mathbf{x}, \mathbf{y}] \Leftrightarrow \text{is-tuple}[\mathbf{y}] \wedge \mathbf{x} \approx \mathbf{y} \wedge \text{is-sorted}[\mathbf{y}]),$

(Proposition (is tuple tuple))  $\forall_{\bar{\mathbf{x}}} \text{is-tuple}[\langle \bar{\mathbf{x}} \rangle],$

(Definition (prepend):  $\neg$ )  $\forall_{\mathbf{x}, \bar{\mathbf{y}}} (\mathbf{x} - \langle \bar{\mathbf{y}} \rangle = \langle \mathbf{x}, \bar{\mathbf{y}} \rangle),$

(Proposition (singleton tuple is singleton tuple))  $\forall_{\mathbf{x}} \text{is-singleton-tuple}[\langle \mathbf{x} \rangle],$

(Definition (is trivial tuple))

$\forall_{\text{is-tuple}[\mathbf{x}]} (\text{is-trivial-tuple}[\mathbf{x}] \Leftrightarrow \text{is-empty-tuple}[\mathbf{x}] \vee \text{is-singleton-tuple}[\mathbf{x}]),$

(Definition (is element): 1)  $\forall_{\mathbf{x}} (\mathbf{x} \notin \langle \rangle),$

(Definition (is element): 2)  $\forall_{\mathbf{x}, \mathbf{y}, \bar{\mathbf{y}}} (\mathbf{x} \in \langle \mathbf{y}, \bar{\mathbf{y}} \rangle \Leftrightarrow (\mathbf{x} = \mathbf{y}) \vee \mathbf{x} \in \langle \bar{\mathbf{y}} \rangle),$

(Definition (deletion of the first occurrence): 1)  $\forall_{\mathbf{a}} (\text{dfo}[\mathbf{a}, \langle \rangle] = \langle \rangle),$

(Definition (deletion of the first occurrence): 2)

$\forall_{\mathbf{a}, \mathbf{x}, \bar{\mathbf{x}}} (\text{dfo}[\mathbf{a}, \langle \mathbf{x}, \bar{\mathbf{x}} \rangle] = \|\langle \bar{\mathbf{x}} \rangle \Leftarrow \mathbf{x} = \mathbf{a}, \mathbf{x} - \text{dfo}[\mathbf{a}, \langle \bar{\mathbf{x}} \rangle] \Leftarrow \text{otherwise}\|),$

(Definition (is longer than): 1)  $\forall_{\bar{y}} (\langle \rangle \not> \langle \bar{y} \rangle)$ ,

(Definition (is longer than): 2)  $\forall_{\mathbf{x}, \bar{\mathbf{x}}} (\langle \mathbf{x}, \bar{\mathbf{x}} \rangle > \langle \rangle)$ ,

(Definition (is longer than): 3)  $\forall_{\mathbf{x}, \bar{\mathbf{x}}, \mathbf{y}, \bar{\mathbf{y}}} (\langle \mathbf{x}, \bar{\mathbf{x}} \rangle > \langle \mathbf{y}, \bar{\mathbf{y}} \rangle \Leftrightarrow \langle \bar{\mathbf{x}} \rangle > \langle \bar{\mathbf{y}} \rangle)$ ,

(Proposition (trivial tuples are sorted))  $\forall_{\bar{\mathbf{x}}} \text{is-trivial-tuple}[\langle \bar{\mathbf{x}} \rangle] \Rightarrow \text{is-sorted}[\langle \bar{\mathbf{x}} \rangle]$ ,

(Proposition (only trivial tuple permuted version of itself))  $\forall_{\bar{\mathbf{x}}, \mathbf{y}} (\text{is-trivial-tuple}[\langle \bar{\mathbf{x}} \rangle] \wedge \langle \mathbf{y} \rangle = \langle \bar{\mathbf{x}} \rangle \Rightarrow \mathbf{y} \approx \langle \bar{\mathbf{x}} \rangle)$ ,

(Proposition (reflexivity of permuted version))  $\forall_{\bar{\mathbf{x}}} (\langle \bar{\mathbf{x}} \rangle \approx \langle \bar{\mathbf{x}} \rangle)$ ,

(Algorithm (sorted))

$\forall_{\mathbf{x}} \text{is-tuple}[\mathbf{x}] \Rightarrow (\text{sorted}[\mathbf{x}] = \|\text{special}[\mathbf{x}] \Leftarrow \text{is-trivial-tuple}[\mathbf{x}], \text{merged}[\text{sorted}[\text{left-split}[\mathbf{x}]], \text{sorted}[\text{right-split}[\mathbf{x}]]] \Leftarrow \text{otherwise}\|)$ ,

(Lemma (closure of special))  $\forall_{\mathbf{x}} \text{is-tuple}[\mathbf{x}] \wedge \text{is-trivial-tuple}[\mathbf{x}] \Rightarrow \text{is-tuple}[\text{special}[\mathbf{x}]]$ ,

(Lemma (splits are tuples): 1)  $\forall_{\mathbf{x}} \text{is-tuple}[\mathbf{x}] \wedge \text{is-trivial-tuple}[\mathbf{x}] \Rightarrow \text{is-tuple}[\text{left-split}[\mathbf{x}]]$ ,

(Lemma (splits are tuples): 2)  $\forall_{\mathbf{x}} \text{is-tuple}[\mathbf{x}] \wedge \text{is-trivial-tuple}[\mathbf{x}] \Rightarrow \text{is-tuple}[\text{right-split}[\mathbf{x}]]$ ,

(Lemma (splits are shorter): 1)  $\forall_{\mathbf{x}} (\text{is-tuple}[\mathbf{x}] \wedge \neg \text{is-trivial-tuple}[\mathbf{x}] \Rightarrow \langle \mathbf{x} \rangle > \text{left-split}[\mathbf{x}])$ ,

(Lemma (splits are shorter): 2)  $\forall_{\mathbf{x}} (\text{is-tuple}[\mathbf{x}] \wedge \neg \text{is-trivial-tuple}[\mathbf{x}] \Rightarrow \langle \mathbf{x} \rangle > \text{right-split}[\mathbf{x}])$ ,

(Lemma (closure of merge))  $\forall_{\substack{\mathbf{x} \\ \mathbf{y}}} \text{is-tuple}[\mathbf{x}] \wedge \text{is-tuple}[\mathbf{y}] \Rightarrow \text{is-tuple}[\text{merged}[\mathbf{x}, \mathbf{y}]]$ .

We try to prove (Theorem (correctness of sort)) by well-founded induction on  $\mathbf{X}$ .

Well-founded induction:

Assume:

(1)  $\text{is-tuple}[\langle \bar{X}_0 \rangle]$ .

Well-Founded Induction Hypothesis:

(2)  $\forall_{\substack{\mathbf{x} \\ \mathbf{x}_1}} \text{is-tuple}[\mathbf{x}] \wedge \langle \bar{X}_0 \rangle > \mathbf{x}_1 \Rightarrow \text{is-sorted-version}[\mathbf{x}_1, \text{sorted}[\mathbf{x}_1]]$

We have to show:

(3)  $\text{is-sorted-version}[\langle \bar{X}_0 \rangle, \text{sorted}[\langle \bar{X}_0 \rangle]]$ .

We try to prove (3) by case distinction using (Algorithm (sorted)). However, the proof fails in at least one of the cases.

Case 1:

(4)  $\text{is-trivial-tuple}[\langle \bar{X}_0 \rangle]$ .

Hence, we have to prove

(5)  $\text{is-sorted-version}[\langle \overline{X_0} \rangle, \text{special}[\langle \overline{X_0} \rangle]]$ .

Formula (4), by (Proposition (only trivial tuple permuted version of itself)), implies:

(10)  $\forall_{\mathbf{Y}} ((\mathbf{Y} = \langle \overline{X_0} \rangle) \Rightarrow \mathbf{Y} \approx \langle \overline{X_0} \rangle)$ .

Formula (1), by ([Lemma \(Closure of Special\)](#)), implies:

(12)  $\text{is-tuple}[\text{special}[\langle \overline{X_0} \rangle]]$ .

By (1), Formula (5), using (Definition (is sorted version)), is implied by:

(13)  $\text{is-tuple}[\text{special}[\langle \overline{X_0} \rangle]] \wedge \text{special}[\langle \overline{X_0} \rangle] \approx \langle \overline{X_0} \rangle \wedge \text{is-sorted}[\text{special}[\langle \overline{X_0} \rangle]]$ .

Not all the conjunctive parts of (13) can be proved.

Proof of (13.1)  $\text{is-tuple}[\text{special}[\langle \overline{X_0} \rangle]]$ :

Formula (13.1) is true because it is identical to (12).

Proof of (13.2)  $\text{special}[\langle \overline{X_0} \rangle] \approx \langle \overline{X_0} \rangle$ :

Formula (13.3), using (10), is implied by:

(14)  $\text{special}[\langle \overline{X_0} \rangle] = \langle \overline{X_0} \rangle$ .

The proof of (14) fails. (The prover "QR" was unable to transform the proof situation.)

Proof of (13.4)  $\text{is-sorted}[\text{special}[\langle \overline{X_0} \rangle]]$ :

Pending proof of (13.4).

Case 2:

(6)  $\neg \text{is-trivial-tuple}[\langle \overline{X_0} \rangle]$ .

Hence, we have to prove

(8)  $\text{is-sorted-version}[\langle \overline{X_0} \rangle, \text{merged}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]], \text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]]]$

Pending proof of (8).

□

## Notebook 2

Comment on Notebook 2: Note that in the knowledge base (i.e. the formulae listed under "assumptions"), the specification (Lemma (conjecture15): conjecture15) is now contained, which describes the specification automatically generated (from the failing proof in Notebook 1) for the function 'special'. The proof then proceeds as in Notebook 1 but succeeds to get over the point at which the first proof was stuck.

Prove:

(Theorem (correctness of sort))  $\forall_{\text{is-tuple}[\mathbf{x}]} \text{is-sorted-version}[\mathbf{x}, \text{sorted}[\mathbf{x}]]$ ,

under the assumptions:

(Definition (is sorted): 1)  $\text{is-sorted}[\langle \rangle]$ ,

(Definition (is sorted): 2)  $\forall_{\mathbf{x}} \text{is-sorted}[\langle \mathbf{x} \rangle]$ ,

(Definition (is sorted): 3)  $\forall_{\mathbf{x}, \mathbf{y}, \bar{\mathbf{z}}} (\text{is-sorted}[\langle \mathbf{x}, \mathbf{y}, \bar{\mathbf{z}} \rangle] \Leftrightarrow \mathbf{x} \geq \mathbf{y} \wedge \text{is-sorted}[\langle \mathbf{y}, \bar{\mathbf{z}} \rangle])$ ,

.... and all the formulae in the assumptions of Notebook 1,

(Lemma (closure of merge))  $\forall_{\substack{\text{is-tuple}[\mathbf{x}] \\ \text{is-tuple}[\mathbf{y}]}} \text{is-tuple}[\text{merged}[\mathbf{x}, \mathbf{y}]]$ ,

(Lemma (conjecture15): conjecture15)  $\forall_{\substack{\mathbf{x1} \\ \text{is-tuple}[\mathbf{x1}]}} (\text{is-trivial-tuple}[\mathbf{x1}] \Rightarrow (\text{special}[\mathbf{x1}] = \mathbf{x1}))$ .

We try to prove (Theorem (correctness of sort)) by applying several proof methods for sequences.

We try to prove (Theorem (correctness of sort)) by well-founded induction on  $\mathbf{X}$ .

Well-founded induction:

Assume:

(1)  $\text{is-tuple}[\langle \overline{\mathbf{X}_0} \rangle]$ .

Well-Founded Induction Hypothesis:

(2)  $\forall_{\text{is-tuple}[\mathbf{x2}]} (\langle \overline{\mathbf{X}_0} \rangle > \mathbf{x2} \Rightarrow \text{is-sorted-version}[\mathbf{x2}, \text{sorted}[\mathbf{x2}]])$

We have to show:

(3)  $\text{is-sorted-version}[\langle \overline{\mathbf{X}_0} \rangle, \text{sorted}[\langle \overline{\mathbf{X}_0} \rangle]]$ .

We try to prove (3) by case distinction using (Algorithm (sorted)). However, the proof fails in at least one of the cases.

Case 1:

(4)  $\text{is-trivial-tuple}[\langle \overline{\mathbf{X}_0} \rangle]$ .

Hence, we have to prove

(5)  $\text{is-sorted-version}[\langle \overline{\mathbf{X}_0} \rangle, \text{special}[\langle \overline{\mathbf{X}_0} \rangle]]$ .

Formula (4), by (Proposition (trivial tuples are sorted)), implies:

(9)  $\text{is-sorted}[\langle \overline{\mathbf{X}_0} \rangle]$ .

Formula (4), by (Proposition (only trivial tuple permuted version of itself)), implies:

(10)  $\forall_{\mathbf{Y}} ((\mathbf{Y} = \langle \overline{\mathbf{X}_0} \rangle) \Rightarrow \mathbf{Y} \approx \langle \overline{\mathbf{X}_0} \rangle)$ .

Formula (1) and (4), by (Lemma (closure of special)), implies:

(11)  $\text{is-tuple}[\text{special}[\langle \overline{\mathbf{X}_0} \rangle]]$ .

Formula (1) and (4), by (Lemma (conjecture15): conjecture15), implies:

(13)  $\text{special}[\langle \overline{\mathbf{X}_0} \rangle] = \langle \overline{\mathbf{X}_0} \rangle$ .

Formula (5), using (13), is implied by:

(21)  $\text{is-sorted-version}[\langle \overline{\mathbf{X}_0} \rangle, \langle \overline{\mathbf{X}_0} \rangle]$ .

Formula (21), using (Definition (is sorted version)), is implied by:

(22)  $\text{is-tuple}[\langle \overline{\mathbf{X}_0} \rangle] \wedge \langle \overline{\mathbf{X}_0} \rangle \approx \langle \overline{\mathbf{X}_0} \rangle \wedge \text{is-sorted}[\langle \overline{\mathbf{X}_0} \rangle]$ .

We prove the individual conjunctive parts of (22):

Proof of (22.1)  $\text{is-tuple}[\langle \overline{X_0} \rangle]$ :

Formula (22.1) is true because it is identical to (1).

Proof of (22.2)  $\langle \overline{X_0} \rangle \approx \langle \overline{X_0} \rangle$ :

Formula (22.2) is true by (10).

Proof of (22.3)  $\text{is-sorted}[\langle \overline{X_0} \rangle]$ :

Formula (22.3) is true because it is identical to (9).

Case 2:

(6)  $\neg \text{is-trivial-tuple}[\langle \overline{X_0} \rangle]$ .

Hence, we have to prove

(8)  $\text{is-sorted-version}[\langle \overline{X_0} \rangle, \text{merged}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]], \text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]]]$

From (6), by (2), (Lemma (splits are tuples): 1), (Lemma (splits are tuples): 2), (Lemma (splits are shorter): 1), (Lemma (splits are shorter): 1) and (Lemma (splits are shorter): 2), we obtain:

(23)  $\text{is-sorted-version}[\text{left-split}[\langle \overline{X_0} \rangle], \text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]]]$ ,

(24)  $\text{is-sorted-version}[\text{right-split}[\langle \overline{X_0} \rangle], \text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]]$ ,

From (23), by (Definition (is sorted version)), we obtain:

(25)

$\text{is-tuple}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]]] \wedge \text{left-split}[\langle \overline{X_0} \rangle] \approx \text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]] \wedge \text{is-sorted}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]]]$

From (24), by (Definition (is sorted version)), we obtain:

(26)  $\text{is-tuple}[\text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]] \wedge \text{right-split}[\langle \overline{X_0} \rangle] \approx \text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]] \wedge \text{is-sorted}[\text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]]$

From (1) and (8), using (Definition (is sorted version)), is implied by:

(41)  $\text{is-tuple}[\text{merged}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]], \text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]]] \wedge \text{merged}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]], \text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]] \approx \langle \overline{X_0} \rangle \wedge \text{is-sorted}[\text{merged}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]], \text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]]]$

Not all the conjunctive parts of (41) can be proved.

Proof of (41.1)  $\text{is-tuple}[\text{merged}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]], \text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]]]$ :

(41.1), by (Lemma (closure of merge)) is implied by:

(42)  $\text{is-tuple}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]]] \wedge \text{is-tuple}[\text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]]$ .

We prove the individual conjunctive parts of (42):

Proof of (42.1)  $\text{is-tuple}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]]]$ :

Formula (42.1) is true because it is identical to (25.1).

Proof of (42.2)  $\text{is-tuple}[\text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]]$ :

Formula (42.2) is true because it is identical to (26.1).

Proof of (41.3) `merged[sorted[left-split[⟨X̄₀⟩]], sorted[right-split[⟨X̄₀⟩]]] ≈ ⟨X̄₀⟩`:

The proof of (41.3) fails. (The prover "QR" was unable to transform the proof situation.)

Proof of (41.4)

`is-sorted[merged[sorted[left-split[⟨X̄₀⟩]], sorted[right-split[⟨X̄₀⟩]]]`:

Pending proof of (41.4).

□

### Notebook 3

Comment on Notebook 3: Note that in the knowledge base (i.e. the formulae listed under "assumptions"), the additional specification (Lemma (conjecture44): conjecture44) is now contained, which describes part of the specification automatically generated (from the failing proof in Notebook 2) for the functions 'left', 'right', and 'merge'. The proof then proceeds as in Notebook 2 but succeeds to get over the point at which the second proof was stuck.

Prove:

(Theorem (correctness of sort))  $\forall_{\text{is-tuple}[\mathbf{x}]} \text{is-sorted-version}[\mathbf{x}, \text{sorted}[\mathbf{x}]],$

under the assumptions:

(Definition (is sorted): 1) `is-sorted[⟨⟩]`,

(Definition (is sorted): 2)  $\forall_{\mathbf{x}} \text{is-sorted}[\langle \mathbf{x} \rangle],$

(Definition (is sorted): 3)  $\forall_{\mathbf{x}, \mathbf{y}, \bar{\mathbf{z}}} (\text{is-sorted}[\langle \mathbf{x}, \mathbf{y}, \bar{\mathbf{z}} \rangle] \Leftrightarrow \mathbf{x} \geq \mathbf{y} \wedge \text{is-sorted}[\langle \mathbf{y}, \bar{\mathbf{z}} \rangle]),$

... and all the assumptions of Notebook 1 ...

(Lemma (closure of merge))  $\forall_{\substack{\text{is-tuple}[\mathbf{x}] \\ \text{is-tuple}[\mathbf{y}]}} \text{is-tuple}[\text{merged}[\mathbf{x}, \mathbf{y}]],$

(Lemma (conjecture15): conjecture15)

$\forall_{\substack{\mathbf{x1} \\ \text{is-tuple}[\mathbf{x1}]}} (\text{is-trivial-tuple}[\mathbf{x1}] \wedge \text{is-sorted}[\mathbf{x1}] \Rightarrow (\text{special}[\mathbf{x1}] = \mathbf{x1})),$

(Lemma (conjecture44): conjecture44)

$\forall_{\substack{\mathbf{x2}, \mathbf{x3}, \mathbf{x4} \\ \text{is-tuple}[\mathbf{x4}]}} (\text{is-tuple}[\mathbf{x2}] \wedge \text{left-split}[\mathbf{x4}] \approx \mathbf{x2} \wedge$   
 $\text{is-sorted}[\mathbf{x2}] \wedge \text{is-tuple}[\mathbf{x3}] \wedge \text{right-split}[\mathbf{x4}] \approx \mathbf{x3} \wedge$   
 $\text{is-sorted}[\mathbf{x3}] \wedge \neg \text{is-trivial-tuple}[\mathbf{x4}] \Rightarrow \text{merged}[\mathbf{x2}, \mathbf{x3}] \approx \mathbf{x4)$

We try to prove (Theorem (correctness of sort)) by well-founded induction on  $\mathbf{x}$ .

Well-founded induction:

Assume:

(1)  $\text{is-tuple}[\langle \overline{X_0} \rangle]$ .

Well-Founded Induction Hypothesis:

(2)  $\forall_{\text{is-tuple}[\mathbf{x3}]} (\langle \overline{X_0} \rangle > \mathbf{x3} \Rightarrow \text{is-sorted-version}[\mathbf{x3}, \text{sorted}[\mathbf{x3}]])$

We have to show:

(3)  $\text{is-sorted-version}[\langle \overline{X_0} \rangle, \text{sorted}[\langle \overline{X_0} \rangle]]$ .

We try to prove (3) by case distinction using (Algorithm (sorted)). However, the proof fails in at least one of the cases.

Case 1:

(4)  $\text{is-trivial-tuple}[\langle \overline{X_0} \rangle]$ .

Hence, we have to prove

(5)  $\text{is-sorted-version}[\langle \overline{X_0} \rangle, \text{special}[\langle \overline{X_0} \rangle]]$ .

Formula (4), by (Proposition (trivial tuples are sorted)), implies:

(9)  $\text{is-sorted}[\langle \overline{X_0} \rangle]$ .

Formula (4), by (Proposition (only trivial tuple permuted version of itself)), implies:

(10)  $\forall_{\mathbf{Y}} ((\mathbf{Y} = \langle \overline{X_0} \rangle) \Rightarrow \mathbf{Y} \approx \langle \overline{X_0} \rangle)$ .

Formula (1) and (4), by (Lemma (closure of special)), implies:

(11)  $\text{is-tuple}[\text{special}[\langle \overline{X_0} \rangle]]$ .

Formula (1) and (4), by (Lemma (conjecture15): conjecture15), implies:

(13)  $\text{special}[\langle \overline{X_0} \rangle] = \langle \overline{X_0} \rangle$ .

Formula (5), using (13), is implied by:

(21)  $\text{is-sorted-version}[\langle \overline{X_0} \rangle, \langle \overline{X_0} \rangle]$ .

Formula (21), using (Definition (is sorted version)), is implied by:

(22)  $\text{is-tuple}[\langle \overline{X_0} \rangle] \wedge \langle \overline{X_0} \rangle \approx \langle \overline{X_0} \rangle \wedge \text{is-sorted}[\langle \overline{X_0} \rangle]$ .

We prove the individual conjunctive parts of (22):

Proof of (22.1)  $\text{is-tuple}[\langle \overline{X_0} \rangle]$ :

Formula (22.1) is true because it is identical to (1).

Proof of (22.2)  $\langle \overline{X_0} \rangle \approx \langle \overline{X_0} \rangle$ :

Formula (22.2) is true by (10).

Proof of (22.3)  $\text{is-sorted}[\langle \overline{X_0} \rangle]$ :

Formula (22.3) is true because it is identical to (9).

Case 2:

(6)  $\neg \text{is-trivial-tuple}[\langle \overline{X_0} \rangle]$ .

Hence, we have to prove

(8)  $\text{is-sorted-version}[\langle \overline{X_0} \rangle, \text{merged}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]], \text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]]]$ .

From (6), by (2), (Lemma (splits are tuples): 1), (Lemma (splits are tuples): 2), (Lemma (splits are shorter): 1), (Lemma (splits are shorter): 1) and (Lemma (splits are shorter): 2), we obtain:

(23)  $\text{is-sorted-version}[\text{left-split}[\langle \overline{X_0} \rangle], \text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]]]$ ,

(24)  $\text{is-sorted-version}[\text{right-split}[\langle \overline{X_0} \rangle], \text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]]$ ,

From (23), by (Definition (is sorted version)), we obtain:

(25)

$\text{is-tuple}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]]] \wedge \text{left-split}[\langle \overline{X_0} \rangle] \approx \text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]] \wedge \text{is-sorted}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]]]$ .

From (24), by (Definition (is sorted version)), we obtain:

(26)  $\text{is-tuple}[\text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]] \wedge \text{right-split}[\langle \overline{X_0} \rangle] \approx \text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]] \wedge \text{is-sorted}[\text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]]$ .

From (1) and (8), using (Definition (is sorted version)), is implied by:

(41)  $\text{is-tuple}[\text{merged}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]], \text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]]] \wedge \text{merged}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]], \text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]] \approx \langle \overline{X_0} \rangle \wedge \text{is-sorted}[\text{merged}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]], \text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]]]$

Not all the conjunctive parts of (41) can be proved.

Proof of (41.1)  $\text{is-tuple}[\text{merged}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]], \text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]]]$ :

(41.1), by (Lemma (closure of merge)) is implied by:

(42)  $\text{is-tuple}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]]] \wedge \text{is-tuple}[\text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]]$ .

We prove the individual conjunctive parts of (42):

Proof of (42.1)  $\text{is-tuple}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]]]$ :

Formula (42.1) is true because it is identical to (25.1).

Proof of (42.2)  $\text{is-tuple}[\text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]]$ :

Formula (42.2) is true because it is identical to (26.1).

Proof of (41.2)  $\text{merged}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]], \text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]] \approx \langle \overline{X_0} \rangle$ :

Formula (41.2), using (Lemma (conjecture44): conjecture44), is implied by:

(44)

$\text{is-tuple}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]]] \wedge \text{left-split}[\langle \overline{X_0} \rangle] \approx \text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]] \wedge \text{is-sorted}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]]] \wedge \text{is-tuple}[\text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]] \wedge \text{right-split}[\langle \overline{X_0} \rangle] \approx \text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]] \wedge \text{is-sorted}[\text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]] \wedge \neg \text{is-trivial-tuple}[\langle \overline{X_0} \rangle]$

We prove the individual conjunctive parts of (44):

Proof of (44.1) `is-tuple[sorted[left-split[⟨X̄₀⟩]]]`:

Formula (44.1) is true because it is identical to (25.1).

Proof of (44.2) `left-split[⟨X̄₀⟩] ≈ sorted[left-split[⟨X̄₀⟩]]`:

Formula (44.2) is true because it is identical to (25.1).

Proof of (44.3) `is-sorted[sorted[left-split[⟨X̄₀⟩]]]`:

Formula (44.3) is true because it is identical to (25.3).

Proof of (44.4) `is-tuple[sorted[right-split[⟨X̄₀⟩]]]`:

Formula (44.4) is true because it is identical to (26.1).

Proof of (44.5) `right-split[⟨X̄₀⟩] ≈ sorted[right-split[⟨X̄₀⟩]]`:

Formula (44.5) is true because it is identical to (26.2).

Proof of (44.6) `is-sorted[sorted[right-split[⟨X̄₀⟩]]]`:

Formula (44.6) is true because it is identical to (26.2).

Proof of (44.7) `¬ is-trivial-tuple[⟨X̄₀⟩]`:

Formula (44.7) is true because it is identical to (6).

Proof of (41.3)

`is-sorted[merged[sorted[left-split[⟨X̄₀⟩]], sorted[right-split[⟨X̄₀⟩]]]`:

The proof of (41.3) fails. (The prover "QR" was unable to transform the proof situation.)

□

## Notebook 4

Comment on Notebook 4: Note that in the knowledge base (i.e. the formulae listed under "assumptions"), the additional specification (Lemma (conjecture46): conjecture46) is now contained, which describes the second part of the specification automatically generated (from the failing proof in Notebook 3) for the functions 'left', 'right', and 'merge'. The proof then proceeds as in Notebook 3 but succeeds to get over the point at which the third proof was stuck and, actually, proceeds until the successful end.

Prove:

(Theorem (correctness of sort))  $\forall_{\text{is-tuple}[\mathbf{x}]} \text{is-sorted-version}[\mathbf{x}, \text{sorted}[\mathbf{x}]],$

under the assumptions:

(Definition (is sorted): 1) `is-sorted[⟨⟩]`,

(Definition (is sorted): 2)  $\forall_{\mathbf{x}} \text{is-sorted}[\langle \mathbf{x} \rangle],$

(Definition (is sorted): 3)  $\forall_{\mathbf{x}, \mathbf{y}, \bar{\mathbf{z}}} (\text{is-sorted}[\langle \mathbf{x}, \mathbf{y}, \bar{\mathbf{z}} \rangle] \Leftrightarrow \mathbf{x} \geq \mathbf{y} \wedge \text{is-sorted}[\langle \mathbf{y}, \bar{\mathbf{z}} \rangle]),$

... and all the assumptions appearing in Notebook 1 ...

(Lemma (closure of merge))  $\forall_{\substack{\text{is-tuple}[\mathbf{x}] \\ \text{is-tuple}[\mathbf{y}]}} \text{is-tuple}[\text{merged}[\mathbf{x}, \mathbf{y}]],$

(Lemma (conjecture15): conjecture15)

$$\forall_{\substack{\mathbf{x1} \\ \text{is-tuple}[\mathbf{x1}]}} (\text{is-trivial-tuple}[\mathbf{x1}] \wedge \text{is-sorted}[\mathbf{x1}] \Rightarrow (\text{special}[\mathbf{x1}] = \mathbf{x1})),$$

(Lemma (conjecture44): conjecture44)

$$\forall_{\substack{\mathbf{x2}, \mathbf{x3}, \mathbf{x4} \\ \text{is-tuple}[\mathbf{x4}]}} (\text{is-tuple}[\mathbf{x2}] \wedge \text{left-split}[\mathbf{x4}] \approx \mathbf{x2} \wedge \text{is-sorted}[\mathbf{x2}] \wedge \text{is-tuple}[\mathbf{x3}] \wedge \text{right-split}[\mathbf{x4}] \approx \mathbf{x3} \wedge \text{is-sorted}[\mathbf{x3}] \wedge \neg \text{is-trivial-tuple}[\mathbf{x4}] \Rightarrow \text{merged}[\mathbf{x2}, \mathbf{x3}] \approx \mathbf{x4})$$

(Lemma (conjecture46): conjecture46)

$$\forall_{\substack{\mathbf{x5}, \mathbf{x6}, \mathbf{x7} \\ \text{is-tuple}[\mathbf{x7}]}} (\text{is-tuple}[\mathbf{x5}] \wedge \text{left-split}[\mathbf{x7}] \approx \mathbf{x5} \wedge \text{is-sorted}[\mathbf{x5}] \wedge \text{is-tuple}[\mathbf{x6}] \wedge \text{right-split}[\mathbf{x7}] \approx \mathbf{x6} \wedge \text{is-sorted}[\mathbf{x6}] \wedge \neg \text{is-trivial-tuple}[\mathbf{x7}] \Rightarrow \text{is-sorted}[\text{merged}[\mathbf{x5}, \mathbf{x6}]])$$

We prove (Theorem (correctness of sort)) by well-founded induction on  $X$ .

Well-founded induction:

Assume:

$$(1) \text{is-tuple}[\langle \overline{X_0} \rangle].$$

Well-Founded Induction Hypothesis:

$$(2) \forall_{\text{is-tuple}[\mathbf{x4}]} (\langle \overline{X_0} \rangle > \mathbf{x4} \Rightarrow \text{is-sorted-version}[\mathbf{x4}, \text{sorted}[\mathbf{x4}]])$$

We have to show:

$$(3) \text{is-sorted-version}[\langle \overline{X_0} \rangle, \text{sorted}[\langle \overline{X_0} \rangle]].$$

We prove (3) by case distinction using (Algorithm (sorted)).

Case 1:

$$(4) \text{is-trivial-tuple}[\langle \overline{X_0} \rangle].$$

Hence, we have to prove

$$(5) \text{is-sorted-version}[\langle \overline{X_0} \rangle, \text{special}[\langle \overline{X_0} \rangle]].$$

Formula (4), by (Proposition (trivial tuples are sorted)), implies:

$$(9) \text{is-sorted}[\langle \overline{X_0} \rangle].$$

Formula (4), by (Proposition (only trivial tuple permuted version of itself)), implies:

$$(10) \forall_{\mathbf{Y}} ((\mathbf{Y} = \langle \overline{X_0} \rangle) \Rightarrow \mathbf{Y} \approx \langle \overline{X_0} \rangle).$$

Formula (1) and (4), by (Lemma (closure of special)), implies:

$$(11) \text{is-tuple}[\text{special}[\langle \overline{X_0} \rangle]].$$

Formula (1) and (4), by (Lemma (conjecture15): conjecture15), implies:

$$(13) \text{special}[\langle \overline{X_0} \rangle] = \langle \overline{X_0} \rangle.$$

Formula (5), using (13), is implied by:

$$(21) \text{is-sorted-version}[\langle \overline{X_0} \rangle, \langle \overline{X_0} \rangle].$$

Formula (21), using (Definition (is sorted version)), is implied by:

$$(22) \text{is-tuple}[\langle \overline{X_0} \rangle] \wedge \langle \overline{X_0} \rangle \approx \langle \overline{X_0} \rangle \wedge \text{is-sorted}[\langle \overline{X_0} \rangle].$$

We prove the individual conjunctive parts of (22):

Proof of (22.1)  $\text{is-tuple}[\langle \overline{X_0} \rangle]$ :

Formula (22.1) is true because it is identical to (1).

Proof of (22.2)  $\langle \overline{X_0} \rangle \approx \langle \overline{X_0} \rangle$ :

Formula (22.2) is true by (10).

Proof of (22.3)  $\text{is-sorted}[\langle \overline{X_0} \rangle]$ :

Formula (22.3) is true because it is identical to (9).

Case 2:

$$(6) \neg \text{is-trivial-tuple}[\langle \overline{X_0} \rangle].$$

Hence, we have to prove

$$(8) \text{is-sorted-version}[\langle \overline{X_0} \rangle, \text{merged}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]], \text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]]]$$

From (6), by (2), (Lemma (splits are tuples): 1), (Lemma (splits are tuples): 2), (Lemma (splits are shorter): 1), (Lemma (splits are shorter): 1) and (Lemma (splits are shorter): 2), we obtain:

$$(23) \text{is-sorted-version}[\text{left-split}[\langle \overline{X_0} \rangle], \text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]]],$$

$$(24) \text{is-sorted-version}[\text{right-split}[\langle \overline{X_0} \rangle], \text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]],$$

From (23), by (Definition (is sorted version)), we obtain:

$$(25)$$

$$\text{is-tuple}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]]] \wedge \text{left-split}[\langle \overline{X_0} \rangle] \approx \text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]] \wedge \text{is-sorted}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]]]$$

From (24), by (Definition (is sorted version)), we obtain:

$$(26) \text{is-tuple}[\text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]] \wedge \text{right-split}[\langle \overline{X_0} \rangle] \approx \text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]] \wedge \text{is-sorted}[\text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]]$$

From (1) and (8), using (Definition (is sorted version)), is implied by:

$$(41) \text{is-tuple}[\text{merged}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]], \text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]]] \wedge \text{merged}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]], \text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]] \approx \langle \overline{X_0} \rangle \wedge \text{is-sorted}[\text{merged}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]], \text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]]]$$

We prove the individual conjunctive parts of (41):

Proof of (41.1)  $\text{is-tuple}[\text{merged}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]], \text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]]]$ :

(41.1), by (Lemma (closure of merge)) is implied by:

$$(42) \text{is-tuple}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]]] \wedge \text{is-tuple}[\text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]].$$

We prove the individual conjunctive parts of (42):

Proof of (42.1)  $\text{is-tuple}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]]]$ :

Formula (42.1) is true because it is identical to (25.1).

Proof of (42.2)  $\text{is-tuple}[\text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]]$ :

Formula (42.2) is true because it is identical to (26.1).

Proof of (41.2)  $\text{merged}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]], \text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]] \approx \langle \overline{X_0} \rangle$ :

Formula (41.2), using (Lemma (conjecture44): conjecture44), is implied by:

(44)

$$\begin{aligned} & \text{is-tuple}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]]] \wedge \text{left-split}[\langle \overline{X_0} \rangle] \approx \text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]] \wedge \\ & \text{is-sorted}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]]] \wedge \text{is-tuple}[\text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]] \wedge \\ & \text{right-split}[\langle \overline{X_0} \rangle] \approx \text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]] \wedge \\ & \text{is-sorted}[\text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]] \wedge \neg \text{is-trivial-tuple}[\langle \overline{X_0} \rangle] \end{aligned}$$

We prove the individual conjunctive parts of (44):

Proof of (44.1)  $\text{is-tuple}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]]]$ :

Formula (44.1) is true because it is identical to (25.1).

Proof of (44.2)  $\text{left-split}[\langle \overline{X_0} \rangle] \approx \text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]]$ :

Formula (44.2) is true because it is identical to (25.1).

Proof of (44.3)  $\text{is-sorted}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]]]$ :

Formula (44.3) is true because it is identical to (25.3).

Proof of (44.4)  $\text{is-tuple}[\text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]]$ :

Formula (44.4) is true because it is identical to (26.1).

Proof of (44.5)  $\text{right-split}[\langle \overline{X_0} \rangle] \approx \text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]$ :

Formula (44.5) is true because it is identical to (26.2).

Proof of (44.6)  $\text{is-sorted}[\text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]]$ :

Formula (44.6) is true because it is identical to (26.2).

Proof of (44.7)  $\neg \text{is-trivial-tuple}[\langle \overline{X_0} \rangle]$ :

Formula (44.7) is true because it is identical to (6).

Proof of (41.3)

$\text{is-sorted}[\text{merged}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]], \text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]]]$ :

Formula (41.3), using (Lemma (conjecture46): conjecture46), is implied by:

(52)

$$\begin{aligned} & \text{is-tuple}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]]] \wedge \text{left-split}[\langle \overline{X_0} \rangle] \approx \text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]] \wedge \\ & \text{is-sorted}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]]] \wedge \text{is-tuple}[\text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]] \wedge \\ & \text{right-split}[\langle \overline{X_0} \rangle] \approx \text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]] \wedge \\ & \text{is-sorted}[\text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]] \wedge \neg \text{is-trivial-tuple}[\langle \overline{X_0} \rangle] \end{aligned}$$

We prove the individual conjunctive parts of (52):

Proof of (52.1)  $\text{is-tuple}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]]]$ :

Formula (52.1) is true because it is identical to (25.1).

Proof of (52.2)  $\text{left-split}[\langle \overline{X_0} \rangle] \approx \text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]]$ :

Formula (52.2) is true because it is identical to (25..2).

Proof of (52.3)  $\text{is-sorted}[\text{sorted}[\text{left-split}[\langle \overline{X_0} \rangle]]]$ :

Formula (52.3) is true because it is identical to (25.3).

Proof of (52.4)  $\text{is-tuple}[\text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]]$ :

Formula (52.4) is true because it is identical to (26.1).

Proof of (52.5)  $\text{right-split}[\langle \overline{X_0} \rangle] \approx \text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]$ :

Formula (52.5) is true because it is identical to (26.2).

Proof of (52.6)  $\text{is-sorted}[\text{sorted}[\text{right-split}[\langle \overline{X_0} \rangle]]]$ :

Formula (52.6) is true because it is identical to (26.3).

Proof of (52.7)  $\neg \text{is-trivial-tuple}[\langle \overline{X_0} \rangle]$ :

Formula (52.7) is true because it is identical to (6).

□