Diagonally symmetric polynomials of the roots of systems of polynomial equations

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Fundamental Theorem of Symmetric Polynomials:

Fix d. For the polynomial equations

$$x^d + c_1 x^{d-1} + \cdots + c_{d-1} x + c_d = 0$$

the symmetric polynomials of the roots (in $\mathbb{C}$) are exactly the polynomials in the coefficients $c_1, \ldots, c_{d-1}, c_d$.

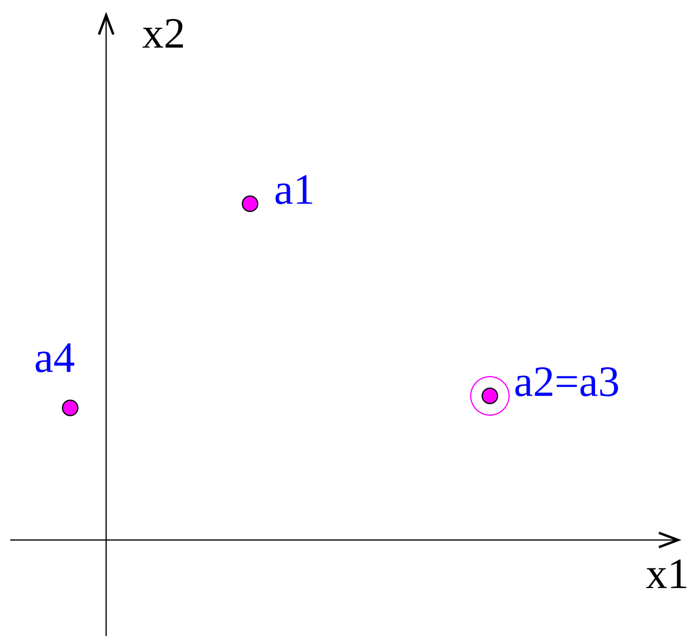
What about systems of polynomial equations in several variables with finitely many solutions?

The diagonally symmetric polynomials are the polynomials in the entries of the matrix

$$\begin{bmatrix} x_1(a_1) & x_1(a_2) & \cdots & x_1(a_n) \\ x_2(a_1) & x_2(a_2) & \cdots & x_2(a_n) \\ \vdots & \vdots & & \vdots \\ x_r(a_1) & x_r(a_2) & \cdots & x_r(a_n) \end{bmatrix}$$

that are invariant under all permutations of the columns.

They are the polynomial functions of multisets of n points, a_1, a_2, ..., a_n, of the affine r–dimensional space !



Ex: the *diagonally symmetric power sums*

$$p_\alpha = \sum_{i=1}^{n} \mathbf{x}^\alpha(a_i) = \sum_{i=1}^{n} (x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_r^{\alpha_r})(a_i).$$

These objects appear naturally when studying systems of equations with finitely many solutions, *e.g.* as coefficients of the Chow form, as traces of monomials, or multidimensional residues . . .

They are not new !!! They were first introduced by Cayley, MacMahon, . . . in the XIX$^{\text{th}}$ century.

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ON THE SYMMETRIC FUNCTIONS OF THE ROOTS OF CERTAIN SYSTEMS OF TWO EQUATIONS.

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SUPPOSE in general that $\phi = 0$, $\psi = 0$, &c. denote a system of $(n-1)$ equations between the n variables $(x, y, z, \ldots)$, where the functions ϕ, ψ, &c. are quantics (i.e. rational and integral homogeneous functions) of the variables. Any values $(x_1, y_1, z_1, \ldots)$ satisfying the equations, are said to constitute a set of roots of the system; the roots of the same set are, it is clear, only determinate to a common factor *près*, i.e. only the ratios *inter se* and not the absolute magnitudes of the roots of a set are determinate. The number of sets, or the degree of the system, is equal to the product of the degrees of the component equations. Imagine a function of the roots which remains unaltered when any two sets $(x_1, y_1, z_1, \ldots)$ and $(x_2, y_2, z_2, \ldots)$ are interchanged (that is, when x_1 and x_2, y_1 and y_2, &c. are simultaneously interchanged), and which is besides homogeneous of the same degree as regards each entire set of roots although

The case of Gröbner bases

Let $\mathbf{x} = (x_1, \ldots, x_r)$. Choose some monomial order, and monomials $\mathbf{x}^{\alpha_1}$, $\mathbf{x}^{\alpha_2}$, . . . , $\mathbf{x}^{\alpha_r}$, with among them a pure power of each variable.

Consider all systems

$$\begin{cases} F_1 & = & \mathbf{x}^{\alpha_1} & + & \textit{smaller terms} \\ F_2 & = & \mathbf{x}^{\alpha_2} & + & \textit{smaller terms} \\ & \vdots & & & \\ F_k & = & \mathbf{x}^{\alpha_k} & + & \textit{smaller terms} \end{cases}$$

that are a Gröbner basis.

THEOREM (E.B.): the diagonally symmetric polynomials of the roots of these systems are polynomial functions of the coefficients.

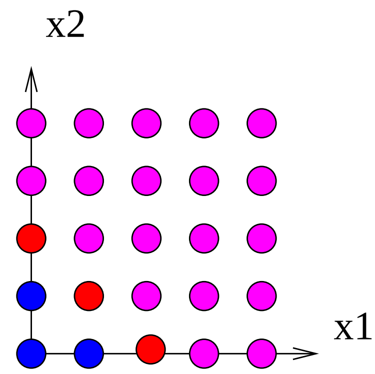
An example:

Lexicographic degree order,

$x_1 > x_2$.

leading monomials:

$\mathbf{x}^{2,0}$, $\mathbf{x}^{1,1}$, $\mathbf{x}^{0,2}$.



$$\begin{cases} F_1 = x_1^2 + b\,x_1x_2 + c\,x_2^2 + d\,x_1 + e\,x_2 + f, \\ F_2 = x_1x_2 + g\,x_2^2 + h\,x_1 + j\,x_2 + k, \\ F_3 = x_2^2 + q\,x_1 + s\,x_2 + t. \end{cases}$$

is a Gröbner basis iff

$$\begin{cases} f = -c\,s + e\,s + d\,j + \textit{other terms} \ldots, \\ k = -g\,h^2 + j\,h - g^3\,q^2 + \textit{other terms} \ldots, \\ t = -h^2 + q\,d + s\,h + \textit{other terms} \ldots \end{cases}$$

Then the system has $n = 3$ zeros a_1, a_2, a_3 in $\mathbb{C}^2$. One has, for instance, that the diagonally symmetric power sum

$$p_{11} = \mathbf{x}^{1,1}(a_1) + \mathbf{x}^{1,1}(a_2) + \mathbf{x}^{1,1}(a_3)$$

admits as an expression

$$\begin{gathered} -b\,h^2 + d\,h - j\,h - g\,s^2 - c\,q\,h \\ +3\,e\,q + g\,j\,q - 3\,b\,j\,q - 3\,c\,q\,s - g^2\,q\,s - d\,g\,q \\ +c\,q^2\,g - b\,g^2\,q^2 + 2\,b\,g\,q\,h + 3\,b\,g\,q\,s + g\,h\,s + j\,s. \end{gathered}$$

If the zeros are all simple, one can find back the coefficients of the *reduced* Gröbner basis

$$\begin{cases} F_1 = x_1^2 & +d\,x_1 + e\,x_2 + f, \\ F_2 = x_1x_2 & +h\,x_1 + j\,x_2 + k, \\ F_3 = x_2^2 & +q\,x_1 + s\,x_2 + t. \end{cases}$$

by means of *Lagrange interpolation*–like formulas, *e.g.*:

$$d = -\frac{\begin{vmatrix} 1 & x_2(a_1) & x_1^2(a_1) \\ 1 & x_2(a_2) & x_1^2(a_2) \\ 1 & x_2(a_3) & x_1^2(a_3) \end{vmatrix}}{\begin{vmatrix} 1 & x_1(a_1) & x_2(a_1) \\ 1 & x_1(a_2) & x_2(a_2) \\ 1 & x_1(a_3) & x_2(a_3) \end{vmatrix}}$$

that can be also written as *quotients* of diagonally symmetric polynomials:

$$d = -\frac{\begin{vmatrix} 3 & p_{01} & p_{20} \\ p_{10} & p_{11} & p_{30} \\ p_{01} & p_{02} & p_{21} \end{vmatrix}}{\begin{vmatrix} 3 & p_{10} & p_{01} \\ p_{10} & p_{20} & p_{11} \\ p_{01} & p_{11} & p_{02} \end{vmatrix}}.$$

Gröbner bases with as many equations as unknowns

Choose a monomial order. Consider all systems

$$\begin{cases} F_1 & = & x_1^{d_1} & + & \textit{smaller terms} \\ F_2 & = & x_2^{d_2} & + & \textit{smaller terms} \\ & \vdots & & & \\ F_r & = & x_r^{d_r} & + & \textit{smaller terms} \end{cases}$$

A formula from the Theory of multidimensional residues, due to Aizenberg and Kytmanov, provides the power sums as the coefficients in some series expansion:

$$\frac{x_1 \cdot x_2 \cdots x_r}{F_1 \cdot F_2 \cdots F_r} \cdot \left| \left(\frac{\partial F_i}{dx_j} \right) \right| \quad = \quad \sum_{\alpha \in \mathbb{N}^r} \frac{p_\alpha}{\mathbf{x}^\alpha} + \quad \cdots$$

that corresponds to *diagonally symmetric analogs of Newton's recursion formulas between coefficients and power sums.*

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