Sphere Packings and Association Schemes

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Part 1. Introduction.

§1. Brief Review on Sphere Packings

There are several new breakthroughs on sphere packings in the last few years.

(1) Solution of Kepler's conjecture by Thomas Hales.

(2) Determination of the kissing number in 4 dimensions by Oleg Musin.

(3) Development on sphere packing problems in 8 and 24 dimensions by Henry Cohn, Noam Elkies and Abhinav Kumar, In particular the optimalty of Leech lattice among lattices.

(1) T. Hales (Announcement, 1998)

$$\left(\begin{matrix}\text{The best density of} \\ \text{sphere packing in } \mathbb{R}^3\end{matrix}\right) = \frac{\pi}{\sqrt{18}} = 0.74\ldots\ldots$$

$\left(\frac{\pi}{\sqrt{12}} \text{ for } \mathbb{R}^2\right)$

(2) O. Musin (preprint, 2003)

(3) Cohn-Elkies : New upper bounds on sphere packings, I. Ann. of Math. 157 (2003), 689–714

Cohn-Kumar : Optimality and uniqueness of the Leech lattice among lattices. (preprint, 2004)

- $$\left(\begin{matrix}\text{The density of sphere} \\ \text{packing in } \mathbb{R}^{24}\end{matrix}\right)$$

$$\leqslant \left(1+10^{-30}\right)\left(\begin{matrix}\text{The density of sphere} \\ \text{packing of Leech lattice}\end{matrix}\right)$$

- The Leech lattice gives the best sphere packings among lattices.

$f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is <u>admissible</u>, if $\exists \delta > 0$ and $\exists c > 0$ such that

$$|f(x)|, |\hat{f}(x)| < c\,(1+\|x\|)^{-n-\delta}$$

(for $\forall x \in \mathbb{R}^n$)

<u>Thm</u> (Cohn-Elkies)

If $\exists f : \mathbb{R}^n \longrightarrow \mathbb{R}$, <u>admissible</u> with

(1) $f(0) = \hat{f}(0) > 0$

(2) $f(x) \leq 0$, for $\|x\| > r$ $(x \in \mathbb{R}^n)$

(3) $\hat{f}(t) \geq 0$, for $\forall t \in \mathbb{R}^n$

$\Longrightarrow$ (The density of sphere packing in $\mathbb{R}^n$)

$$\leqslant \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} \left(\frac{r}{2}\right)^n$$

The area of S^{n-1} $(= |S^{n-1}|)$

<u>Conjecture</u> (Cohn-Elkies)

For $n = 24$, $\exists f : \mathbb{R}^{24} \longrightarrow \mathbb{R}$ admissible satisfying above (1), (2), (3), <u>for $r = 2$</u>

(For $n = 8$, $\exists f : \mathbb{R}^8 \longrightarrow \mathbb{R}$ admissible satisfying (1), (2), (3), for $r = \sqrt{2}$.)

For example, see expository article by

F. Pfender and G. M. Ziegler: Kissing Numbers, Sphere Packings, and Some Unexpected Proofs, Notices of Amer. Math. Soc., 51 (2004), 873-883.

We had two Workshops on Sphere Packings in Kyushu University. First one in Nov. 2004, and the second one in May-June, 2005. Oleg Musin and Henry Cohn were main lecturers respectively.

Proceedings of the two workshops are available in web.

http://www.math.kyushu-u.ac.jp/coe/report/mhf2.cgi

§2. Kissing number problem in R^n.

$n = 2$. $k(2) = 6$.

$n = 3$. $k(3) = 12$. The 13 sphere problem: Newton-Gregory dispute, 1694.
First rigorous proof is due to Schütte-van der Waerden, 1953.

$n = 8$. and $n = 24$. $k(8) = 240$, and $k(24) = 196560$. (Odlyzko-Sloane, Levenshtein, 1977.)

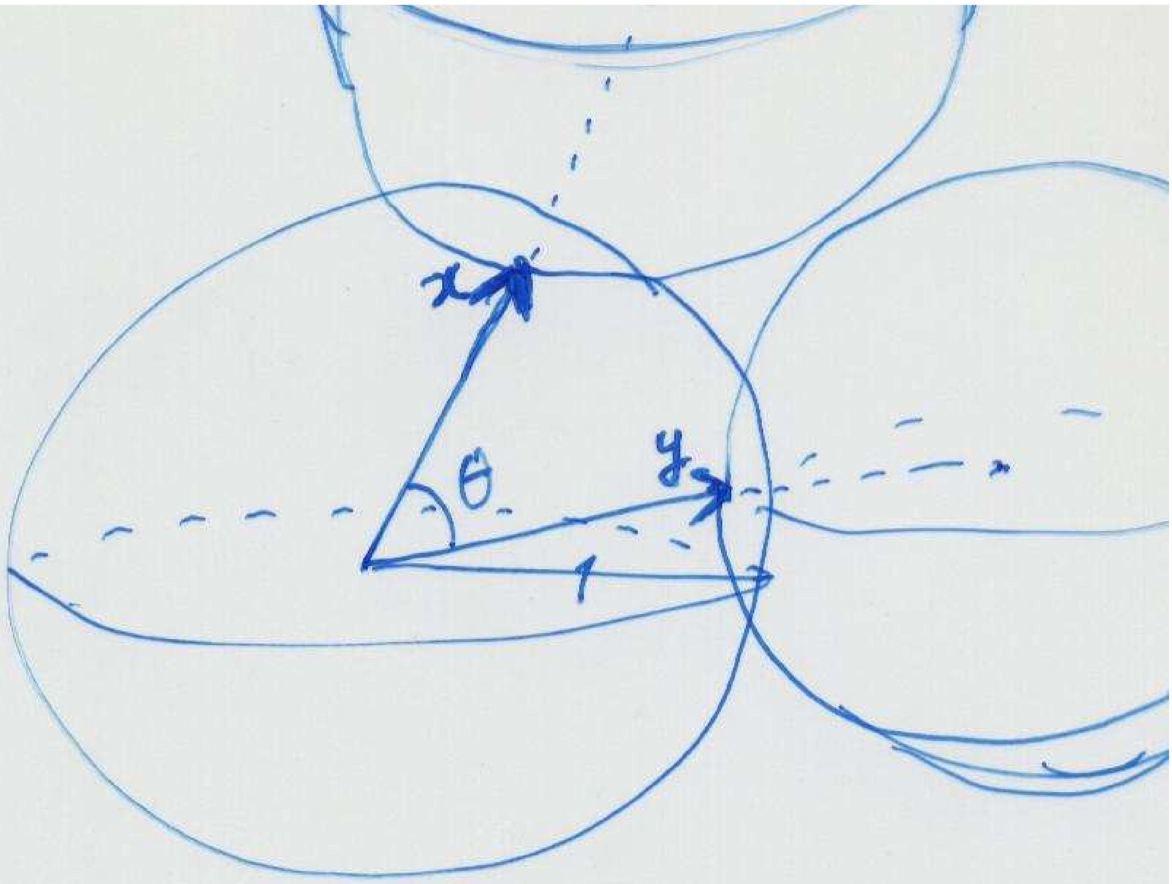
$n = 4$. by Oleg Musin: $k(4) = 24$. (Previously it was known only either $k(4) = 24$ or 25.)

Remarks.

(1) The uniqueness of the kissing configuration of 8 and 24 dimensions were proved by Bannai-Sloane (1981).

(2) The uniqueness of the kissing configuration of 4 dimension is still open.

(3) W.Y. Hsiang has preprint (at least before 2001) which claims the solution of $k(4) = 24$ and the uniqueness of the kissing configuration of 4 dimension. However, it seems that his proofs are not accepted by experts.



x and y are at least $60°$ apart

$$\Longleftrightarrow \quad x \cdot y \leq \frac{1}{2}$$

$$k(n) = \max_{X \subset S^{n-1}} |X|$$

with $x \cdot y \leq \frac{1}{2}$

for $\forall_{x, y \in X}$ $(x \neq y)$

$X \subset S^{n-1} \ (\subset \mathbb{R}^n)$

Suppose that $x \cdot y \leq \frac{1}{2}$, for $\forall x, y \in X$ $(x \neq y)$ — or $(= \alpha)$

If $\exists$ polynomial $f(x)$ such that

(1) $f(x) \leq 0$, $\forall x \in [-1, \frac{1}{2}]$ — or $(= \alpha$

(2) $f(x) = a_0 Q_0(x) + a_1 Q_1(x) + \cdots + a_\ell Q_\ell$

with $a_0 > 0$, $a_i \geq 0$ $(i \geq 1)$

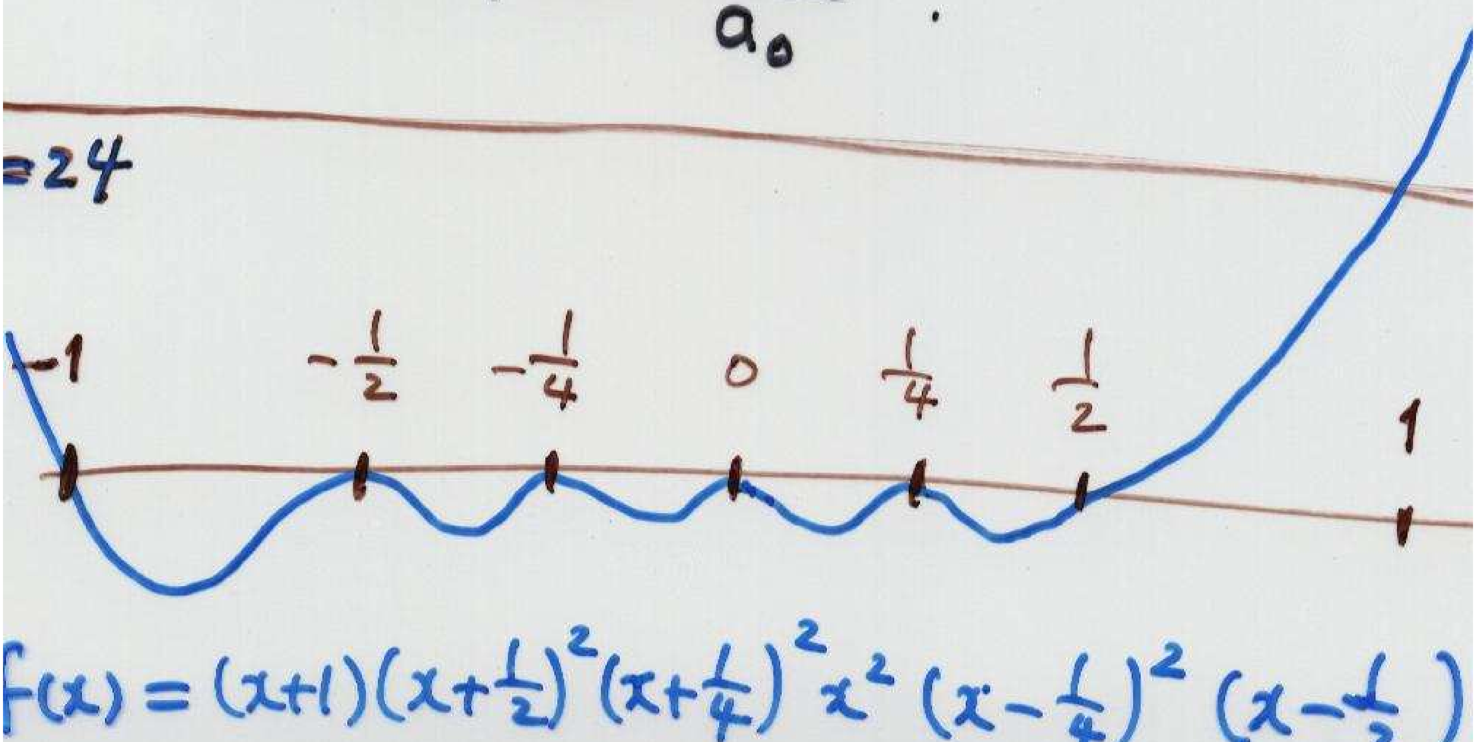
$Q_i(x) \ (= C_i^{\frac{n}{2}-1}(x)) =$ Gegenbauer

(i.e., orthogonal polynomials with

$\omega(x) = (1-x^2)^{\frac{n-3}{2}}$ on $[-1, 1]$

$$\Longrightarrow \ |X| \leq \frac{f(1)}{a_0}.$$

$=24$



$f(x) = (x+1)(x+\frac{1}{2})^2(x+\frac{1}{4})^2 x^2 (x-\frac{1}{4})^2 (x-\frac{1}{2})$

- X is t-design

$\Longleftrightarrow$

$$\frac{1}{|S^{n-1}|}\int_{S^{n-1}} f(x)\,d\omega = \frac{1}{|X|}\sum_{x\in X} f(x)$$

for $\forall$ polynomial $f(x) = f(x_1, x_2, \cdots, x_n)$ of degree $\leq t$.

- $s = \left|\{x\cdot y \mid x, y \in X,\ x \neq y\}\right|$

(= The number of distinct inner prod
in X

Thm (DGS)

$t \geq 2s-2 \Longrightarrow X$ has a structure of
Q-poly. assoc. scheme

- $t \leq 2s$, in general
- $t = 2s \Longrightarrow X =$ tight spherical $2s$-desi
- $t = 2s-1$, $X = -X$

§3. Optimal codes and Universally optimal codes

A finite set $X \subset S^{n-1} \subset R^n$ is called a spherical code (or code) in R^n.
A spherical code X in R^n is called optimal code if

$$\begin{aligned} &\mathrm{Min}\{d(x,y) | x,y \in X, x \neq y\} \\ &\geq \mathrm{Min}\{d(x,y) | x,y \in Y, x \neq y\} \end{aligned}$$

for any spherical code Y in R^n with $|X| = |Y|$.

Examples

(1) 6 points of a regular hexagon in R^2 is optimal (and unique). $t=5, \ s=3$

(2) 240 roots of type E_8 in R^8 is optimal (and unique). $t=7, \ s=4$

(3) 196560 min. vectors in Leech lattice in R^{24} is optimal (and unique). $t=11, \ s=6$

$t=3, \ s=4$

(4) 12 roots of type A_3 in R^3 is NOT optimal. 12 vertices of a regular icosahedron in R^3 is optimal (and unique). $t=5, \ s=3$

(5) 24 roots of type D_4 in R^4 is believed to be optimal (and unique), but it is still an open question whether it is actually optimal or not. (Determination of the kissing number $k(4) = 24$ is weaker than the optimality of the D_4 set.) $t=5, \ s=4$

Many codes which are conjectured to be optimal are known, but generally it is dfficult to prove the optimality rigorously. In R^3, optimal sets are determined for $|X| \le 12$ and $|X| = 24$, but still open for the other values of $|X|$.

We now consider universally optimal

codes, the concept defined by Cohn-Kumar.

Definitions. (i) A function $f : (0, 4] \longrightarrow \boldsymbol{R}$ is called <u>completely monotonic</u>, if f is in class C^∞ and $(-1)^k f^{(k)} \geq 0$ for all $k = 0, 1, 2, \ldots$.

(ii) A function $\alpha : [-1, 1) \longrightarrow \boldsymbol{R}$ is called <u>absolutely monotonic</u>, if α is in class C^∞ and $\alpha^{(k)} \geq 0$ for all $k = 0, 1, 2, \ldots$.

(Note that they are related by $\alpha(t) = f(2 - 2t)$.)

Remark. $f(r) = r^{-s}$ with $s > 0$ is completely monotonic.

Definitions. A spherical code X in R^n is called universally optimal, if

$$\mathrm{Min}\{f(||x-y||^2)|x,y \in X, x \neq y\}$$
$$\geq \mathrm{Min}\{f(||x-y||^2)|x,y \in Y, x \neq y\}$$

for any completely monotonic function f and for any spherical code Y in R^n with $|X| = |Y|$.

Equivalently, A spherical code X in R^n is called universally optimal, if

$$\mathrm{Min}\{\alpha(<x,y>)|x,y \in X, x \neq y\}$$
$$\geq \mathrm{Min}\{\alpha(<x,y>)|x,y \in Y, x \neq y\}$$

for any ~~completely~~ absolutely function α and for any spherical code Y in R^n with $|X| = |Y|$.

Remark. Universally optimal code X is optimal. Take the function

$$f(r) = r^{-s} \quad \text{with} \quad s \longrightarrow \infty.$$

Theorem (Cohn-Kumar) A spherical code X in R^n is universally optimal, if X is a t-design and s-distance set with $t \geq 2s - 1$ (or $t \geq 2s - 2$ if $X = -X$.)

Theorem (Cohn-Kumar) 120 points of regular 600 cell in R^4 is universally optimal. $t = 11$, $s = 7$

Theorem (Cohn-Elkies-Kumar) 24 roots of type D_4 in R^4 is NOT universally optimal. $t = 5$, $s = 4$

Conjecture (Cohn-Kumar) For each n, there are only finitely many universally optimal codes X in R^n.

To find universally optimal codes, and to classify them will be very interesting problems (Cohn-Kumar)

According to Cohn, Ballinger-Cohn-Glausiracusa-Morris, searched for candidates of universally optimal codes in R^n with $n \leq 100$ systematically, by computer experiments, and found the following two new candidates, which are association schemes.

(1) 40 point class 4 association scheme in R^{10}, and

(2) 64 point class 3 association scheme in R^{14}.

Cohn asked whether each of these association schemes are unique, i.e., whether they are characterized by the parameters (intersection numbers). The main purpose of my talk is to answer Cohn's question. That is, each of them is in fact unique!

Part 2. Uniqueness of Certain Association Schemes. (joint work with Etsuko Bannai and Hideo Bannai.)

§4. Uniqueness of the 40 point class 3 association scheme in $\boldsymbol{R}^{10}$.

This association scheme was first described by Ballinger-Cohn-Glausiracusa-Morris as follows.

The relation matrix is given by the

following matrix of size 40 by 40:

$$R = \sum_{i=1}^{4} i A_i = \begin{bmatrix}
01112222 & 34443444 & 34443444 & 34443444 & 34443444 \\
10112222 & 43444344 & 43444344 & 43444344 & 43444344 \\
11012222 & 44344434 & 44344434 & 44344434 & 44344434 \\
11102222 & 44434443 & 44434443 & 44434443 & 44434443 \\
22220111 & 44434443 & 44344434 & 43444344 & 34443444 \\
22221011 & 34443444 & 43444344 & 44344434 & 44434443 \\
22221101 & 43444344 & 34443444 & 44434443 & 44344434 \\
22221110 & 44344434 & 44434443 & 34443444 & 43444344 \\
34444344 & 01112222 & 44344443 & 44434344 & 44344344 \\
43444434 & 10112222 & 44434434 & 34444434 & 34444443 \\
44344443 & 11012222 & 43443444 & 43444443 & 44433444 \\
44433444 & 11102222 & 34444344 & 44343444 & 43444434 \\
34444344 & 22220111 & 44434434 & 43444443 & 43444434 \\
43444434 & 22221011 & 44344443 & 44343444 & 44433444 \\
44344443 & 22221101 & 34444344 & 44434344 & 34444443 \\
44433444 & 22221110 & 43443444 & 34444434 & 44344344 \\
34444434 & 44434434 & 01112222 & 43444434 & 44434344 \\
43444344 & 44344443 & 10112222 & 44433444 & 34444434 \\
44343444 & 34444344 & 11012222 & 34444443 & 43444443 \\
44434443 & 43443444 & 11102222 & 44344344 & 44343444 \\
34444434 & 44344443 & 22220111 & 44344344 & 43444443 \\
43444344 & 44434434 & 22221011 & 34444443 & 44343444 \\
44343444 & 43443444 & 22221101 & 44433444 & 44434344 \\
44434443 & 34444344 & 22221110 & 43444434 & 34444434 \\
34444443 & 43444443 & 44344344 & 01112222 & 44434434 \\
43443444 & 44343444 & 34444443 & 10112222 & 44344443 \\
44344344 & 44434344 & 44433444 & 11012222 & 34444344 \\
44434434 & 34444434 & 43444434 & 11102222 & 43443444 \\
34444443 & 44434344 & 43444434 & 22220111 & 44344443 \\
43443444 & 34444434 & 44433444 & 22221011 & 44434434 \\
44344344 & 43444443 & 34444443 & 22221101 & 43443444 \\
44434434 & 44343444 & 44344344 & 22221110 & 34444344 \\
34443444 & 43444434 & 43444443 & 44344443 & 01112222 \\
43444443 & 44433444 & 44343444 & 44434434 & 10112222 \\
44344434 & 34444443 & 44434344 & 43443444 & 11012222 \\
44434344 & 44344344 & 34444434 & 34444344 & 11102222 \\
34443444 & 44344344 & 44434344 & 44434434 & 22220111 \\
43444443 & 34444443 & 34444434 & 44344443 & 22221011 \\
44344434 & 44433444 & 43444443 & 34444344 & 22221101 \\
44434344 & 43444434 & 44343444 & 43443444 & 22221110
\end{bmatrix}$$

where A_0, A_1, A_2, A_3, A_4 are the adjacency matrices corresponding to the relations R_0, R_1, R_2, R_3, R_4. This is an imprimitive association scheme with two kinds of blocks (system of imprimitivities) of size 4 and size 8.

The following equations are satisfied.

$$
\begin{aligned}
&A_1^2 = 3A_0 + 2A_1, \qquad A_2^2 = 4A_0 + 4A_1,\\
&A_3^2 = 8A_0 + 2A_2 + 2A_4,\\
&A_4^2 = 24A_0 + 16A_1 + 18A_2 + 12A_3 + 14A_4,\\
&A_1A_2 = 3A_2, \qquad A_1A_3 = A_4,\\
&A_1A_4 = 3A_3 + 2A_4, \quad A_2A_3 = A_3 + A_4,\\
&A_2A_4 = 3A_3 + 3A_4,\\
&A_3A_4 = 8A_1 + 6A_2 + 6A_3 + 4A_4.
\end{aligned}
$$

from the above equations, we can read off the parameters (intersection numbers) $p_{i,j}^k$ which are defined by $A_iA_j = \sum_k p_{i,j}^k A_k$.

This also gives a proof that these 40 point set forms an association scheme.

The first and the second eigen matrices are given by

$$
P = \begin{bmatrix} 1 & 3 & 4 & 8 & 24 \\ 1 & -1 & 0 & -4 & 4 \\ 1 & -1 & 0 & 2 & -2 \\ 1 & 3 & 4 & -2 & -6 \\ 1 & 3 & -4 & 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 10 & 20 & 4 & 5 \\ 1 & -\frac{10}{3} & -\frac{20}{3} & 4 & 5 \\ 1 & 0 & 0 & 4 & -5 \\ 1 & -5 & 5 & -1 & 0 \\ 1 & \frac{5}{3} & -\frac{5}{3} & -1 & 0 \end{bmatrix}
$$

One of the main purposes of my talk s to prove the following:

Theorem 2.1(BBB) The 40 point class 4 association scheme in R^{10} described above is uniquely determined by the parameters.

We give a brief sketch of the proof of Theorem 2.1 later in Section 6.

B. Group theoretical description of s association scheme is available Matan Ziv-Av and M. Klin, and Abdukhalikov, using GAP.

§5. Uniqueness of the class 3 association scheme of 64 points in R^{14}

It seems that this association scheme first appeared in de Caen- van Dam, 1999.

$X = F_8 \times F_8$, and the R_i $(1 \le i \le 3)$ are defined as follows:

For (α, x) and (β, y) in $X = F_8 \times F_8$,
$((\alpha, x), (\beta, y)) \in R_0$ if and only if $\alpha = \beta$ and $x = y$.
$((\alpha, x), (\beta, y)) \in R_1$ if and only if $\alpha \ne \beta$ and $x = y$.
$((\alpha, x), (\beta, y)) \in R_2$ if and only if $x \ne y$ and either $\alpha + \beta = (x+y)^3$ or $\alpha + \beta = xy(x+y)$, and
$((\alpha, x), (\beta, y)) \in R_3$ otherwise.

Then we have:

$$A_1^2 = 7A_0 + 6A_1, \qquad A_2^2 = 14A_0 + 2A_1 + 4A_3,$$
$$A_3^2 = 42A_0 + 30A_1 + 24A_2 + 28A_3$$
$$A_1A_2 = A_2 + 2A_3, \qquad A_1A_3 = 6A_2 + 5A_3,$$
$$A_2A_3 = 12A_1 + 12A_2 + 8A_3.$$

and

$$P = Q = \begin{bmatrix} 1 & 7 & 14 & 42 \\ 1 & 7 & -2 & -6 \\ 1 & -1 & -6 & 6 \\ 1 & -1 & 2 & -2 \end{bmatrix}.$$

So, we can represent the 64 points of X in the unit sphere in R^{14}, and $(\alpha, x), (\beta, y) \in R_i$ $(0 \leq i \leq 3)$ if and only if the Euclidean inner products $< (\alpha, x), (\beta, y) >$ are $1, -\frac{1}{7}, -\frac{3}{7}, \frac{1}{7}$, respectively.

Theorem 3.1 (BBB) This 64 point class 3 association scheme in R^{14} described above is uniquely determined by the parameters.

3. Group theoretical description of this
ociation scheme is also available.
Ziv-Av. and M. Klin and K. Abdukhalikov.

§6. Sketch of proof of Theorem 2.1.

Let X be an association scheme with the same parameters as the 40 point class 4 association scheme (in R^{10}). Let $X = \{v_1, \ldots, v_{40}\} \subset S^9 \subset R^{10}$. Then, $(v_i, v_j) \in R_k$ (with $0 \leq k \leq 4$) if and only if the Euclidean inner product $< v_i, v_j >= 1, -\frac{1}{3}, 0, -\frac{1}{2}, \frac{1}{6}$, respectively.

(We denote $a = -\frac{1}{2}, b = \frac{1}{6}, c = -\frac{1}{3}$ in what follows where 0 and 1 remain themselves.)

The main idea of the proof of Theorem 2.1 is to show that, if we denotes the Gram matrix

$$G = [< v_i, v_j >]_{1 \leq i \leq 40, \ 1 \leq j \leq 40},$$

then G is determined uniquely up to orthogonal transformation in R^{10}.

Since the imprimitivities of an association schemes are determined by the parameters $p_{i,j}^k$ of the association scheme only (see e.g., B-I), without loss of generality, we may assume that Gram matrix G of the association scheme is given as follows,

$$G = \begin{bmatrix} G_{1,1} & \cdots & G_{1,10} \\ \vdots & \ddots & \vdots \\ G_{10,1} & \cdots & G_{10,10} \end{bmatrix}$$

where $G_{i,j}$, $(1 \leq i, j \leq 10)$ are matrices of size 4×4. The diagonal 10 blocks satisfy

$$G_{i,i} = \begin{bmatrix} 1 & c & c & c \\ c & 1 & c & c \\ c & c & 1 & c \\ c & c & c & 1 \end{bmatrix} (1 \leq i \leq 10)$$

and $G_{i,i+1}, G_{i+1,i}$, $(i = 1, 3, 5, 7, 9)$ are the 0 matrice of size 4×4. Also, it is easy to see that we may reorder the vectors in such a way that

$$G_{1,3} = G_{2,3} = \begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix}$$

holds. Here, note that $G_{1,i} (3 \leq i \leq 10)$, as well as $G_{2,i} (3 \leq i \leq 10)$, must be permutation matrices, since the parameter $p_{3,3}^1 = 0$, and R_3 is of valency 8.

So, without loss of generality, we may assume that the 12×12 (most upper left) submatrix of G is of the following form:

$$\begin{bmatrix}
1 & c & c & c & 0 & 0 & 0 & 0 & a & b & b & b \\
c & 1 & c & c & 0 & 0 & 0 & 0 & b & a & b & b \\
c & c & 1 & c & 0 & 0 & 0 & 0 & b & b & a & b \\
c & c & c & 1 & 0 & 0 & 0 & 0 & b & b & b & a \\
0 & 0 & 0 & 0 & 1 & c & c & c & a & b & b & b \\
0 & 0 & 0 & 0 & c & 1 & c & c & b & a & b & b \\
0 & 0 & 0 & 0 & c & c & 1 & c & b & b & a & b \\
0 & 0 & 0 & 0 & c & c & c & 1 & b & b & b & a \\
a & b & b & b & a & b & b & b & 1 & c & c & c \\
b & a & b & b & b & a & b & b & c & 1 & c & c \\
b & b & a & b & b & b & a & b & c & c & 1 & c \\
b & b & b & a & b & b & b & a & c & c & c & 1
\end{bmatrix}$$

The rank of the above 12×12 matrix is 9. So, we can construct first 12 points $v_1, \ldots, v_{12}$ in $\mathbb{R}^9 \subset \mathbb{R}^{10}$.

We can see that they are represented as follows.

$$
\begin{array}{llllllllll}
v_1 = (\frac{\sqrt{3}}{3}, & \frac{\sqrt{3}}{3}, & \frac{\sqrt{3}}{3}, & 0, & 0, & 0, & 0, & 0, & 0, & 0) \\
v_2 = (-\frac{\sqrt{3}}{3}, & -\frac{\sqrt{3}}{3}, & \frac{\sqrt{3}}{3}, & 0, & 0, & 0, & 0, & 0, & 0, & 0) \\
v_3 = (-\frac{\sqrt{3}}{3}, & \frac{\sqrt{3}}{3}, & -\frac{\sqrt{3}}{3}, & 0, & 0, & 0, & 0, & 0, & 0, & 0) \\
v_4 = (\frac{\sqrt{3}}{3}, & -\frac{\sqrt{3}}{3}, & -\frac{\sqrt{3}}{3}, & 0, & 0, & 0, & 0, & 0, & 0, & 0) \\
v_5 = (0, & 0, & 0, & \frac{\sqrt{3}}{3}, & \frac{\sqrt{3}}{3}, & \frac{\sqrt{3}}{3}, & 0, & 0, & 0, & 0) \\
v_6 = (0, & 0, & 0, & -\frac{\sqrt{3}}{3}, & -\frac{\sqrt{3}}{3}, & \frac{\sqrt{3}}{3}, & 0, & 0, & 0, & 0) \\
v_7 = (0, & 0, & 0, & -\frac{\sqrt{3}}{3}, & \frac{\sqrt{3}}{3}, & -\frac{\sqrt{3}}{3}, & 0, & 0, & 0, & 0) \\
v_8 = (0, & 0, & 0, & \frac{\sqrt{3}}{3}, & -\frac{\sqrt{3}}{3}, & -\frac{\sqrt{3}}{3}, & 0, & 0, & 0, & 0)
\end{array}
$$

and

$$
\begin{array}{llllllllll}
v_9 = (-\frac{\sqrt{3}}{6}, & -\frac{\sqrt{3}}{6}, & -\frac{\sqrt{3}}{6}, & -\frac{\sqrt{3}}{6}, & -\frac{\sqrt{3}}{6}, & -\frac{\sqrt{3}}{6}, & \frac{\sqrt{6}}{6}, & \frac{\sqrt{6}}{6}, & \frac{\sqrt{6}}{6}, & 0) \\
v_{10} = (\frac{\sqrt{3}}{6}, & \frac{\sqrt{3}}{6}, & -\frac{\sqrt{3}}{6}, & \frac{\sqrt{3}}{6}, & \frac{\sqrt{3}}{6}, & -\frac{\sqrt{3}}{6}, & -\frac{\sqrt{6}}{6}, & -\frac{\sqrt{6}}{6}, & \frac{\sqrt{6}}{6}, & 0) \\
v_{11} = (\frac{\sqrt{3}}{6}, & -\frac{\sqrt{3}}{6}, & \frac{\sqrt{3}}{6}, & \frac{\sqrt{3}}{6}, & -\frac{\sqrt{3}}{6}, & \frac{\sqrt{3}}{6}, & -\frac{\sqrt{6}}{6}, & \frac{\sqrt{6}}{6}, & -\frac{\sqrt{6}}{6}, & 0) \\
v_{12} = (-\frac{\sqrt{3}}{6}, & \frac{\sqrt{3}}{6}, & \frac{\sqrt{3}}{6}, & -\frac{\sqrt{3}}{6}, & \frac{\sqrt{3}}{6}, & \frac{\sqrt{3}}{6}, & \frac{\sqrt{6}}{6}, & -\frac{\sqrt{6}}{6}, & -\frac{\sqrt{6}}{6}, & 0)
\end{array}
$$

Let v be an element of X, and not among the first 12 points $v_1, \ldots, v_{12}$.

Then v must satisfy the following conditions:
(1),(2),(3), (if v and v_9 do not belong to a same block of size 8). and the conditions:
(1),(2),(3bis), (if v and v_9 belong to a same block of size 8)
(1) v is of inner product $-\frac{1}{2}$ with one of v_1, v_2, v_3, v_4 and of inner product $\frac{1}{6}$ with remaining 3 of them.
(2) v is of inner product $-\frac{1}{2}$ with one of v_5, v_6, v_7, v_8 and of inner product $\frac{1}{6}$ with remaining 3 of them.
(3) v is of inner product $-\frac{1}{2}$ with one of $v_9, v_{10}, v_{11}, v_{12}$ and of inner product $\frac{1}{6}$ with remaining 3 of them.
(3bis) v is of inner product 0 with any of $v_9, v_{10}, v_{11}, v_{12}$.

There are 48 vectors v satisfying the properties (1),(2) and (3), and there are 28 vectors v satsfying the properties (1),(2) and (3bis).
We can describe these 88=12+48+~~12~~ 28 vectors explicitly, and we can calculate the inner products among these

38 vectors explicitly. X must consist of $40=12+24+4$ vectors, namely, 12 first vectors, ~~28~~ vectors satsfying the properties (1),(2) and (3), and 4 vectors satsfying the properties (1),(2) and (3bis). Also, all the inner products among these 40 vectors must be in $\{1, -\frac{1}{3}, 0, -\frac{1}{2}, \frac{1}{6}\}$. We determine that there are only two such 40 point subsets X with the above properties.

Moreover, one of these two sets of 40 vectors is obtained from the other by changing the last (10th) coordinates to their negatives. That is, the sets of these 40 vectors are transformed to each other by an orthogonal transformation. This implies the uniqueness of the 40 point association scheme we are considering. QED.

Remark. The uniqueness of the 64 point class 3 association scheme was obtained by a similar, but more involved, method. We omit the details.

§7. Concluding Remarks.

The methods to prove Theorem 2.1 and Theorem 3.1 are similar in philosophy with the proof of the uniqueness of $(24, 196560, 1/2)$ spherical codes, i.e., the kissing configuration in R^{24} or tight 11-design in $S^{24} \subset R^{24}$. (See Bannai-Sloane, 1981.)
(An alternative proof of this fact was also obtained recently by Munemasa).
The uniqueness of $(23, 4600, 1/3)$-code in R^{23} was also obtained by Bannai-Sloane (1981).

Cohn-Kumar recently gave a proof of the uniqueness of $(22, 891, 1/4)$-code in R^{22}. The result was originally obtained as the uniqueness of the distance-regular graph (regular near polygon) wit the intersection array:

$$\left\{ \begin{matrix} * & 1 & 5 & 21 \\ 0 & 1 & 5 & 21 \\ 42 & 40 & 32 & * \end{matrix} \right\}$$

H. Cuypers (2004) gave a uniqueness proof of $(23, 4600, 1/3$ code in R^{23}, using the uniqueness of $(22, 891, 1/4)$-code in R^{22}.

It seems the uniqueness proof of $(21, 336, 1/5)$-code in R^{21} is still open.

In a recent mail, H. Cohn wrote me that he found two more new candidates of universally optimal spherical codes. Namely, $126 = 72(E_6) + 54(E_6^*)$ points in R^6 and $182 = 126(E_7) + 56(E_7^*)$ points in R^7.

It would be interesting to consider highe dimensional analogues of the two association schemes: 40 point class 4 association scheme in R^{10} and 64 point association scheme in R^{14}.

Challenge! Either construct or prove the non-existence of association scheme with the following parameters.

$$|X| = p = 2713 , \quad d = 4$$

$$B_i = (p_{ij}^k)_{\substack{0 \le j \le 4 \\ 0 \le k \le 4}}$$

$$B_0 = \begin{bmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & 1 & & \\ & 0 & & 1 & \\ & & & & 1 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 678 & 179 & 162 & 160 & 176 \\ 0 & 162 & 164 & 185 & 167 \\ 0 & 160 & 185 & 166 & 167 \\ 0 & 176 & 167 & 167 & 168 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 162 & 164 & 185 & 167 \\ 678 & 164 & 170 & 168 & 175 \\ 0 & 185 & 168 & 167 & 158 \\ 0 & 167 & 175 & 158 & 178 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 160 & 185 & 166 & 167 \\ 0 & 185 & 168 & 167 & 158 \\ 678 & 166 & 167 & 170 & 174 \\ 0 & 167 & 158 & 174 & 179 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 176 & 167 & 167 & 168 \\ 0 & 167 & 175 & 158 & 178 \\ 0 & 167 & 158 & 174 & 179 \\ 678 & 168 & 178 & 179 & 152 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 678 & 678 & 678 & 678 \\ 1 & 4.33683\cdots & 16.00668\cdots & 17.00680\cdots & -38.35031\cdots \\ 1 & 34.78472\cdots & -19.05304\cdots & -21.05501\cdots & 4.32332\cdots \\ 1 & -19.24058\cdots & -25.56322\cdots & 26.99771\cdots & 16.80609\cdots \\ 1 & -20.88097\cdots & 27.60958\cdots & -23.94950\cdots & 16.22088\cdots \end{bmatrix}$$

$$x^4 + x^3 - 1017x^2 - 9665x + 60608$$

$$x^4 + x^3 - 1017x^2 - 1526x + 215249$$

$$x^4 + x^3 - 1017x^2 - 1526x + 231527$$

$$x^4 + x^3 - 1017x^2 + 14752x - 45199$$

Based on the work of

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