# Solving linear systems of differential and difference equations with respect to a part of the unknowns

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Based on a recent paper of S.Abramov and M.Bronstein, published in *Computational Mathematics and Mathematical Phisics*, 2006, **46**, No 2, pp. 218–230.

http://www.ccas.ru/sabramov/publications.htm

## Solutions in specified classes of functions

Let $K$ be a differential field and $S$ a system of linear differential equations

$$y' = Ay, \tag{1}$$

$y = (y_1, \ldots, y_n)^T$, $A = Mat_n(K)$.

Depending on $K$, there are several known algorithms for computing the solutions of such systems in various classes of functions, such as polynomial, rational, hyperexponential, algebraic or Liouvillian functions. But those algorithms only find solutions $y$ whose components are *all* in the specified class.

## A computer algebra problem

In this presentation, we address the following problem: given a subset $\{y_{i_1}, \ldots, y_{i_m}\}$ of the components of $y$ and an appropriate class $F$ (i.e., a $K$-vector space, closed under differentiation) of functions, find all solutions $y$ whose specified components are in the given class (more precisely we are interested in computing those components only).

For example, given a differential system (1), $K = \mathbb{C}(x)$, find all the rational functions that are $y_1$ and $y_2$-components of a solution $y$.

In computer algebra, solutions in the form of a power series are also of interest (an algorithm for constructing such a solution must be able to produce any number of the series terms). An example is provided by regular formal solutions to a differential system, where, e.g., $K$ is the field of rational functions with complex coefficients. In a neighborhood of a fixed point $x_0$, these solutions have the form $(x - x_0)^\lambda \sum_{\nu=0}^{k} \varphi_\nu(x - x_0) \log^\nu(x - x_0)$, where $\varphi_\nu(x - x_0)$ are the Laurent series and $\lambda \in \mathbb{C}$. In a neighborhood of an irregular singular point, the system does not always have a solution with all regular components; in this context, it makes sense to consider the problem discussed in this presentation.

We also can consider the classes of functions that are not represented in a “closed” form even in the broadest sense of this word; for example, we can consider meromorphic functions.

## Algorithm specification

We present an algorithm that produces either one of two possible results:

- The conclusion that if the specified components of a solution belong to the given class $F$, then all the other components also belong to this class.
- A new system of a smaller order in which the specified components of a solution to the initial system and some of their derivatives are the unknowns; if the specified components of the initial system belong to the given class $F$, then all the other components of the new system also belong to this class.

In the rest of the presentation we suppose that $(i_1, \ldots, i_m) = (1, \ldots, m)$. So we shall be concerned with the components $y_1, \ldots, y_m$, $m \leq n$, of the solutions of the system $S$.

## $m$-span, $m$-rank

Let $e_1, \ldots, e_n$ be the standard basis of the $K$-space of linear forms in $y_1, \ldots, y_n$ over $K$:

$$e_1 = y_1, \ldots, e_n = y_n.$$

We can differentiate any linear form $f$ from this space, using the initial system $S$.

Identifying any linear form with the column vector of its coefficients, we obtain for the derivatives along the solutions to the system $S$ the representation

$$f^{(0)} = f, \;\; f^{(i)} = A^T f^{(i-1)} + \frac{d}{dx} f^{(i-1)}, \;\; i = 1, 2, \ldots, \qquad (2)$$

where $\frac{d}{dx} f^{(i-1)}$ is the result of the the componentwise differentiation of the column vector $f^{(i-1)}$.

The dimension $l$ of the $K$-space $Span_m(S)$ of linear forms, generated by $e_1, \dots, e_m$ and by all the linear forms $e_i^{(j)}$, $i = 1, \dots, m$, $j = 1, 2, \dots$, is the *$m$-rank* of $S$.

# Admissible partitions of the $m$-rank

If the non-negative integer $l_1, \ldots, l_m$ are such that

$$\begin{array}{c} e_1,\ e_1^{(1)},\ \ldots\ , e_1^{(l_1-1)}, \\ e_2,\ e_2^{(1)},\ \ldots\ , e_2^{(l_2-1)}, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ e_m,\ e_m^{(1)},\ \ldots\ , e_m^{(l_m-1)}, \end{array} \tag{3}$$

form a basis

$$f_1, \ldots, f_l$$

of $Span_m(S)$ where $l = l_1 + \cdots + l_m$, then the $m$-tuple $(l_1, l_2, \ldots, l_m)$ is an *admissible partition* of the $m$-rank $l$ of the system $S$.

**Example 1**. Consider the system $y' = Ay$ consisting of three equations with constant coefficients in which

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The 2-rank $l$ of this system is equal to three, and the only admissible partition of $l$ is $(2, 1)$. For example, the partition $(1, 2)$ is inadmissible because $e_2' = e_2$. On the other hand, the 3-rank of this system has three admissible partitions: $(2, 1, 0)$, $(0, 1, 2)$, and $(1, 1, 1)$.

## New unknowns $z_1, \ldots, z_l$

The derivative of any form from $f_1, \ldots, f_l$ is a $K$-linear combination of the linear forms $f_1, \ldots, f_l$. Therefore, if we introduce new unknowns $z_1, \ldots, z_l$, that are expressed in terms of $y_1, \ldots, y_n$ by the linear forms $f_1, \ldots, f_l$ (i.e., $z_i = f_i(y_1, \ldots, y_n), i = 1, \ldots, l$), then we get a new system

$$z' = Bz,$$

$z = (z_1, \ldots, z_l)^T$, $B \in Mat_l(K)$.

## Connection between $y' = Ay$ and $z' = Bz$

It is possible to extend the basis $f_1, \dots, f_l$ to a basis

$$f_1, \dots, f_l, f_{l+1}, \dots, f_n$$

of the space of all the linear forms in $y_1, \dots, y_n$. If we use the linear forms from this basis to introduce new unknowns $Z = (z_1, \dots, z_l, z_{l+1}, \dots, z_n)$, then this change of unknowns has the form

$$Z = Hy$$

with a non-singular matrix $H \in Mat_n(K)$. This transforms the original system $S$ into

$$Z' = \begin{pmatrix} B & 0 \\ U & V \end{pmatrix} Z. \tag{4}$$

Note that $m$ of the unknowns of $z_1, \ldots, z_l$ are equal to $y_1, \ldots, y_m$ (since $e_1 = y_1, \ldots, e_m = y_m$ belong to basis (3); the transformation $H$ does not change values of these unknowns).

**Proposition 1** *(i) The projections of the set of solutions to the initial system $S$ and to the system $z' = Bz$ on $y_1, \ldots, y_m$ are identical.*

*(ii) If a solution to the system $z' = Bz$ is such that its components $y_1, \ldots, y_m$ belong to the given class $F$, then all the components of this solution belong to this class.*

*(iii) If the $m$-rank of the initial system is $n$ and this system has a solution with the components $y_1, \ldots, y_m$ belonging to the given class $F$, then all the components of this solution belong to this class.*

## Algorithm

Computation of the $m$-rank of the initial system and the construction of the system $z' = Bz$ play a key role in the solution of the problem.

The algorithm is as follows. If the $m$-rank of the initial system is $n$, then the algorithm concludes that, since the specified components of the solution to the system belong to the given class, then all the components also belong to this class. If the $m$-rank of the initial system is less than $n$, then set up the system $z' = Bz$, find its solutions with all the components belonging to the given class, and produce the corresponding subset of the components of these solutions (removing the derivative components).

## Consequences

1) An amusing consequence of this (taking the specified class to be equal to $\{0\}$) is that a nonzero solution of an irreducible (over $K$) system must have *all* of its components nonzero.

2) The original system has a nonzero solution with zero components $y_1, \ldots, y_m$ iff its $m$-rank is less than $n$.

## The minimal scalar equation for $y_1$

If $m = 1$ then the matrix $B$ is of the form

$$\begin{pmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ g_0 & g_1 & \cdots & g_{l-1} \end{pmatrix}$$

where $l$ is equal to the 1-rank of the initial system. In this case, the set of solutions to the system $z' = Bz$ obviously coincides with the set of the vectors $(u, u', \ldots, u^{(l-1)})^{\mathrm{T}}$, where $u$ ranges over all the solutions to the equation $Lu = 0$ with

$$L = \frac{d^l}{dx^l} - \sum_{i=0}^{l-1} g_i \frac{d^i}{dx^i}.$$

Applying Proposition 1 to the case $m = 1$, we see that any $y_1 = u$, where $u$ is a solution to the equation $Lu = 0$, can be extended to become a solution of the system $y' = Ay$.

Thus, we have the following result.

**Proposition 2** *All the solutions to the equation $Lu = 0$ and only these solutions are the $y_1$-components of all the solutions $y$ of the system $y' = Ay$.*

The statement in the book *W. Wasow. Asymptotic expansions for ordinary differential equations*) to the effect that an equation of order not exceeding $n$ with respect to $y_j$ "usually has many solutions which are not the $j$-th component of a solution of initial system" is true only when the order of this equation is not minimal.

**Example 2**.

$y' = Ay;$

$$A = \begin{pmatrix} 3\,x & -1-2\,x & 2\,x \\ (1-x-x^2)/x^2 & -1/x^2 & (1-x-x^2)/x^2 \\ (1-x-3\,x^3)/x^2 & (x^2-1+2\,x^3)/x^2 & (1-x-2\,x^3)/x^2 \end{pmatrix},$$

$y = (y_1, y_2, y_3)^T, \quad K = \mathbb{Q}(x).$

Assume we are interested in all rational $y_1, y_2$. Since the 2-rank of this system is equal to 3, any solution of $y' = Ay$ with $y_1$ and $y_2$ rational functions must have all its components rational functions.

Using, e.g., Maple shows that there are no nonzero rational solutions, so there are no nonzero solutions for which $y_1$ and $y_2$ are both rational functions.

However, assume that we are interested in all rational $y_2$. Then the corresponding 1-rank is 2, and we construct a new system $z' = Bz$;

$$B = \begin{pmatrix} 0 & 1 \\ (x+3)/(x^3+x^2-x) & (x+3)(1-x)/(x^3+x^2-x) \end{pmatrix},$$

$z = (z_1, z_2)^T$, $z_1 = y_2, z_2 = y_2'$.

The rational solutions of the system $z' = Bz$ can be found by a direct algorithm.

The system is equivalent to the scalar equation $Ly_2 = 0$ where

$$L = (x - x^2 - x^3)\frac{d^2}{dx^2} + (x+3)(1-x)\frac{d}{dx} + x + 3\,.$$

By Proposition 2, the projection of the solution space of $y' = Ay$ on $y_2$ is exactly the solution space of $Ly_2 = 0$, which turns out to be generated by $x - 1$ and $x^{-1} + x^{-2}$. This means that $y_2$ is a rational function for all solutions of $y' = Ay$, and that for any constants $c_1, c_2$, there is a solution $y$ such that $y_2 = c_1(x-1) + c_2(x^{-1} + x^{-2})$. In this particular example, this can be verified by computing the 3-dimensional space of Liouvillian solutions of $y' = Ay$ (which is too unwieldy to print here).

The space of solutions to this system for which $y_2 = 0$ is one-dimensional and is generated by the solution

$$y = e^{x^2/2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} .$$

## Difference systems

The results discussed above are also valid for difference systems if the differentiation is replaced by the unit shift $E$ with respect to the independent variable: $E\varphi(x) = \varphi(x+1)$. The sequential shifts $f^{[0]}, f^{[1]}, \ldots$ of the linear form $f$ along the solutions of the system $Ey = Ay$ are defined similarly to the sequential derivatives in the differential case; by analogy with formulas for the differential case, we have

$$f^{[0]} = f, \quad f^{[i]} = A^T(Ef^{[i-1]}), \quad i = 1, 2, \ldots,$$

where $Ef^{[i-1]}$ is the result of the componentwise shift of the column vector $f^{[i-1]}$ (again, we identify every linear form with the column vector of its coefficients in the basis $y_1, \ldots, y_n$).

**Example 3.**

Consider the system $Ey = Ay$ with

$$A = \begin{pmatrix} 1 & -\frac{1}{x(x+1)^2} & -x & 1 \\ 1 & \frac{x^2+2x-1}{x(x+1)} & x & -1 \\ 1 & 0 & 1 & 0 \\ x+1 & 0 & 0 & \frac{x+1}{x} \end{pmatrix},$$

$y = (y_1, y_2, y_3, y_4)^T, \quad K = \mathbb{Q}(x).$

Assume that we are interested in the rational $y_1$ and $y_2$. The general rational solution of the initial system is the product of an arbitrary constant and the vector $(0,\,0,\,1,\,x)^T$.

The 2-rank of this system is 3, and we can construct the new system $Ez = Bz$, where $B$ is

$$\begin{pmatrix} 0 & 1 & 0 \\ -\frac{x^4+6x^3+13x^2+14x+4}{x(x+1)(x+2)^2} & \frac{2x^4+11x^3+21x^2+17x+4}{x(x+1)(x+2)^2} & \frac{3x^2+6x+2}{x^2(x+1)^3(x+2)} \\ 2 & -1 & \frac{x^3+3x^2+x-2}{x(x+1)^2} \end{pmatrix},$$

$z = (z_1, z_2, z_3)^T$, $z_1 = y_1, z_2 = Ey_1, z_3 = y_2$.

The general rational solution of the latter system is the product of an arbitrary constant and the vector $(1/x,\ 1/(x+1),\ x+1)^T$. Both components $y_1$ and $y_2$ of the solution to the initial system are rational functions if and only if $y_1 = c/x$, $y_2 = c \cdot (x+1)$, where $c$ is a constant.

## Adjustable algorithm

The algorithms for constructing the new system with the matrix $B$ of the order coinciding with the $m$-rank of the initial system in the differential and the difference cases have much in common. They can be described in a unified way using the Ore noncommutative polynomials. For this reason, the algorithm can also be used in the $q$-difference case. In such a generalized form, the algorithm was implemented by D.E. Khmel'nov in Maple. The matrices $B$ for the systems considered in examples for the differential and the difference cases can be constructed by the same procedure, which is adjusted to one or another ring of Ore polynomials using some special parameters (the procedure can be adjusted, e.g., to the differential and the difference cases, respectively).

## Constant coeffs; partial stability of zero solution

The direct application of the Routh–Hurwitz criterion does not help to analyze the stability of the zero solution to the differential system $y' = Ay$ with a constant real matrix $A$ with respect to $y_1, \ldots, y_m$ (this is a well-known problem in classical mechanics).

**Example 4**. Consider the system $y' = Ay$ with

$$A = \begin{pmatrix} -1 & 1 & -2 \\ 4 & 1 & 0 \\ 5/2 & 1 & -1 \end{pmatrix} .$$

Since the characteristic polynomial has a root 1, the solution $y_1 = y_2 = y_3 = 0$ is Lyapunov unstable. Suppose that we want to analyze the stability of this system with respect to $y_1$.

## $\mu$-Transformation method

Applying the $\mu$-transformation method by Vorotnikov and Prokopiev (1978), we define $\mu_1 = y_2 - 2y_3$ in the first equation

$$y_1' = -y_1 + y_2 - 2y_3$$

of the initial system. Then, we obtain

$$\mu_1' = -y_1 - y_2 + 2y_3.$$

Since the 1-rank of the initial system is 2, the unknowns $y_2, y_3$ can be eliminated. We have

$$y_1' = -y_1 + \mu_1,$$

$$\mu_1' = -y_1 - \mu_1.$$

In the general case, it can be needed to use $\mu_1, \mu_2, \ldots$ instead of $\mu$.

The characteristic polynomial of the last system is $-\lambda^2 - 2\lambda - 2$; its roots are $-1 + i$, $-1 - i$. The real part of each of them is negative; therefore, the solution $y_1 = y_2 = y_3 = 0$ to the initial system is asymptotically stable with respect to $y_1$.

## The matrix $B$ instead of the $\mu$-transformation

The consideration of the system $z' = Bz$ described in this presentation can be used instead the $\mu$-transformation. In the example under consideration, we have

$$B = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix},$$

$z_1 = y_1, z_2 = y_1'$, and the roots of the characteristic polynomial are $-1 + i$, $-1 - i$, as in the case of the $\mu$-transformation.

## Structure of the matrix $B$

In the general case, the sparse form of the matrix $B$ is convenient when the characteristic polynomial must be found.

We have seen that if $m = 1$ then the matrix of the system $z' = Bz$ is a companion matrix

$$\begin{pmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ g_0 & g_1 & \cdots & g_{l-1} \end{pmatrix}, \tag{5}$$

Consider now the case $m > 1$.

$B$ is an $l \times l$-matrix of the following form (we consider the case $m = 3$ for simplicity):

$$
\begin{pmatrix}
0 & 1 & & & & & & & & & & \\
 & \ddots & \ddots & & & & & & & & & \\
 & & 0 & 1 & & & & & & & & \\
* & \dots & * & * & * & * & \dots & * & * & \dots & * & * \\
 & & & & 0 & 1 & & & & & & \\
 & & & & & \ddots & \ddots & & & & & \\
 & & & & & & 0 & 1 & & & & \\
* & \dots & * & * & * & \dots & * & * & * & * & \dots & * \\
 & & & & & & & & 0 & 1 & & \\
 & & & & & & & & & \ddots & \ddots & \\
 & & & & & & & & & & 0 & 1 \\
* & \dots & * & * & * & \dots & * & * & * & \dots & * & *
\end{pmatrix}
\tag{6}
$$

where $*$'s denote arbitrary elements of the field $K$. If $A$ in the initial system is a constant $n \times n$ matrix, then $B$ is also constant.

If we compute the characteristic polynomial, then there is even no need for writing out and storing the entire matrix (6); it is sufficient to find only the $m$ rows of this matrix depicted by the asterisks and construct the $m \times m$ matrix $(p_{ij}(\lambda))$.

**Proposition 3** *Suppose that the system $S$ with constant entries has the $m$-rank $l$, ($l \geq m \geq 1$), the space* $\mathrm{Span}_m(S)$ *has basis (3), and the vector of the coefficients of the expansion $e_s^{(l_s)}$ in terms of this basis is $(d_{s1}, \ldots, d_{sl})$. Then, the characteristic polynomial of the matrix $B$ produced by the proposed algorithm coincides, up to a sign, with the determinant of the $m \times m$ matrix $(p_{ij}(\lambda))$, in which*

$$p_{ij} = d_{i,k_{j-1}+1} + d_{i,k_{j-1}+2}\lambda + \cdots + d_{i,k_j}\lambda^{l_j-1} - \lambda^{l_j},$$

*where $k_0 = 0$, $k_j = l_1 + \cdots + l_j$, $j = 1, 2, \ldots, m$.*

Proposition 3 is a generalization of the fact that if matrix

$$\begin{pmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ g_0 & g_1 & \cdots & g_{l-1} \end{pmatrix},$$

is constant, then its characteristic polynomial is equal to $g_0 + g_1\lambda + \cdots + g_{l-1}\lambda^{l-1} - \lambda^l$ up to a sign.

For the case of constant coefficients, G.V. Plotnikov implemented the proposed algorithm for constructing the matrix $B$ and for finding the characteristic polynomial in Maple. Experiments showed that Plotnikov's program computes the characteristic polynomial of a matrix $B$ of order 20 with $m = 3$ at least two and a half times faster than the standard Maple procedures.

## Other uses of the $m$-rank

The quantity that we called $m$-rank is used in mechanics and control theory (although defined in a different manner) for the investigation of the observability via unknowns $y_1, \dots, y_m$ a system $y' = Ay$, $A \in Mat_n(K)$ in terms of the variables $y_1, \dots, y_m$: in this sense the system is observable iff its $m$-rank is $n$.

The concept of an admissible partition of $m$-rank, which was proposeded in this presentation, can find other applications. We consider its application in the theory of autonomous systems.

## Non-linear autonomous systems

Consider an autonomous system

$$
\begin{aligned}
y_1' &= f_1(y_1, \ldots, y_n), \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
y_n' &= f_n(y_1, \ldots, y_n),
\end{aligned}
\tag{7}
$$

such that $y_1(x) = \ldots = y_n(x) = 0$ is its solution, i.e., $f_i(0, \ldots, 0) = 0$, $i = 1, \ldots, n$. Assume that the functions $f_1, \ldots, f_n$ are sufficiently smooth when $|y_i| < r$, $i = 1, \ldots, n$, where $r$ is a positive real number.

## $m$-rank of autonomous systems

For $0 < m \leq n$, define the $m$-rank of system (7) as the $m$-rank (over $\mathbb{R}$ or $\mathbb{C}$) of its linearization $y' = Cy$ ($C$ is a constant $n \times n$-matrix).

**Proposition 4** *Suppose that the $m$-rank of system (7) is $l$ and $(l_1, \dots, l_m)$ an admissible partition of $l$. Then there exist $\rho$, $0 < \rho < r$, such that for any $a_1, \dots, a_m$ satisfying the inequalities $|a_i| < \rho$, $i = 1, \dots, m$, system (7) has a solution $(y_1, \dots, y_n)$ for which*

$$y_i(0) = a_i, \quad i = 1, \dots, m,$$

*and*

$$y_i^{(j)}(0) = 0, \quad i = 1, \dots, m, \ j = 1, \dots, l_i - 1.$$

# Remarks to Proposition 4

Consider the linear system $y' = Ay$ of 3 equations with constant coefficients:

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

whose general solution is

$$y_1 = c_1 e^x + c_2 e^{-x}, \quad y_2 = c_3 e^x, \quad y_3 = c_1 e^x - c_2 e^{-x}.$$

The 2-rank $l$ of this system is 3, and the only admissible partition of $l$ is $(2,1)$. In accordance with Proposition 4 for any $a_1, a_2$ the system has the solution

$$y_1 = \frac{a_1}{2} e^x + \frac{a_1}{2} e^{-x}, \quad y_2 = a_2 e^x; \quad y_3 = \frac{a_1}{2} e^x - \frac{a_1}{2} e^{-x},$$

such that $y_1(0) = a_1, y_1'(0) = 0, y_2(0) = a_2$.

## Remarks to Proposition 4 (continued)

Now consider the partition $(1,2)$ of $l$, which is not admissible. If $a_2 \neq 0$, then $y_2(0) = a_2$ implies $y_2'(0) \neq 0$ (because $y_2'(x) = y_2(x)$).

This demonstrates the importance of the admissibility of a partition of $l$ in the hypothesis of Proposition 4.

In the linear (not necessarily autonomous) case, Proposition 4 is valid in a more general form: there exists a solution such that $y_i^{(j)}(0)$, $i = 1, 2, \ldots, m$, $j = 0, 1, \ldots, l_i - 1$, have arbitrarily specified values.

In the difference case, the modified version of Proposition 4 states that, if the $m$-rank of a difference autonomous system is $l$ and $(l_1, \ldots, l_m)$ is an admissible partition of $l$, then there exists a $\rho$, $0 < \rho < r$, such that, for any $a_1, \ldots, a_m$, satisfying the inequalities $|a_i| < \rho$, $i = 1, 2, \ldots, m$, the system has a solution $(y_1, \ldots, y_n)^T$, for which

$$y_i(0) = a_i, \quad i = 1, \ldots, m,$$

and

$$E^j y_i(0) = 0, \quad i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, l_i - 1,$$

that is,

$$y_i(0) = a_i, \;\; y_i(1) = \cdots = y(l_i - 1) = 0, \quad i = 1 \ldots, m.$$

It is also possible to use $\Delta = E - 1$ instead of $E$, which leads to another solution satisfying the conditions

$$\Delta^j y_i(0) = 0, \quad i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, l_i - 1,$$

that is,

$$y_i(0) = y_i(1) = \cdots = y_i(l_i - 1) = a_i, \quad i = 1, 2, \ldots, m.$$