APPROXIMATE BUCHBERGER-MÖLLER ALGORITHM IN CoCoA

John Abbott
Claudia Fassino
Laura Torrente

Recall exact Buchberger–Möller Algorithm

Input: finite set of points $X \subseteq K^n$

Output: reduced G-basis $G \subset K[x_1, \ldots x_n]$ of ideal of polys vanishing at each point in X.

Method: consider PPs in increasing order

evaluate PP at each point in X

<u>if</u> vector of values is lin. dep. on QB

<u>then</u> get elem. of $LT(G)$

<u>else</u> get new elem of quotient basis, QB.

Call QB the STRUCTURE of the result.

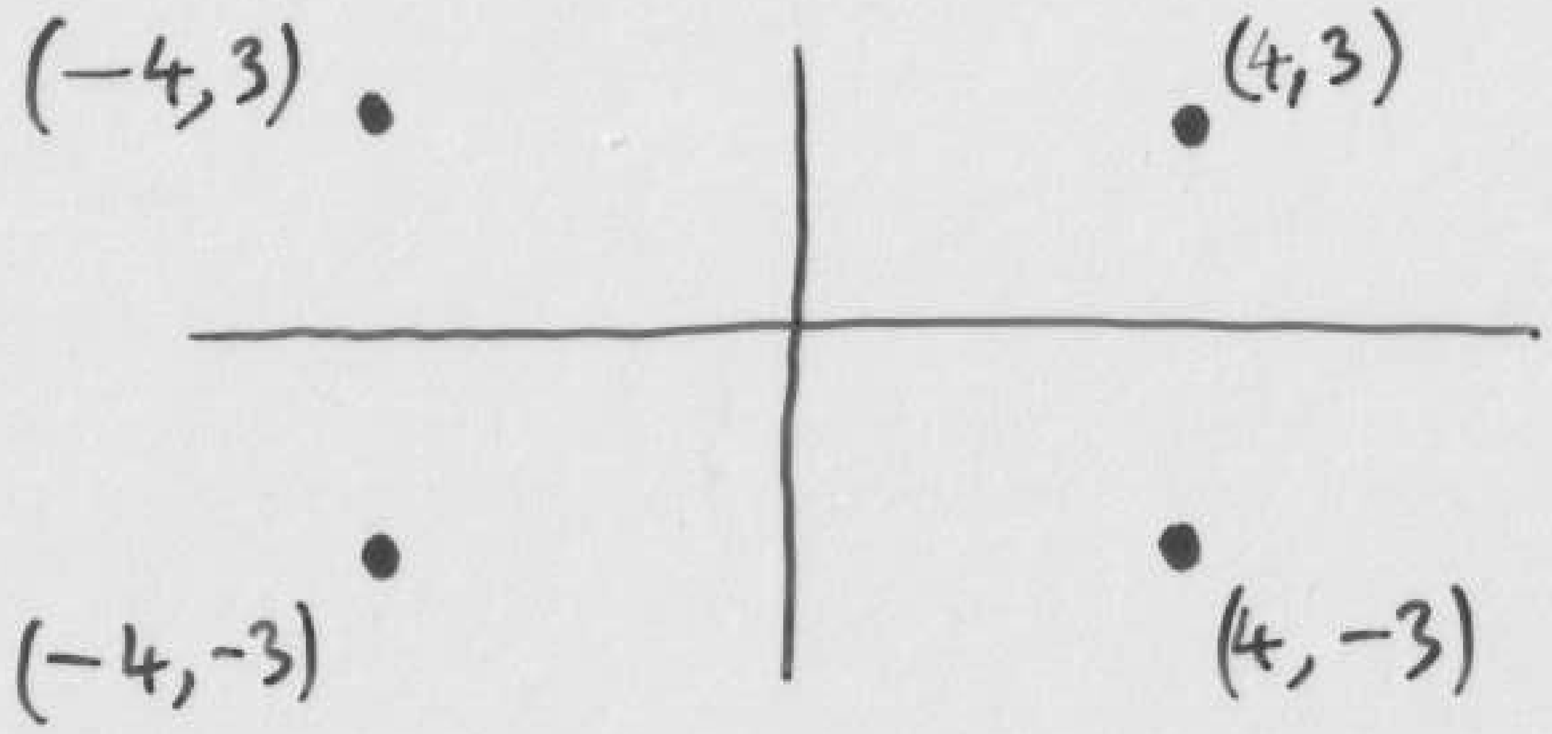
The structure uniquely determines the whole result.

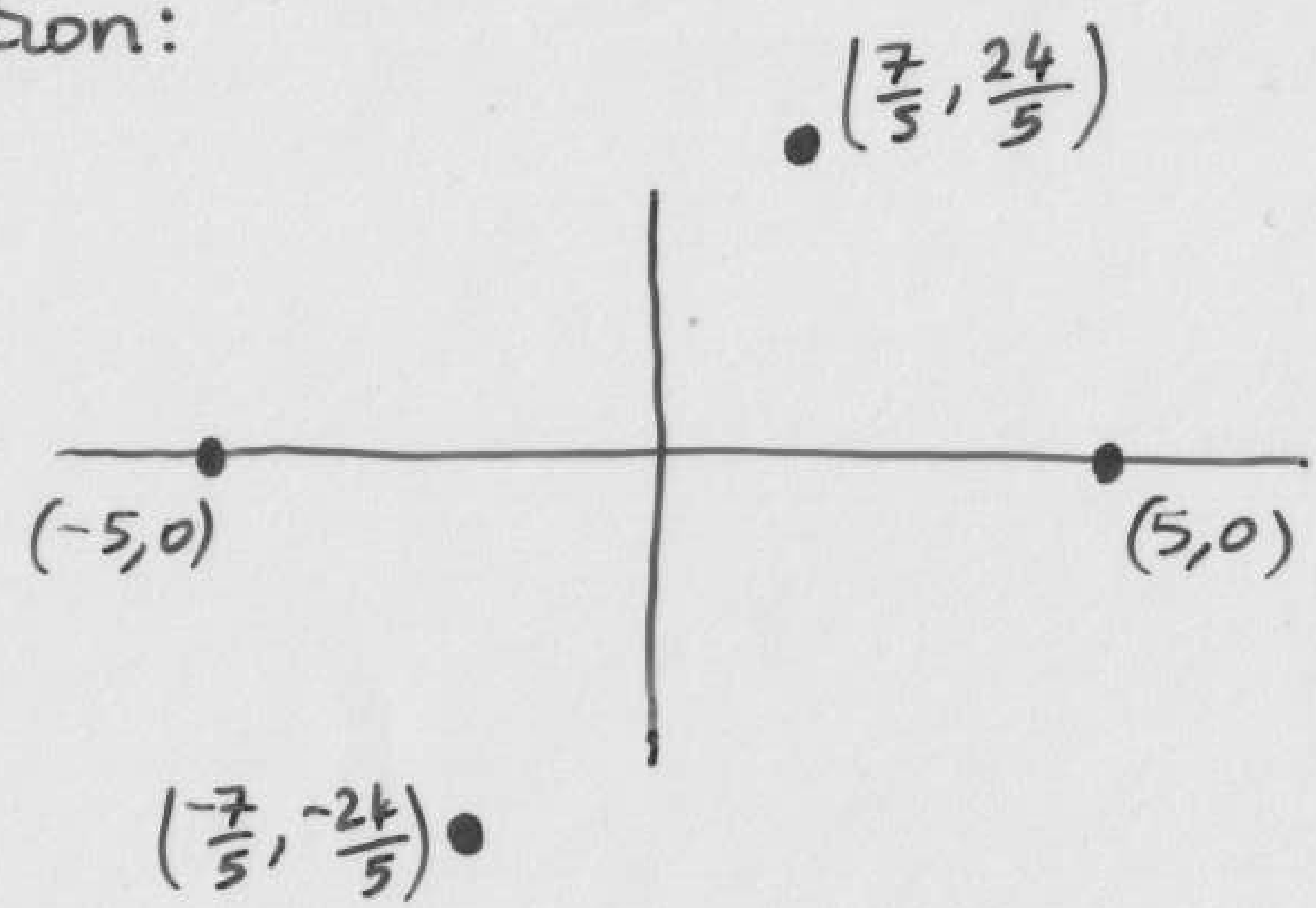
Preservation of Structure

Structure is preserved under scaling and translation

e.g. change of units °F $\longleftrightarrow$ °C

But it is not preserved under rotation:





GBasis $\begin{cases} x^2 - 16 \\ y^2 - 9 \end{cases}$

QB
$$\begin{matrix} y & xy \\ 1 & x \end{matrix}$$

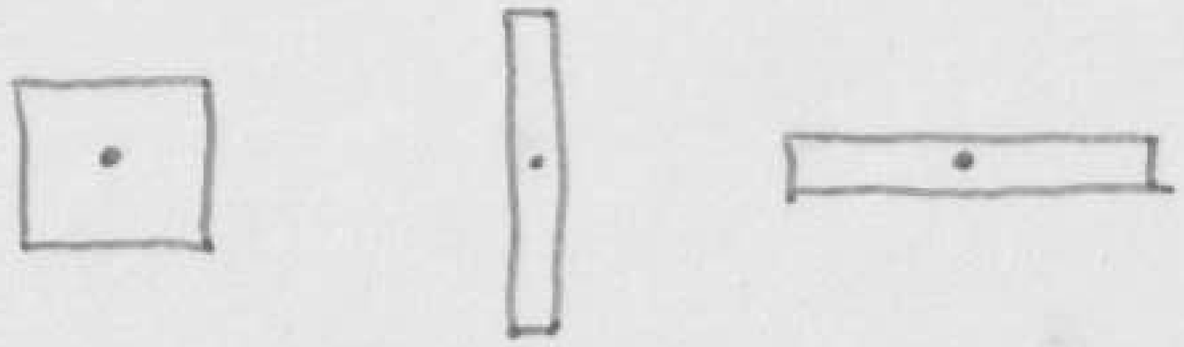
GBasis $\begin{cases} x^2 + y^2 - 25 \\ xy - \frac{7}{24} y^2 \\ y^3 - \frac{576}{25} y \end{cases}$

QB
$$\begin{matrix} y^2 & \\ y & \\ 1 & x \end{matrix}$$

What is an approximate point?

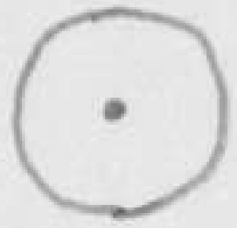
Here are some natural candidates:

error boxes



axially aligned

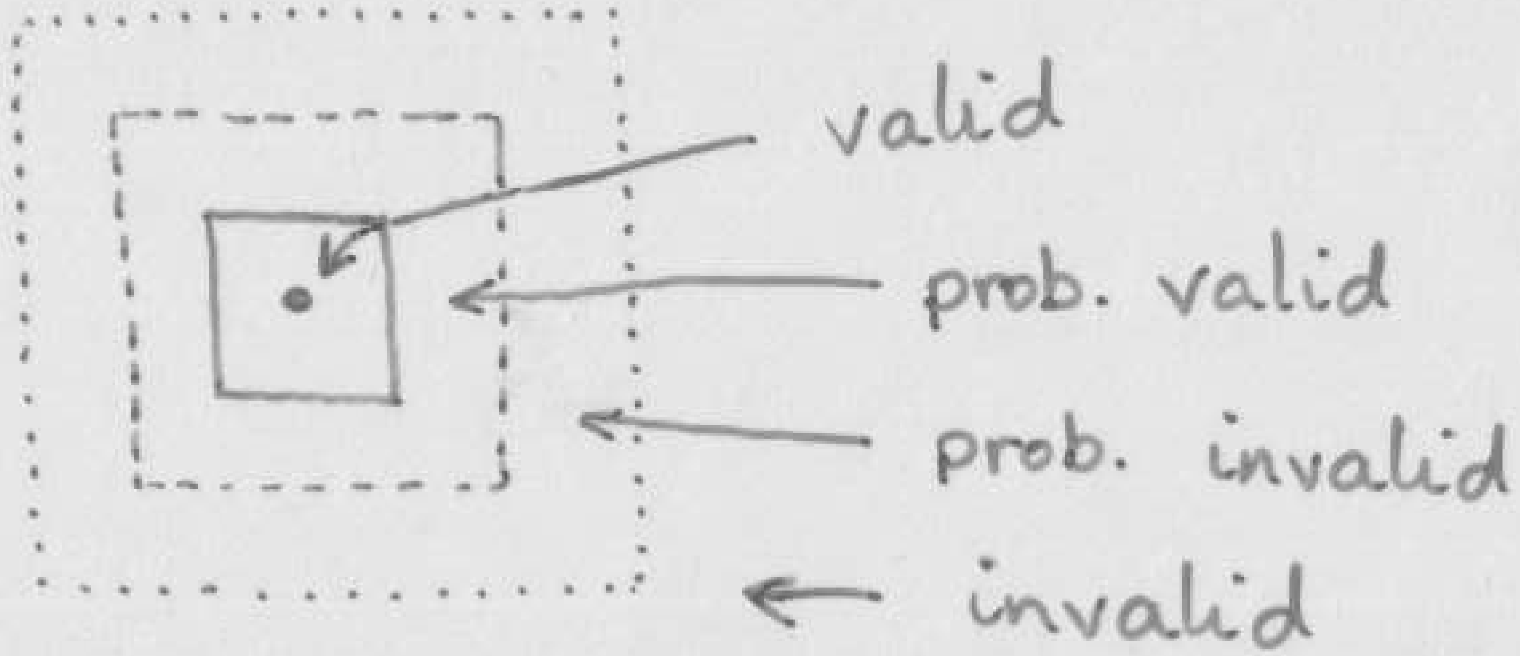
error balls



probability distribution

non-unif. density

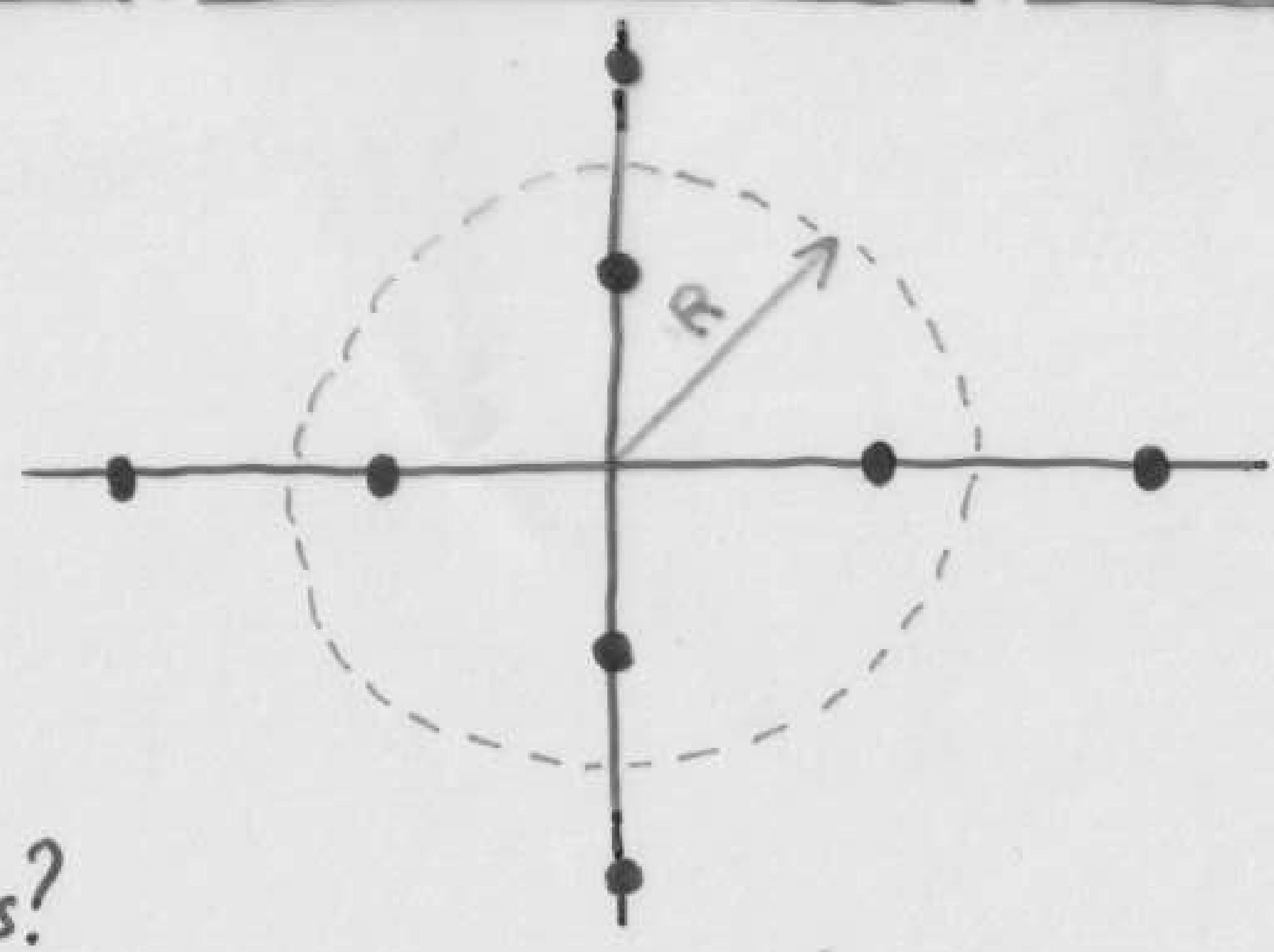
soft-edged boxes



How closely does a poly. pass by a given set of points?

Example: 8 points in $\mathbb{R}^2$

$(1, 0)$	$(2, 0)$
$(0, 1)$	$(0, 2)$
$(-1, 0)$	$(-2, 0)$
$(0, -1)$	$(0, -2)$



Qn: what radius circle best fits these points?

A1 Using the measure $\sum |\text{distance}|$

All radii $R \in [1,2]$ give best fit.

A2 Using measure $\sum |\text{distance}|^2$

Unique optimum $R = 1.5$

intuitively correct

A3 Using measure $\sum |\text{value}|$

All radii $R \in [1,2]$ give best fit.

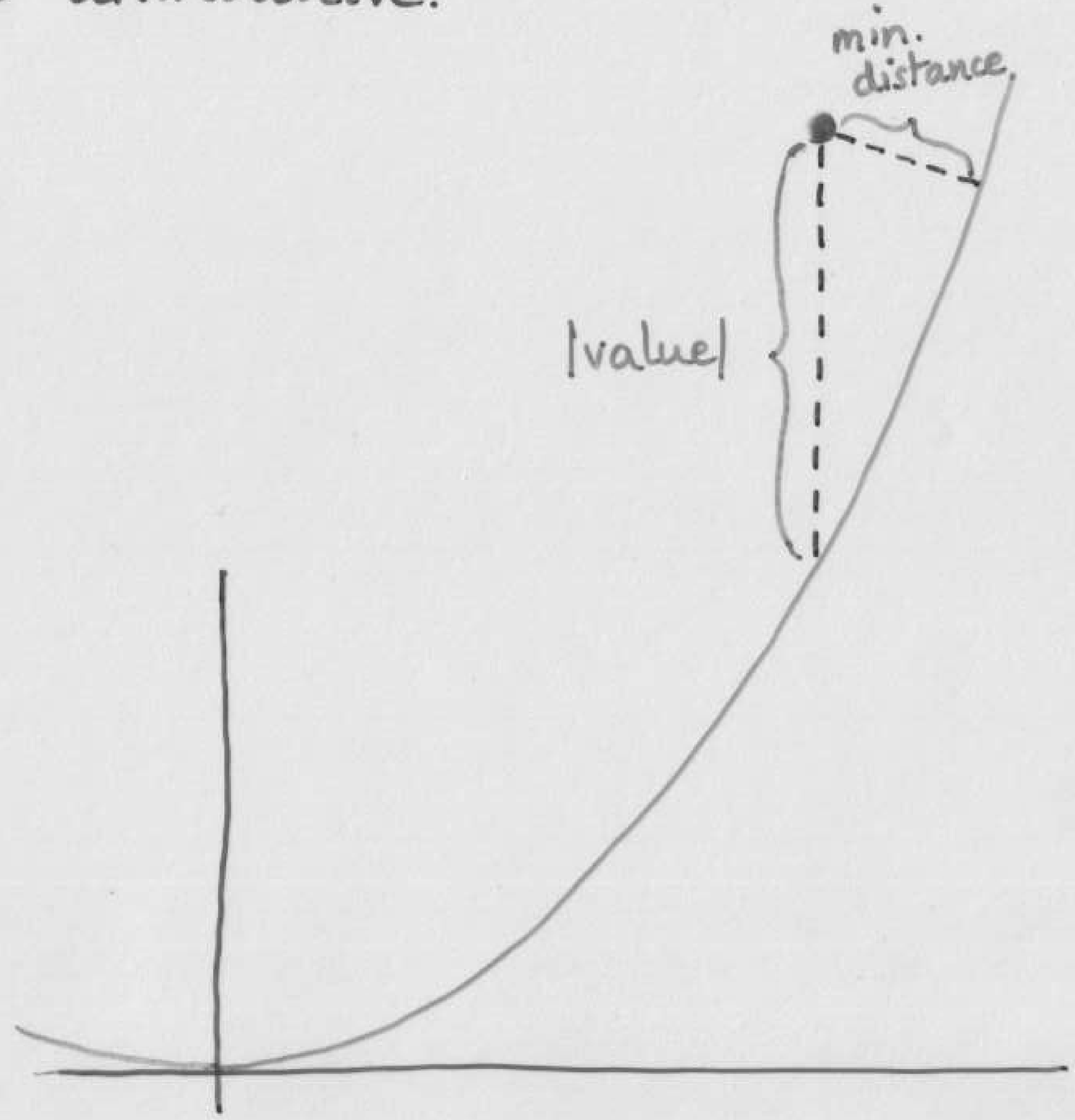
A4 Using measure $\sum |\text{value}|^2$

Unique optimum $R = \sqrt{5/2} \simeq 1.58$

How closely ... ? (cont'd)

We choose **A4.** $\sum |\text{value}|^2$ is our choice because it cooperates with B-M algm.

BUT this measure is unintuitive.



$y - x^2 = 0$

Bigger discrepancy as the curve gets steeper!

Preprocessing of the input points

- coord. values close to zero are forced equal to zero.

  usually gives "cleaner" results.

- coord. values close to each other are collapsed to a single central value;
  if there is $x \longleftrightarrow -x$ symmetry, collapse abs. values (e.g. -9.9 & 10.1).

- if x and y coords have the same units, apply collapsing to both sets of coord values together

  may help identify $x \leftrightarrow y$ symmetry

- close clusters of input points are replaced by a single central point.

  this changes the number of input points.

Preprocessing (cont'd)

Input: $(0.95, 0.94)$ $(1.04, -1.045)$ box width ~ 0.1

$(-0.95, -0.955)$ $(-1.048, 1.051)$

Without: GBasis $\left\{ \begin{array}{l} y^2 + 0.011x + 0.003y - 0.998 \\ x^2 + 0.094xy + 0.004x - 0.003y - 0.987 \end{array} \right.$

With: GBasis $\left\{ \begin{array}{l} y^2 - 0.991 \\ x^2 - 0.991 \end{array} \right.$ a cleaner result.

Note: cannot postprocess GBasis to make it cleaner.

Definition of Approximately Linearly Dependent

Given approx. vector v

basis of approx vectors $v_1, v_2, \dots v_k$

Decide if v is approx. lin. dep. on $v_1, v_2 \cdots v_k$

Criterion

$$\frac{\|v^\perp\|_2}{\|v\|_2} < \varepsilon$$

where $v^\perp$ is orthog. compt. of v w.r.t. $\mathrm{span}\{v_1, \dots v_k\}$

In our case we know where the vectors come from.

Each vector comprises the values of a known poly at known points.

Repr. each vector compt. as 1st order Taylor expn

Use knowledge of error box widths to estimate variation in $v^\perp$, and deduce whether $v^\perp = 0$ is possible.

Approx. B-M Algorithm

SAME as exact B-M algm except:

- test for approx lin. dep.
- principal output is the set QB

factor closed set of PPs & a basis for interpolation

From QB we can compute a set $\overline{GB}$ of polys

- elems of $\overline{GB}$ take on small values at the given points
- $\overline{GB}$ has a similar shape to a Gröbner basis
- we believe $\overline{GB}$ is close to a G-basis of points close to those given. (proved for "non-degenerate" cases).

Example

6 points almost on a circle

$(10,0)$ $(-10,0)$ $(0,10)$ $(0,-10)$ $(7,6.9)$ $(-7.12,-7)$

Without preprocessing:

$$\text{QB} \quad \begin{array}{ll} y^3 & \\ y^2 & \\ y & xy \\ 1 & x \end{array} \qquad \text{GBasis} \begin{cases} x^2 + 0.04xy + y^2 + 0.06x + 0.06y - 100 \\ xy^2 + 0.98y^3 + 0.23xy - 98y \\ y^4 - 0.11y^3 + 51xy - 100y^2 + 11y \end{cases}$$

roots are less than 0.03 from the input points

With preprocessing

$$\text{QB} \quad \begin{array}{ll} y^3 & \\ y^2 & \\ y & xy \\ 1 & x \end{array} \qquad \text{GBasis} \begin{cases} x^2 + 0.03xy + y^2 - 100 \\ xy^2 + 0.97y^3 - 97y \\ y^4 + 51xy - 100y^2 \end{cases}$$

roots = preprocessed points

Approx. B-M Algm. and Border Bases

Idea construct QB one elem. at a time,
at each step choose candidate which
makes the assoc matrix most lin. indep.

When QB is complete, compute coeffs
of polys in BB using linear algebra.

if input points are not well separated then we use least sqs. for BB coeffs.

Definition of most lin. indep.

1. maximize $\|v^\perp\|_2 / \|v\|_2$
2. maximize $\|v^\perp\|_2 - \max \|\Delta v^\perp\|_2$
3. maximize $\|v^\perp\|_2 / \max \|\Delta v^\perp\|_2$

other?