Garden of curves with many automorphisms

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joint work with Massimo Giulietti

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Outline

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For further developments in specific questions and for effective constructions we need the potential of Finite Group Theory.

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- $|\mathrm{Aut}(\mathcal{X})| < 8\mathrm{g}^3$; up to four exceptions. [Henn (1976)]

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(II) The $\mathrm{Roquette}$ curve: $\mathbf{v}(Y^2-(X^q-X))$ with $p>2$, a hyperelliptic curve of genus $g=\frac{1}{2}(q-1)$; $\mathrm{Aut}(\mathcal{X})/\mathrm{M} \cong \mathrm{PSL}(2,\mathrm{q})$ or $\mathrm{Aut}(\mathcal{X})/\mathrm{M} \cong \mathrm{PGL}(2,\mathrm{q})$, where $q=p^r$ and $|M|=2$;

(III) The $\mathrm{Hermitian}$ curve: $\mathbf{v}(Y^n+Y-X^{n+1})$ with $n=p^r$, genus $\frac{1}{2}n(n-1)$, $\mathrm{Aut}(\mathcal{X}) \cong \mathrm{PGU}(3,\mathrm{n})$, n a power of 2. $|\mathrm{Aut}(\mathcal{X})| = (\mathrm{n}^3+1)\mathrm{n}^3(\mathrm{n}^2-1)$.

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(IV) The DLS curve (Deligne-Lusztig curve of Suzuki type): $\mathbf{v}(X^{n_0}(X^n+X)+Y^n+Y)$, with $p=2$, $n_0=2^r\geq 2$, $n=2n_0^2$, $g=n_0(n-1)$, $\mathrm{Aut}(\mathcal{X}) \cong \mathrm{Sz}(n)$ where $\mathrm{Sz}(n)$ is the Suzuki group, $|\mathrm{Aut}(\mathcal{X})| = (n^2+1)n^2(n-1)$

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(V) The DLR curve (the Deligne-Lusztig curve arising from the Ree group):
$\mathbf{v}(Y^{n^2} - [1 + (X^n - X)^{n-1}]Y^n + (X^n - X)^{n-1}Y - X^n(X^n - X)^{n+3n_0})$, with $p = 3$, $n_0 = 3^r$, $n = 3n_0^2$;
$g = \frac{3}{2}n_0(n-1)(n+n_0+1)$; $\mathrm{Aut}(\mathcal{X}) \cong \mathrm{Ree(n)}$ where $\mathrm{Ree}(n)$ is the Ree group, $|\mathrm{Aut}(\mathcal{X})| = (\mathrm{n}^3 + 1)\mathrm{n}^3(\mathrm{n} - 1)$.

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(VI) The G.K curve:
$\mathbf{v}(Y^{n^3+1} + (X^n + X)(\sum_{i=0}^{n}(-1)^{i+1}X^{i(n-1)})^{n+1})$, a curve of genus $g = \frac{1}{2}(n^3+1)(n^2-2)+1$ with $\mathrm{Aut}(\mathcal{X})$ containing a subgroup isomorphic to $\mathrm{SU}(3,n)$, $n = p^r$.
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- Problem 3: $\exists$ simple or almost simple groups, other than those in the examples (II),. . . (VI), occurring as an automorphism group of a maximal curve?

Problems on zero p-rank curves with very large p-group of automorphisms

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- If $\mathrm{Aut}(\mathcal{X})$ fixes no point and $|S| > pg/(p-1)$ then $\mathcal{X}$ is one of the curves (II) ... (VI). (Giulietti-K. 2010).

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Important partial classifications (under further conditions) were given by Hering, Herzog, Aschbacher, and more recently by Guralnick-Pries-Stevenson.

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Furthermore, either $G = O(G) \rtimes S_2$, *or* $G/O(G) \cong \mathrm{SL}(2, 3)$, *or* $G/O(G) \cong \mathrm{GL}(2, 3)$, *or* $G/O(G) \cong \mathcal{G}_{48}$.

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(c) *G fixes a point of $\mathcal{X}$, and $G = S_2 \rtimes H$, with a subgroup H of odd order.*

Corollary

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- *If G is a solvable, then the Hurwitz bound holds for G; more precisely $|G| \leq 72(g-1)$.*
- *If G is not solvable, then G is known and the possible genera of $\mathcal{X}$ are computed from the order of its commutator subgroup G' provided that G is large enough, namely whenever $|G| \geq 24g(g-1)$.*

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- Problem 7 (essentially) solved by Guralnick-Malmskog-Pries 2012.

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Theorem (Giulietti-K. 2009)

Let $p = 2$. Let $\mathcal{X}$ be an $\mathbb{F}_{q^2}$-maximal curve of genus $g \geq 2$. Then $\mathrm{Aut}(\mathcal{X})$ acts on $\mathcal{X}(\mathbb{F}_{q^2})$ as a transitive permutation group if and only if $\mathcal{X}$ is the Hermitian curve $\mathbf{v}(Y^n + Y - X^{n+1})$, with $q = n$.

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Let $p = 2$. *Let* $\mathcal{X}$ *be an* $\mathbb{F}_{q^2}$*-maximal curve of genus* $g \geq 2$. *Then* $\mathrm{Aut}(\mathcal{X})$ *acts on* $\mathcal{X}(\mathbb{F}_{q^2})$ *as a transitive permutation group if and only if* $\mathcal{X}$ *is the Hermitian curve* $\mathbf{v}(Y^n + Y - X^{n+1})$, *with* $q = n$.

- Problem 8: Prove a similar characterization theorem for the other "classical" maximal curves.

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$$|S| \leq \begin{cases} 4(\gamma-1) \quad \text{for} \quad p=2,\ \gamma>1 \\ \frac{p}{p-2}(\gamma-1) \quad \text{for} \quad p\neq 2,\ \gamma>1, \\ g-1 \quad \text{for} \quad \gamma=1. \end{cases}$$

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Hypothesis (I): $|S| > 2(g-1)$ *(and* $|S| \geq 8$*),*

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Problem 10: Find examples where S has not maximal class.

Case $p = 3$, Examples for small genera

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- If $|S| = 27$ then $S = UT(3, 3)$ and $\mathcal{X} = \mathbf{v}((X^3 - X)(Y^3 - Y) + c, Z^3 - Z - X^3Y + YX^3)$ with $c \in \mathbb{K}^*$, $g(\mathcal{X}) = 10$.

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- For $|S| = 81$ an explicit example: $S \cong \mathrm{Syl}_3(\mathrm{Sym}_9)$, $\mathcal{X} = \mathbf{v}((X^3 - X)(Y^3 - Y) + c, U^3 - U - X,$ $(U - Y)(W^3 - W) - 1, (U - (Y + 1))(T^3 - T) - 1)$ with $c \in \mathbb{K}^*$, $g(\mathcal{X}) = 28$.

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Corollary

F_N is an extremal function field w.r. Nakajima's bound.

Remark

If $N \trianglelefteq S$ *and* $[S : N] \geq 9$ *then* $\mathcal{X}/N$ *is also an extremal curve w.r.t. Nakajima's bound.*

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These groups and surjections form an inverse system.

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Problem 10: What about the arising profinite group (limit of the this inverse system)?

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- *Every central involution of S is inductive.*

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  - (ib) $S = (E \times \langle u \rangle) \rtimes \langle w \rangle$ *where* E *is cyclic group of order* $g-1$ *and* u *and* w *are involutions.*
- $|S| = 2g+2$, *and* $S = A \rtimes B$, A *is an elementary abelian subgroup of index* 2 *and* $B = 2$;
- *Every central involution of* S *is inductive.*

Involution $u \in Z(S)$ is inductive:= $S/\langle u \rangle$, viewed as a subgroup of $\mathrm{Aut}(\bar{\mathcal{X}})$ of the quotient curve $\bar{\mathcal{X}} = \mathcal{X}/\langle u \rangle$ satisfies the hypotheses of the theorem.

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Problem 11: Construct infinite family of curves of type (ib).