Good covering codes from algebraic curves

Massimo Giulietti

University of Perugia (Italy)

Special Semester on Applications of Algebra and Number Theory
Workshop 2: Algebraic curves over finite fields
Linz, 14 November 2013

$(\mathbb{F}_q^n, d)$ d Hamming distance $C \subset \mathbb{F}_q^n$

covering codes

$(\mathbb{F}_q^n, d)$ d Hamming distance $C \subset \mathbb{F}_q^n$

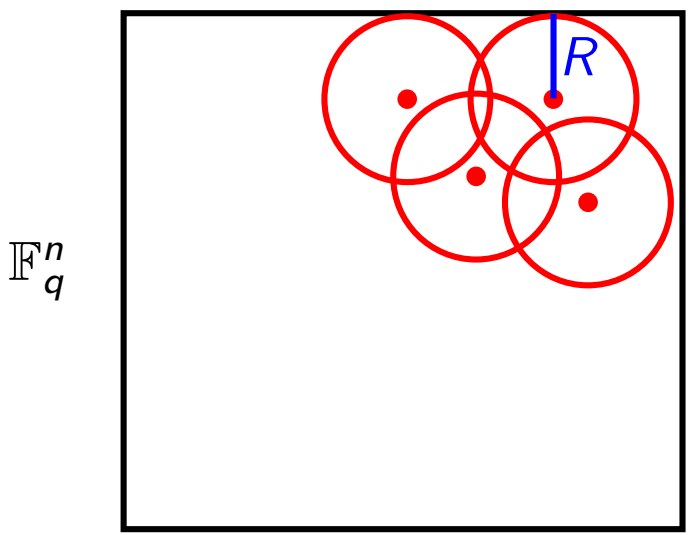
- covering radius of C

$$R(C) := \max_{v \in \mathbb{F}_q^n} d(v, C)$$

$(\mathbb{F}_q^n, d)$ d Hamming distance $C \subset \mathbb{F}_q^n$

- covering radius of C

$$R(C) := \max_{v \in \mathbb{F}_q^n} d(v, C)$$

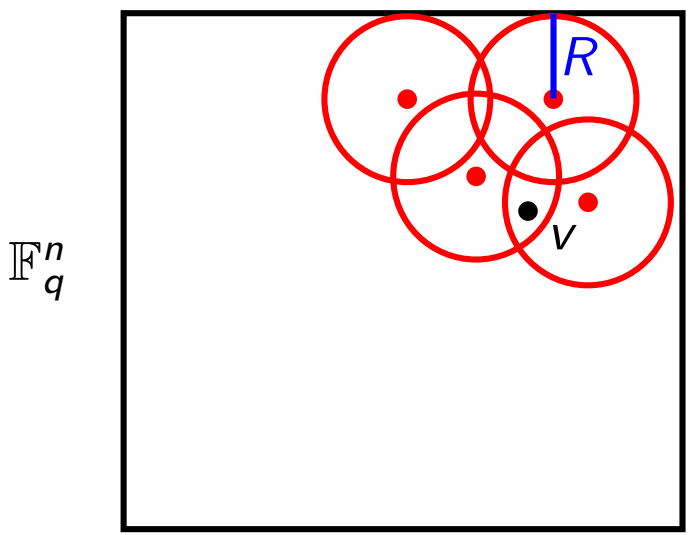


covering codes

$(\mathbb{F}_q^n, d)$ d Hamming distance $C \subset \mathbb{F}_q^n$

- covering radius of C

$$R(C) := \max_{v \in \mathbb{F}_q^n} d(v, C)$$



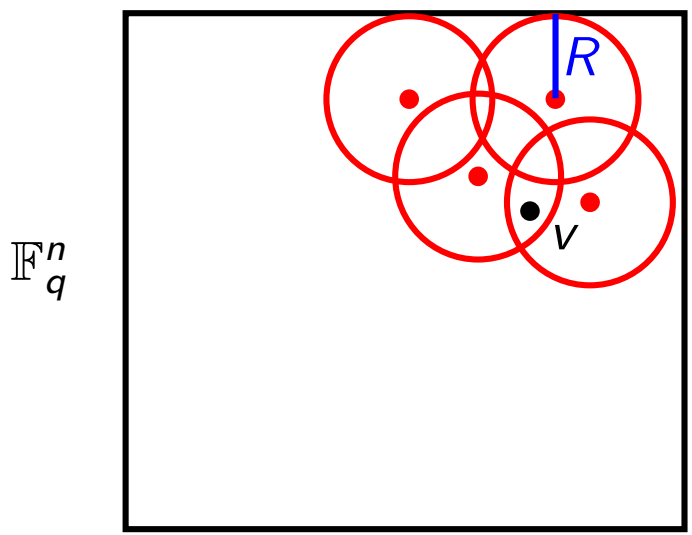
- covering density of C

$$\mu(C) := \#C \cdot \frac{\text{size of a sphere of radius } R(C)}{q^n}$$

$(\mathbb{F}_q^n, d)$ d Hamming distance $C \subset \mathbb{F}_q^n$

- covering radius of C

$$R(C) := \max_{v \in \mathbb{F}_q^n} d(v, C)$$



- covering density of C

$$\mu(C) := \#C \cdot \frac{\text{size of a sphere of radius } R(C)}{q^n} \geq 1$$

$k = \dim C \qquad r = n - k$

$$\mu(C) = \frac{1 + n(q-1) + \binom{n}{2}(q-1)^2 + \ldots + \binom{n}{R(C)}(q-1)^{R(C)}}{q^r}$$

$k = \dim C \qquad r = n - k$

$$\mu(C) = \frac{1 + n(q-1) + \binom{n}{2}(q-1)^2 + \ldots + \binom{n}{R(C)}(q-1)^{R(C)}}{q^r}$$

$$\ell(r,q)_R \quad := \quad \text{min } n \text{ for which there exists } C \subset \mathbb{F}_q^n \text{ with } R(C) = R, \quad n - \dim(C) = r$$

$k = \dim C \qquad r = n - k$

$$\mu(C) = \frac{1 + n(q-1) + \binom{n}{2}(q-1)^2 + \ldots + \binom{n}{R(C)}(q-1)^{R(C)}}{q^r}$$

$$\ell(r,q)_{R,d} := \quad \min\ n \text{ for which there exists } C \subset \mathbb{F}_q^n \text{ with } R(C) = R, \quad n - \dim(C) = r, \quad d(C) = d$$

$k = \dim C \qquad r = n - k$

$$\mu(C) = \frac{1 + n(q-1) + \binom{n}{2}(q-1)^2 + \ldots + \binom{n}{R(C)}(q-1)^{R(C)}}{q^r}$$

$$\ell(r,q)_{R,d} := \quad \min\, n \text{ for which there exists } C \subset \mathbb{F}_q^n \text{ with } R(C) = R, \quad n - \dim(C) = r, \quad d(C) = d$$

$R = 2, \quad d = 4 \qquad$ (quasi-perfect codes)

$R = r - 1, \quad d = r + 1 \qquad$ (MDS codes)

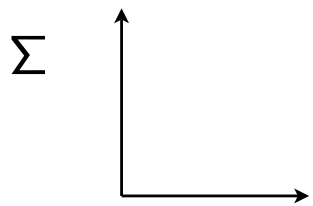
q odd

$$\ell(3, q)_{2,4}$$

in geometrical terms...

$$\Sigma = \Sigma(2, q)$$

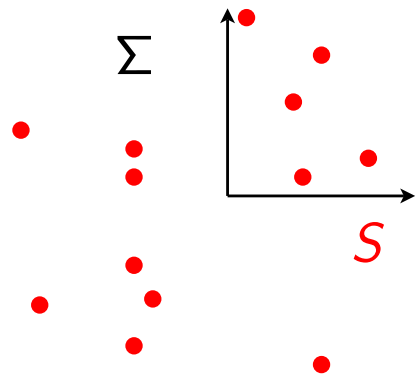
Galois plane over the finite field $\mathbb{F}_q$



in geometrical terms...

$$\Sigma = \Sigma(2, q)$$

Galois plane over the finite field $\mathbb{F}_q$

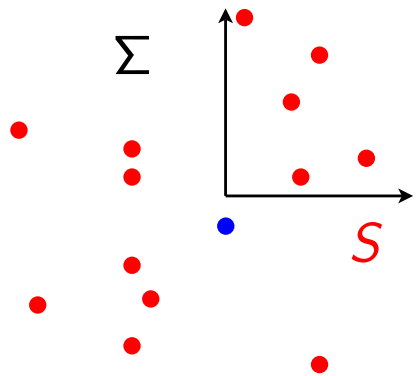


- $S \subset \Sigma$ is a saturating set if every point in $\Sigma \setminus S$ is collinear with two points in S

in geometrical terms...

$$\Sigma = \Sigma(2, q)$$

Galois plane over the finite field $\mathbb{F}_q$

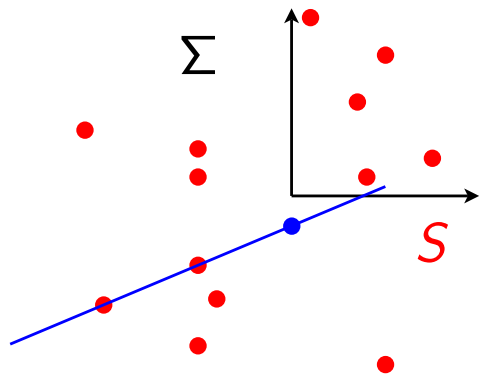


- $S \subset \Sigma$ is a saturating set if every point in $\Sigma \setminus S$ is collinear with two points in S

in geometrical terms...

$$\Sigma = \Sigma(2, q)$$

Galois plane over the finite field $\mathbb{F}_q$

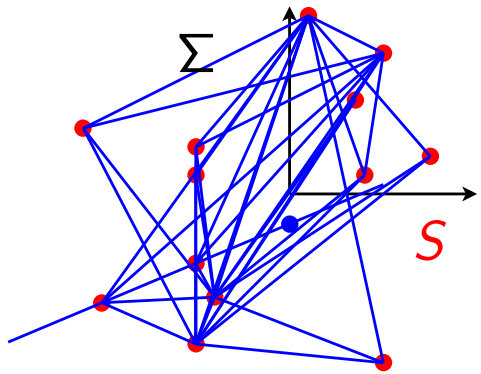


- $S \subset \Sigma$ is a saturating set if every point in $\Sigma \setminus S$ is collinear with two points in S

in geometrical terms...

$$\Sigma = \Sigma(2, q)$$

Galois plane over the finite field $\mathbb{F}_q$

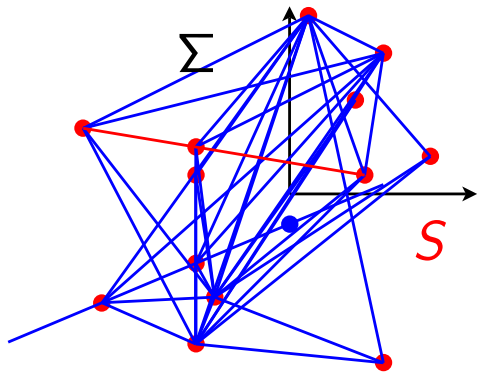


- $S \subset \Sigma$ is a saturating set if every point in $\Sigma \setminus S$ is collinear with two points in S

in geometrical terms...

$$\Sigma = \Sigma(2, q)$$

Galois plane over the finite field $\mathbb{F}_q$

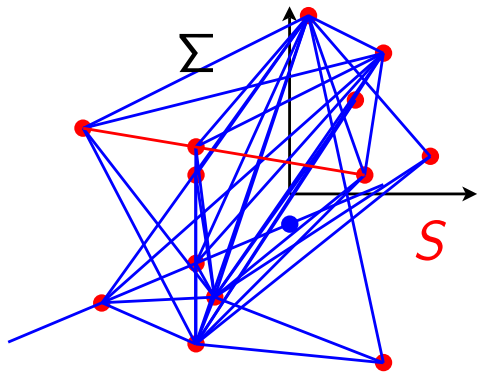


- $S \subset \Sigma$ is a saturating set if every point in $\Sigma \setminus S$ is collinear with two points in S
- a complete cap is a saturating set which does not contain 3 collinear points

in geometrical terms...

$$\Sigma = \Sigma(2, q)$$

Galois plane over the finite field $\mathbb{F}_q$



- $S \subset \Sigma$ is a saturating set if every point in $\Sigma \setminus S$ is collinear with two points in S
- a complete cap is a saturating set which does not contain 3 collinear points

$$\ell(3, q)_{2,4} = \text{minimum size of a complete cap in } \mathbb{P}^2(\mathbb{F}_q)$$

- TLB:

$$\#S > \sqrt{2q} + 1$$

plane complete caps

- TLB:

$$\#S > \sqrt{2q} + 1$$

- in $\mathbb{P}^2(\mathbb{F}_q)$ there exists a complete cap S of size

$$\#S \leq D\sqrt{q}\log^C q$$

(Kim-Vu, 2003)

plane complete caps

- TLB:
$$\#S > \sqrt{2q} + 1$$
- in $\mathbb{P}^2(\mathbb{F}_q)$ there exists a complete cap S of size
$$\#S \leq D\sqrt{q}\log^C q$$
(Kim-Vu, 2003)

- for every q prime $q < 67000$ there exists a complete cap S of size
$$\#S \leq \sqrt{q}\log q$$
(Bartoli-Davydov-Faina-Marcugini-Pambianco, 2012)

plane complete caps

- TLB:

$$\#S > \sqrt{2q} + 1$$

- in $\mathbb{P}^2(\mathbb{F}_q)$ there exists a complete cap S of size

$$\#S \leq D\sqrt{q}\log^C q$$

(Kim-Vu, 2003)

- for every q prime $q < 67000$ there exists a complete cap S of size

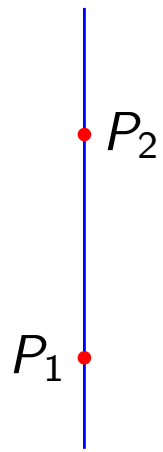
$$\#S \leq \sqrt{q}\log q$$

(Bartoli-Davydov-Faina-Marcugini-Pambianco, 2012)

- *naive* construction method:

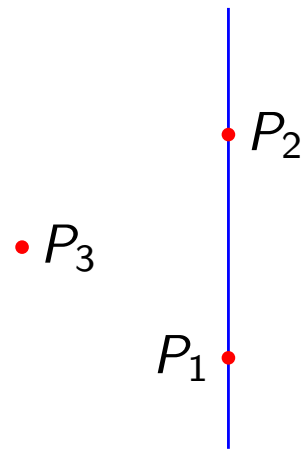
- *naive* construction method:

$$S = \{P_1,\ P_2,$$



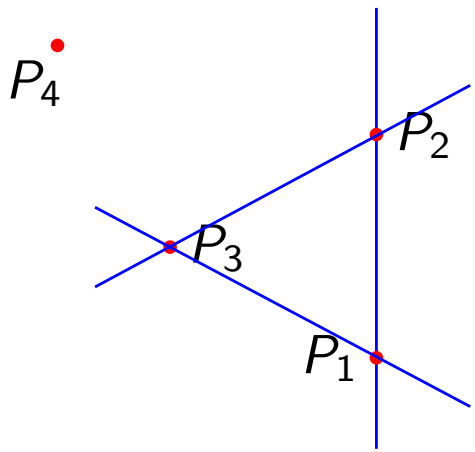
- *naive* construction method:

$$S = \{P_1,\ P_2,\ P_3,$$



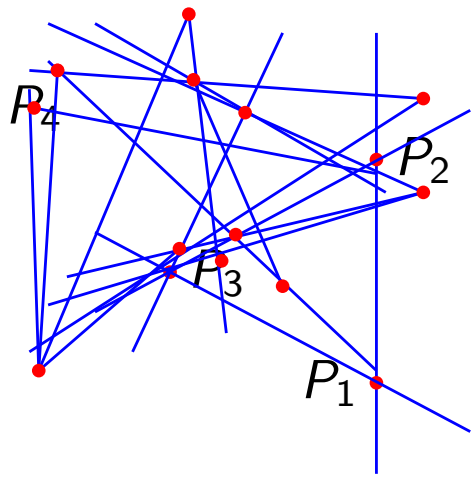
- *naive* construction method:

$$S = \{P_1,\ P_2,\ P_3,\ P_4,$$



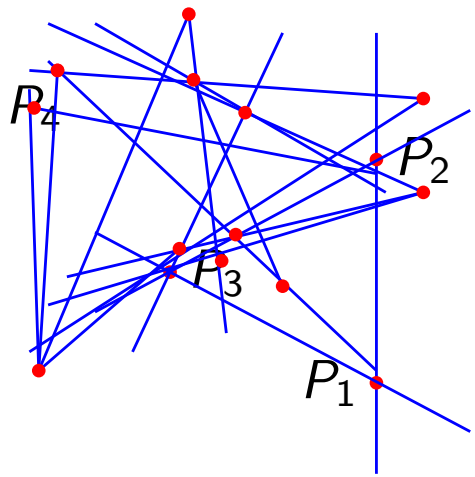
- *naive* construction method:

$$S = \{P_1,\ P_2,\ P_3,\ P_4, \ldots,$$

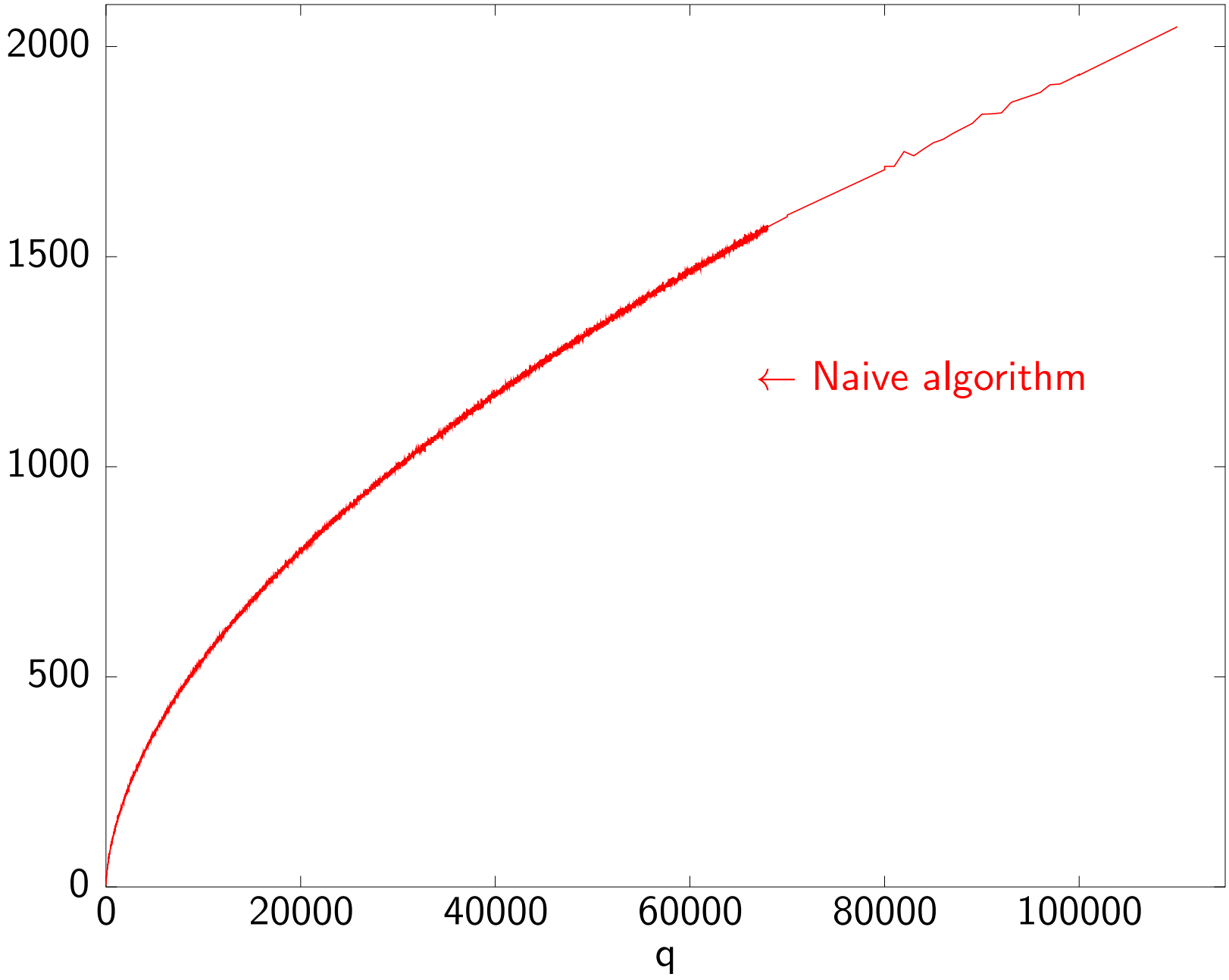


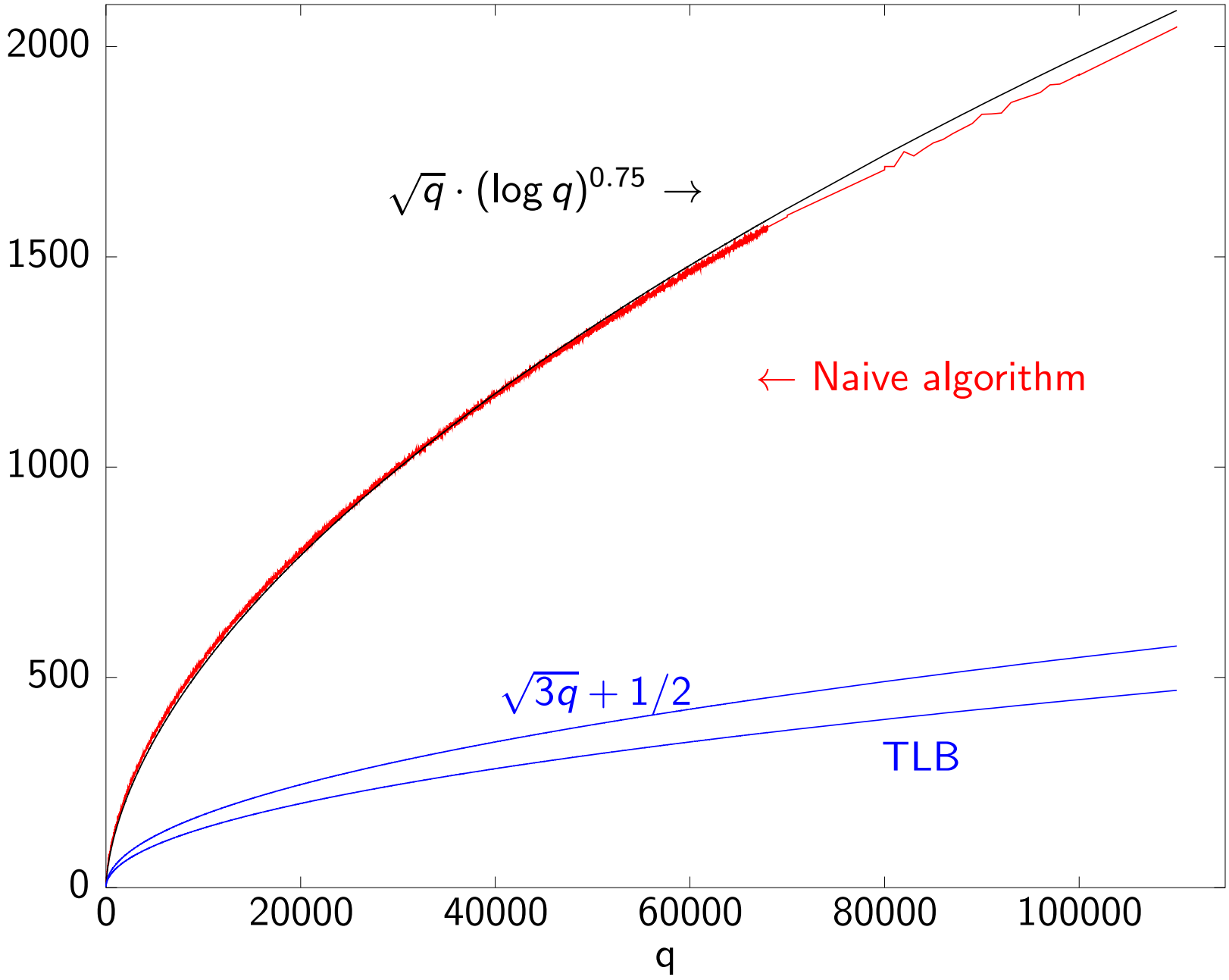
- *naive* construction method:

$$S = \{P_1,\ P_2,\ P_3,\ P_4, \ldots,\ P_n\}$$



- *naive vs. theoretical*

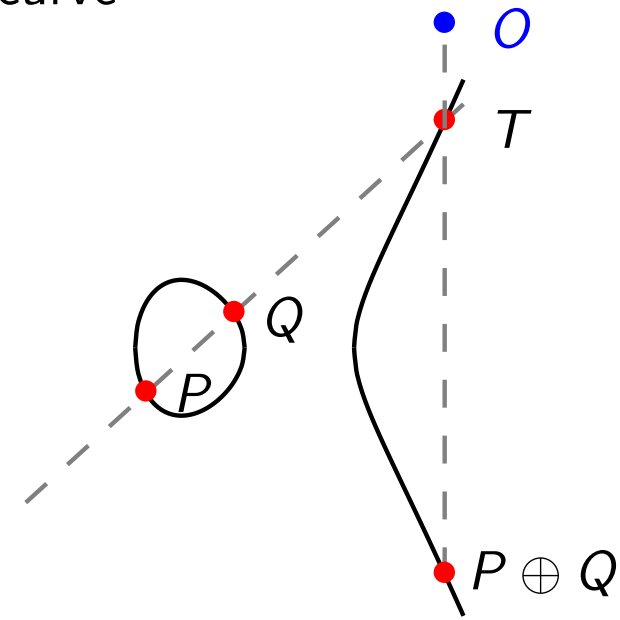

2000
1500
1000
500
0
0
20000
40000
60000
80000
100000
q
← Naive algorithm


2000
1500
1000
500
0
0
20000
40000
60000
80000
100000
q
$\sqrt{q} \cdot (\log q)^{0.75} \rightarrow$
$\leftarrow$ Naive algorithm
$\sqrt{3q} + 1/2$
TLB

$\mathcal{X}$ plane irreducible cubic curve

cubic curves

$\mathcal{X}$ plane irreducible cubic curve



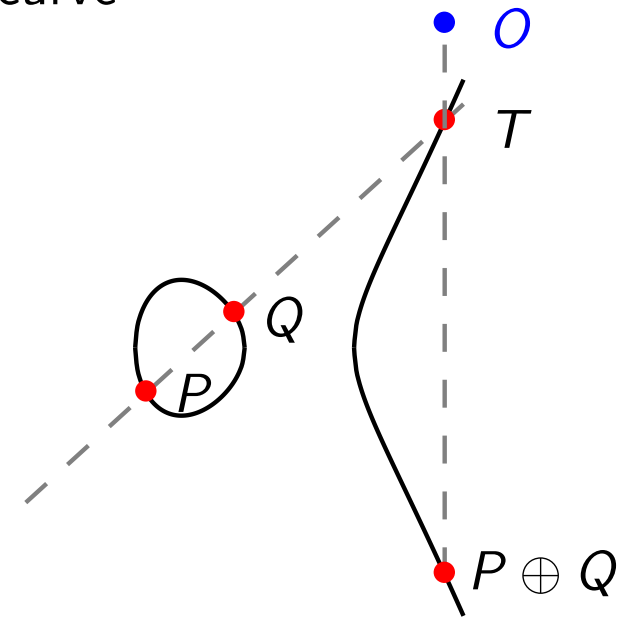
$G = \mathcal{X}(\mathbb{F}_q) \setminus \mathrm{Sing}(\mathcal{X})$

- if O is an inflection point of $\mathcal{X}$, then P, Q, $T \in G$ are collinear if and only if

$$P \oplus Q \oplus T = O$$

cubic curves

$\mathcal{X}$ plane irreducible cubic curve



$G = \mathcal{X}(\mathbb{F}_q) \setminus \mathrm{Sing}(\mathcal{X})$

- if O is an inflection point of $\mathcal{X}$, then P, Q, $T \in G$ are collinear if and only if

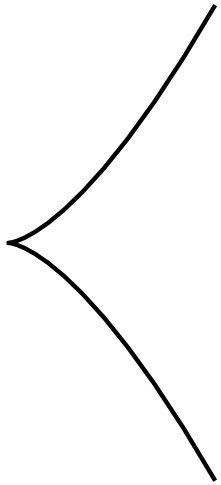
$$P \oplus Q \oplus T = O$$

- for a subgroup K of index m with $(3, m) = 1$, no 3 points in a coset

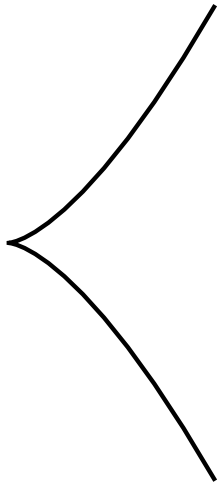
$$S = K \oplus Q, \qquad Q \notin K$$

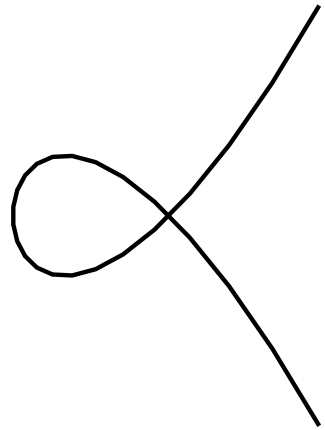
are collinear

classification ($p > 3$)

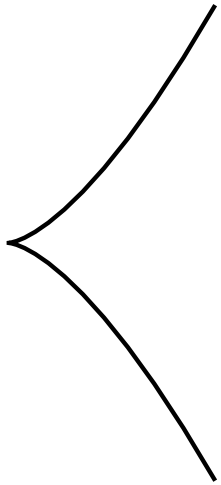


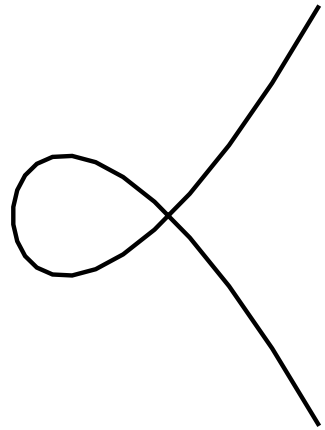
classification ($p > 3$)

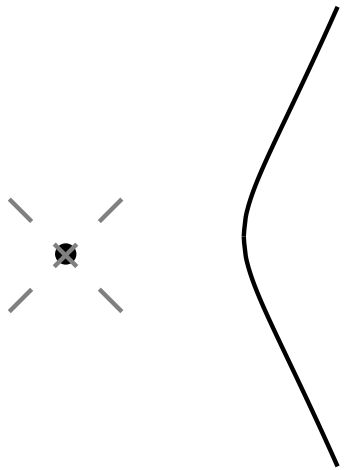




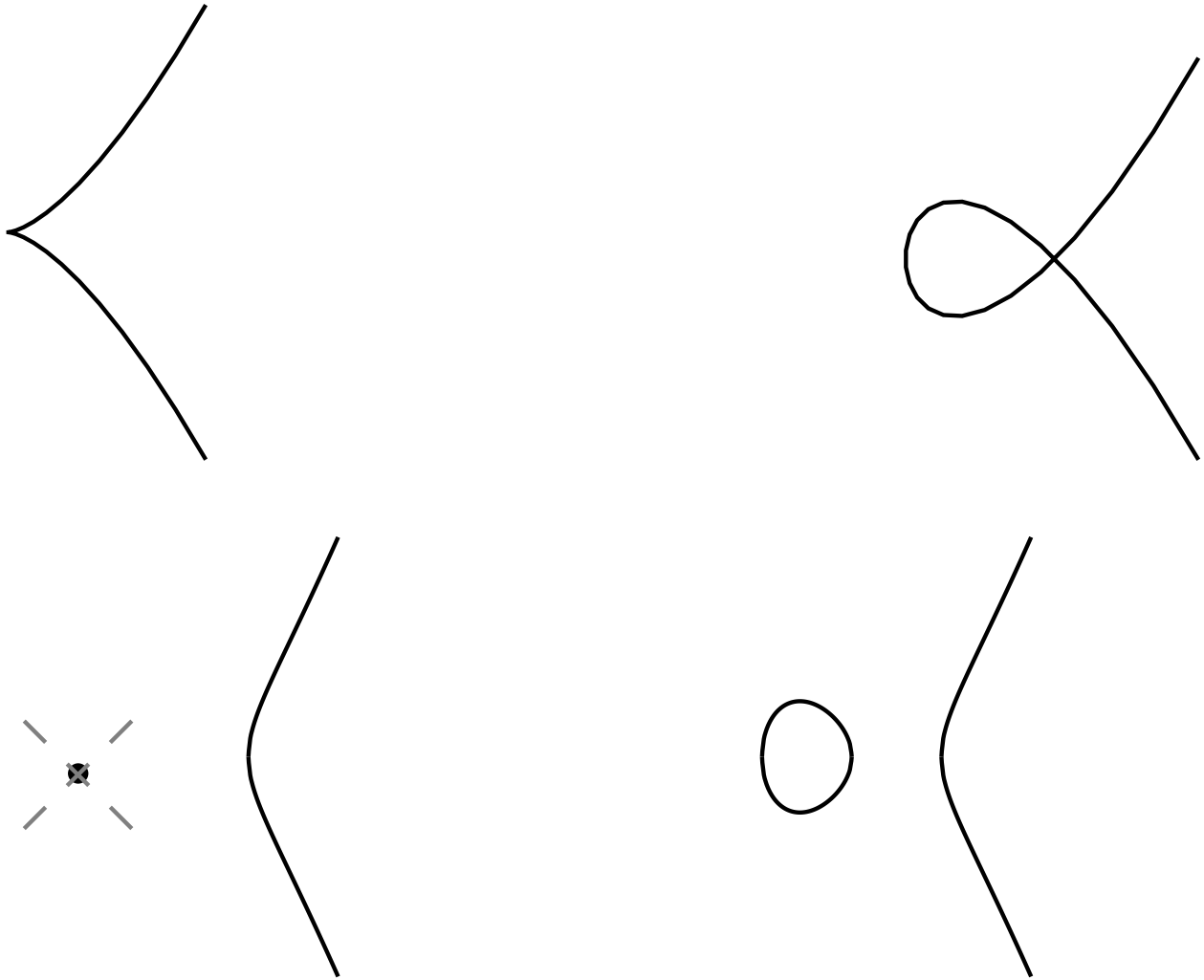
classification ($p > 3$)



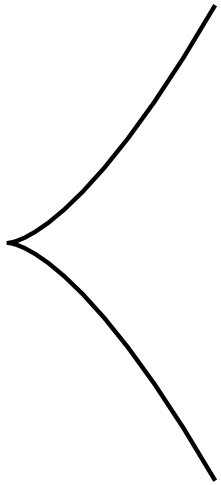




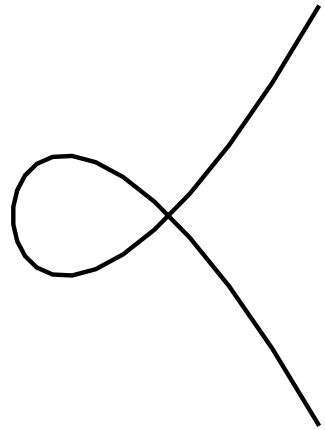
classification ($p > 3$)



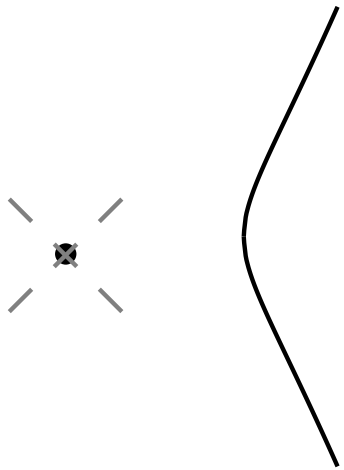
classification ($p > 3$)



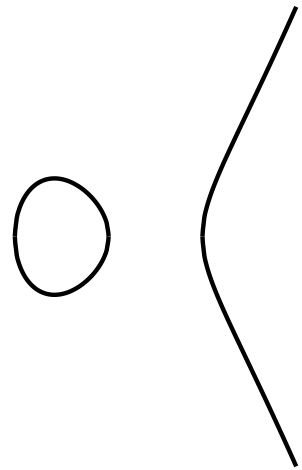
$Y = X^3$



$XY = (X-1)^3$



$Y(X^2 - \beta) = 1$



$Y^2 = X^3 + AX + B$

how to prove completeness?

- S parametrized by polynomials defined over $\mathbb{F}_q$

$$S = \{(f(t), g(t)) \mid t \in \mathbb{F}_q\} \subset \mathbb{A}^2(\mathbb{F}_q)$$

how to prove completeness?

- S parametrized by polynomials defined over $\mathbb{F}_q$

$$S = \{(f(t), g(t)) \mid t \in \mathbb{F}_q\} \subset \mathbb{A}^2(\mathbb{F}_q)$$

- $P = (a, b)$ collinear with two points in S if there exist $x, y \in \mathbb{F}_q$ with

$$\det \begin{pmatrix} a & b & 1 \\ f(x) & g(x) & 1 \\ f(y) & g(y) & 1 \end{pmatrix} = 0$$

how to prove completeness?

- S parametrized by polynomials defined over $\mathbb{F}_q$

$$S = \{(f(t), g(t)) \mid t \in \mathbb{F}_q\} \subset \mathbb{A}^2(\mathbb{F}_q)$$

- $P = (a, b)$ collinear with two points in S if there exist $x, y \in \mathbb{F}_q$ with $F_{a,b}(x, y) = 0$, where

$$F_{a,b}(x, y) := \det \begin{pmatrix} a & b & 1 \\ f(x) & g(x) & 1 \\ f(y) & g(y) & 1 \end{pmatrix}$$

how to prove completeness?

- S parametrized by polynomials defined over $\mathbb{F}_q$

$$S = \{(f(t), g(t)) \mid t \in \mathbb{F}_q\} \subset \mathbb{A}^2(\mathbb{F}_q)$$

- $P = (a, b)$ collinear with two points in S if there exist $x, y \in \mathbb{F}_q$ with $F_{a,b}(x, y) = 0$, where

$$F_{a,b}(x, y) := \det \begin{pmatrix} a & b & 1 \\ f(x) & g(x) & 1 \\ f(y) & g(y) & 1 \end{pmatrix}$$

- $P = (a, b)$ collinear with two points in S if the algebraic curve

$$\mathcal{C}_P : F_{a,b}(X, Y) = 0$$

has a suitable $\mathbb{F}_q$-rational point (x, y)

how to prove completeness?

- S parametrized by polynomials defined over $\mathbb{F}_q$

$$S = \{(f(t), g(t)) \mid t \in \mathbb{F}_q\} \subset \mathbb{A}^2(\mathbb{F}_q)$$

- $P = (a, b)$ collinear with two points in S if there exist $x, y \in \mathbb{F}_q$ with $F_{a,b}(x, y) = 0$, where

$$F_{a,b}(x, y) := \det \begin{pmatrix} a & b & 1 \\ f(x) & g(x) & 1 \\ f(y) & g(y) & 1 \end{pmatrix}$$

- $P = (a, b)$ collinear with two points in S if the algebraic curve

$$\mathcal{C}_P : F_{a,b}(X, Y) = 0$$

has a suitable $\mathbb{F}_q$-rational point (x, y)

cuspidal case: $Y = X^3$

- G is an elementary abelian p-group $\qquad q = p^h$

cuspidal case: $Y = X^3$

- G is an elementary abelian p-group $\qquad q = p^h$

$$K = \{(t^p - t, (t^p - t)^3) \mid t \in \mathbb{F}_q\}$$

cuspidal case: $Y = X^3$

- G is an elementary abelian p-group $\qquad q = p^h$

$$K = \{(t^p - t, (t^p - t)^3) \mid t \in \mathbb{F}_q\}$$

$$S = \{\underbrace{(\overbrace{t^p - t + \overline{t}}^{f(t)}, \overbrace{(t^p - t + \overline{t})^3}^{g(t)})}_{P_t} \mid t \in \mathbb{F}_q\}$$

cuspidal case: $Y = X^3$

- G is an elementary abelian p-group $\qquad q = p^h$

$$K = \{(t^p - t, (t^p - t)^3) \mid t \in \mathbb{F}_q\}$$

$$S = \{\underbrace{(\overbrace{t^p - t + \bar{t}}^{f(t)}, \overbrace{(t^p - t + \bar{t})^3}^{g(t)})}_{P_t} \mid t \in \mathbb{F}_q\}$$

- $P = (a, b)$ is collinear with P_x and P_y if and only if

$$\begin{aligned} F_{a,b}(x, y) := \quad & a + (x^p - x + \bar{t})(y^p - y + \bar{t})^2 + \\ & (x^p - x + \bar{t})^2(y^p - y + \bar{t}) - b((x^p - x + \bar{t})^2 \\ & +(x^p - x + \bar{t})(y^p - y + \bar{t}) + (y^p - y + \bar{t})^2) = 0 \end{aligned}$$

cuspidal case: $Y = X^3$

- G is an elementary abelian p-group $\qquad q = p^h$

$$K = \{(t^p - t, (t^p - t)^3) \mid t \in \mathbb{F}_q\}$$

$$S = \{\underbrace{(\overbrace{t^p - t + \bar{t}}^{f(t)}, \overbrace{(t^p - t + \bar{t})^3}^{g(t)})}_{P_t} \mid t \in \mathbb{F}_q\}$$

- $P = (a, b)$ is collinear with P_x and P_y if and only if

$$\begin{aligned} F_{a,b}(x, y) := \quad & a + (x^p - x + \bar{t})(y^p - y + \bar{t})^2 + \\ & (x^p - x + \bar{t})^2(y^p - y + \bar{t}) - b((x^p - x + \bar{t})^2 \\ & +(x^p - x + \bar{t})(y^p - y + \bar{t}) + (y^p - y + \bar{t})^2) = 0 \end{aligned}$$

- the curve $\mathcal{C}_P$ then is $F_{a,b}(X, Y) = 0$

applying Segre's criterion

(Segre, 1962)

if there exists a point $P \in \mathcal{C}$ and a tangent ℓ of $\mathcal{C}$ at P such that

- ℓ counts once among the tangents of $\mathcal{C}$ at P,
- the intersection multiplicity of $\mathcal{C}$ and ℓ at P equals $\deg(\mathcal{C})$,
- $\mathcal{C}$ has no linear components through P,

then $\mathcal{C}$ is irreducible.

applying Segre's criterion

(Segre, 1962)

if there exists a point $P \in \mathcal{C}$ and a tangent ℓ of $\mathcal{C}$ at P such that

- ℓ counts once among the tangents of $\mathcal{C}$ at P,
- the intersection multiplicity of $\mathcal{C}$ and ℓ at P equals $\deg(\mathcal{C})$,
- $\mathcal{C}$ has no linear components through P,

then $\mathcal{C}$ is irreducible.

$$\begin{aligned} F_{a,b}(X,Y) := \quad & a + (X^p - X + \bar{t})(Y^p - Y + \bar{t})^2 + \\ & (X^p - X + \bar{t})^2(Y^p - Y + \bar{t}) - b((X^p - X + \bar{t})^2 \\ & +(X^p - X + \bar{t})(Y^p - Y + \bar{t}) + (Y^p - Y + \bar{t})^2) = 0 \end{aligned}$$

applying Segre's criterion

(Segre, 1962)

if there exists a point $P \in \mathcal{C}$ and a tangent ℓ of $\mathcal{C}$ at P such that

- ℓ counts once among the tangents of $\mathcal{C}$ at P,
- the intersection multiplicity of $\mathcal{C}$ and ℓ at P equals $\deg(\mathcal{C})$,
- $\mathcal{C}$ has no linear components through P,

then $\mathcal{C}$ is irreducible.

$$\begin{aligned} F_{a,b}(X,Y) := \quad & a + (X^p - X + \bar{t})(Y^p - Y + \bar{t})^2 + \\ & (X^p - X + \bar{t})^2(Y^p - Y + \bar{t}) - b((X^p - X + \bar{t})^2 \\ & +(X^p - X + \bar{t})(Y^p - Y + \bar{t}) + (Y^p - Y + \bar{t})^2) = 0 \end{aligned}$$

at $P = X_\infty$ the tangents are $\ell : Y = \beta$ with $\beta^p - \beta + \bar{t} = b$

applying Segre's criterion

(Segre, 1962)

if there exists a point $P \in \mathcal{C}$ and a tangent ℓ of $\mathcal{C}$ at P such that

- ℓ counts once among the tangents of $\mathcal{C}$ at P,
- the intersection multiplicity of $\mathcal{C}$ and ℓ at P equals $\deg(\mathcal{C})$,
- $\mathcal{C}$ has no linear components through P,

then $\mathcal{C}$ is irreducible.

$$\begin{aligned} F_{a,b}(X,Y) := \quad & a + (X^p - X + \overline{t})(Y^p - Y + \overline{t})^2 + \\ & (X^p - X + \overline{t})^2(Y^p - Y + \overline{t}) - b((X^p - X + \overline{t})^2 \\ & +(X^p - X + \overline{t})(Y^p - Y + \overline{t}) + (Y^p - Y + \overline{t})^2) = 0 \end{aligned}$$

at $P = X_\infty$ the tangents are $\ell : Y = \beta$ with $\beta^p - \beta + \overline{t} = b$

$$F_{a,b}(X,\beta) = a - b^3$$

applying Segre's criterion

(Segre, 1962)

if there exists a point $P \in \mathcal{C}$ and a tangent ℓ of $\mathcal{C}$ at P such that

- ℓ counts once among the tangents of $\mathcal{C}$ at P,
- the intersection multiplicity of $\mathcal{C}$ and ℓ at P equals $\deg(\mathcal{C})$,
- $\mathcal{C}$ has no linear components through P,

then $\mathcal{C}$ is irreducible.

$$\begin{aligned} F_{a,b}(X,Y) := \quad & a + (X^p - X + \bar{t})(Y^p - Y + \bar{t})^2 + \\ & (X^p - X + \bar{t})^2(Y^p - Y + \bar{t}) - b((X^p - X + \bar{t})^2 \\ & +(X^p - X + \bar{t})(Y^p - Y + \bar{t}) + (Y^p - Y + \bar{t})^2) = 0 \end{aligned}$$

at $P = X_\infty$ the tangents are $\ell : Y = \beta$ with $\beta^p - \beta + \bar{t} = b$

$$F_{a,b}(X,\beta) = a - b^3$$

if $P \notin \mathcal{X}$

- $\mathcal{C}_P$ is irreducible of genus $g \leq 3p^2 - 3p + 1$

applying Segre's criterion

(Segre, 1962)

if there exists a point $P \in \mathcal{C}$ and a tangent ℓ of $\mathcal{C}$ at P such that

- ℓ counts once among the tangents of $\mathcal{C}$ at P,
- the intersection multiplicity of $\mathcal{C}$ and ℓ at P equals $\deg(\mathcal{C})$,
- $\mathcal{C}$ has no linear components through P,

then $\mathcal{C}$ is irreducible.

$$\begin{aligned} F_{a,b}(X,Y) := \quad & a + (X^p - X + \bar{t})(Y^p - Y + \bar{t})^2 + \\ & (X^p - X + \bar{t})^2(Y^p - Y + \bar{t}) - b((X^p - X + \bar{t})^2 \\ & +(X^p - X + \bar{t})(Y^p - Y + \bar{t}) + (Y^p - Y + \bar{t})^2) = 0 \end{aligned}$$

at $P = X_\infty$ the tangents are $\ell : Y = \beta$ with $\beta^p - \beta + \bar{t} = b$

$$F_{a,b}(X,\beta) = a - b^3$$

if $P \notin \mathcal{X}$

- $\mathcal{C}_P$ is irreducible of genus $g \leq 3p^2 - 3p + 1$
- $\mathcal{C}_P$ has at least $q + 1 - (6p^2 - 6p + 2)\sqrt{q}$ points

cuspidal case: $Y = X^3$

- G is elementary abelian, isomorphic to $(\mathbb{F}_q, +)$

cuspidal case: $Y = X^3$

- G is elementary abelian, isomorphic to $(\mathbb{F}_q, +)$

$$S = \{\underbrace{(L(t) + \bar{t}, (L(t) + \bar{t})^3)}_{P_t} \mid t \in \mathbb{F}_q\}$$

$$L(T) = \prod_{\alpha \in M} (T - \alpha), \quad M < (\mathbb{F}_q, +), \quad \#M = m$$

cuspidal case: $Y = X^3$

- G is elementary abelian, isomorphic to $(\mathbb{F}_q, +)$

$$S = \{\underbrace{(L(t) + \bar{t}, (L(t) + \bar{t})^3)}_{P_t} \mid t \in \mathbb{F}_q\}$$

$$L(T) = \prod_{\alpha \in M} (T - \alpha), \quad M < (\mathbb{F}_q, +), \quad \#M = m$$

- $P = (a, b)$ is collinear with P_x and P_y if and only if

$$\begin{aligned} F_{a,b}(x, y) := \quad & a + (L(x) + \bar{t})(L(y) + \bar{t})^2 + \\ & (L(x) + \bar{t})^2(L(y) + \bar{t}) - b((L(x) + \bar{t})^2 \\ & +(L(x) + \bar{t})(L(y) + \bar{t}) + (L(y) + \bar{t})^2) = 0 \end{aligned}$$

cuspidal case: $Y = X^3$

- G is elementary abelian, isomorphic to $(\mathbb{F}_q, +)$

$$S = \{\underbrace{(L(t) + \bar{t}, (L(t) + \bar{t})^3)}_{P_t} \mid t \in \mathbb{F}_q\}$$

$$L(T) = \prod_{\alpha \in M} (T - \alpha), \quad M < (\mathbb{F}_q, +), \quad \#M = m$$

- $P = (a, b)$ is collinear with P_x and P_y if and only if

$$\begin{aligned} F_{a,b}(x, y) := \quad & a + (L(x) + \bar{t})(L(y) + \bar{t})^2 + \\ & (L(x) + \bar{t})^2(L(y) + \bar{t}) - b((L(x) + \bar{t})^2 \\ & +(L(x) + \bar{t})(L(y) + \bar{t}) + (L(y) + \bar{t})^2) = 0 \end{aligned}$$

  if $P \notin \mathcal{X}$

- $\mathcal{C}_P$ is irreducible of genus $g \leq 3m^2 - 3m + 1$

cuspidal case: $Y = X^3$

- G is elementary abelian, isomorphic to $(\mathbb{F}_q, +)$

$$S = \{\underbrace{(L(t) + \bar{t}, (L(t) + \bar{t})^3)}_{P_t} \mid t \in \mathbb{F}_q\}$$

$$L(T) = \prod_{\alpha \in M} (T - \alpha), \quad M < (\mathbb{F}_q, +), \quad \#M = m$$

- $P = (a, b)$ is collinear with P_x and P_y if and only if

$$\begin{aligned} F_{a,b}(x, y) := \quad & a + (L(x) + \bar{t})(L(y) + \bar{t})^2 + \\ & (L(x) + \bar{t})^2(L(y) + \bar{t}) - b((L(x) + \bar{t})^2 \\ & +(L(x) + \bar{t})(L(y) + \bar{t}) + (L(y) + \bar{t})^2) = 0 \end{aligned}$$

  if $P \notin \mathcal{X}$

- $\mathcal{C}_P$ is irreducible of genus $g \leq 3m^2 - 3m + 1$
- $\mathcal{C}_P$ has at least $q + 1 - (6m^2 - 6m + 2)\sqrt{q}$ points

(Szőnyi, 1985 - Anbar, Bartoli, G., Platoni, 2013)

let $P = (a, b)$ be a point in $\mathbb{A}^2(\mathbb{F}_q) \setminus \mathcal{X}$; if

$$m < \sqrt[4]{q/36}$$

then there is a secant of S passing through P.

(Szőnyi, 1985 - Anbar, Bartoli, G., Platoni, 2013)

let $P = (a, b)$ be a point in $\mathbb{A}^2(\mathbb{F}_q) \setminus \mathcal{X}$; if

$$m < \sqrt[4]{q/36}$$

then there is a secant of S passing through P.

- m is a power of p

(Szőnyi, 1985 - Anbar, Bartoli, G., Platoni, 2013)

let $P = (a, b)$ be a point in $\mathbb{A}^2(\mathbb{F}_q) \setminus \mathcal{X}$; if

$$m < \sqrt[4]{q/36}$$

then there is a secant of S passing through P.

- m is a power of p
- the points in $\mathcal{X} \setminus S$ need to be dealt with

(Szőnyi, 1985 - Anbar, Bartoli, G., Platoni, 2013)

let $P = (a, b)$ be a point in $\mathbb{A}^2(\mathbb{F}_q) \setminus \mathcal{X}$; if

$$m < \sqrt[4]{q/36}$$

then there is a secant of S passing through P.

- m is a power of p
- the points in $\mathcal{X} \setminus S$ need to be dealt with

theorem

if $m < \sqrt[4]{q/36}$, *then there exists a complete cap in* $\mathbb{A}^2(\mathbb{F}_q)$ *with size*

$$m + \frac{q}{m} - 3$$

(Szőnyi, 1985 - Anbar, Bartoli, G., Platoni, 2013)

let $P = (a, b)$ be a point in $\mathbb{A}^2(\mathbb{F}_q) \setminus \mathcal{X}$; if

$$m < \sqrt[4]{q/36}$$

then there is a secant of S passing through P.

- m is a power of p
- the points in $\mathcal{X} \setminus S$ need to be dealt with

theorem

if $m < \sqrt[4]{q/36}$, *then there exists a complete cap in* $\mathbb{A}^2(\mathbb{F}_q)$ *with size*

$$m + \frac{q}{m} - 3 \sim p^{1/4} \cdot q^{3/4}$$

nodal case: $XY = (X-1)^3$

- G is isomorphic to $(\mathbb{F}_q^*, \cdot)$

$$G \to \mathbb{F}_q^* \qquad \left(v, \frac{(v-1)^3}{v}\right) \mapsto v$$

nodal case: $XY = (X-1)^3$

- G is isomorphic to $(\mathbb{F}_q^*, \cdot)$

$$G \to \mathbb{F}_q^* \qquad \left(v, \frac{(v-1)^3}{v}\right) \mapsto v$$

- the subgroup of index m (m a divisor of $q-1$):

$$K = \left\{ \left(t^m, \frac{(t^m-1)^3}{t^m}\right) \mid t \in \mathbb{F}_q^* \right\}$$

nodal case: $XY = (X-1)^3$

- G is isomorphic to $(\mathbb{F}_q^*, \cdot)$

$$G \to \mathbb{F}_q^* \qquad \left(v, \frac{(v-1)^3}{v}\right) \mapsto v$$

- the subgroup of index m (m a divisor of $q-1$):

$$K = \left\{ \left(t^m, \frac{(t^m-1)^3}{t^m}\right) \mid t \in \mathbb{F}_q^* \right\}$$

- a coset:

$$S = \left\{ \left(\bar{t}t^m, \frac{(\bar{t}t^m-1)^3}{\bar{t}t^m}\right) \mid t \in \mathbb{F}_q^* \right\}$$

nodal case: $XY = (X-1)^3$

- G is isomorphic to $(\mathbb{F}_q^*, \cdot)$

$$G \to \mathbb{F}_q^* \qquad \left(v, \frac{(v-1)^3}{v}\right) \mapsto v$$

- the subgroup of index m (m a divisor of $q-1$):

$$K = \left\{ \left(t^m, \frac{(t^m-1)^3}{t^m}\right) \mid t \in \mathbb{F}_q^* \right\}$$

- a coset:

$$S = \left\{ \underbrace{\left(\overbrace{\bar{t}t^m}^{f(t)}, \overbrace{\frac{(\bar{t}t^m-1)^3}{\bar{t}t^m}}^{g(t)} \right)}_{P_t} \mid t \in \mathbb{F}_q^* \right\}$$

nodal case: $XY = (X-1)^3$

- G is isomorphic to $(\mathbb{F}_q^*, \cdot)$

$$G \to \mathbb{F}_q^* \qquad \left(v, \frac{(v-1)^3}{v}\right) \mapsto v$$

- the subgroup of index m (m a divisor of $q-1$):

$$K = \left\{ \left(t^m, \frac{(t^m-1)^3}{t^m}\right) \mid t \in \mathbb{F}_q^* \right\}$$

- a coset:

$$S = \left\{ \underbrace{\left(\overbrace{\bar{t}t^m}^{f(t)}, \overbrace{\frac{(\bar{t}t^m-1)^3}{\bar{t}t^m}}^{g(t)} \right)}_{P_t} \mid t \in \mathbb{F}_q^* \right\}$$

- the curve $\mathcal{C}_P$:

$$\begin{aligned} F_{a,b}(X,Y) \quad = \quad & a(\bar{t}^3X^{2m}Y^m + \bar{t}^3X^mY^{2m} - 3\bar{t}^2X^mY^m + 1) \\ & -b\bar{t}^2X^mY^m - \bar{t}^4X^{2m}Y^{2m} + 3\bar{t}^2X^mY^m \\ & -\bar{t}X^m - \bar{t}Y^m = 0 \end{aligned}$$

(Anbar-Bartoli-G.-Platoni, 2013)

let P be a point in $\mathbb{A}^2(\mathbb{F}_q) \setminus \mathcal{X}$; if

$$m < \sqrt[4]{q/36}$$

then there is a secant of S passing through P

(Anbar-Bartoli-G.-Platoni, 2013)

let P be a point in $\mathbb{A}^2(\mathbb{F}_q) \setminus \mathcal{X}$; if

$$m < \sqrt[4]{q/36}$$

then there is a secant of S passing through P

- m is a divisor of $q - 1$

(Anbar-Bartoli-G.-Platoni, 2013)

let P be a point in $\mathbb{A}^2(\mathbb{F}_q) \setminus \mathcal{X}$; if

$$m < \sqrt[4]{q/36}$$

then there is a secant of S passing through P

- m is a divisor of $q-1$
- some points from $\mathcal{X} \setminus S$ need to be added to S

(Anbar-Bartoli-G.-Platoni, 2013)

let P be a point in $\mathbb{A}^2(\mathbb{F}_q) \setminus \mathcal{X}$; if

$$m < \sqrt[4]{q/36}$$

then there is a secant of S passing through P

- m is a divisor of $q-1$
- some points from $\mathcal{X} \setminus S$ need to be added to S

theorem

if m is a divisor of $q-1$ with $m < \sqrt[4]{q/36}$, and in addition $(m, \frac{q-1}{m}) = 1$, then there exists a complete cap in $\mathbb{A}^2(\mathbb{F}_q)$ with size

$$m + \frac{q-1}{m} - 3$$

(Anbar-Bartoli-G.-Platoni, 2013)

let P be a point in $\mathbb{A}^2(\mathbb{F}_q) \setminus \mathcal{X}$; if

$$m < \sqrt[4]{q/36}$$

then there is a secant of S passing through P

- m is a divisor of $q-1$
- some points from $\mathcal{X} \setminus S$ need to be added to S

theorem

if m is a divisor of $q-1$ with $m < \sqrt[4]{q/36}$, and in addition $(m, \frac{q-1}{m}) = 1$, then there exists a complete cap in $\mathbb{A}^2(\mathbb{F}_q)$ with size

$$m + \frac{q-1}{m} - 3 \qquad \sim q^{3/4}$$

isolated double point case: $Y(X^2 - \beta) = 1$

- G cyclic of order $q + 1$

isolated double point case: $Y(X^2 - \beta) = 1$

- G cyclic of order $q + 1$

(Anbar-Bartoli-G.-Platoni, 2013)

if m is a divisor of $q + 1$ with $m < \sqrt[4]{q/36}$, and in addition $(m, \frac{q+1}{m}) = 1$, then there exists a complete cap in $\mathbb{A}^2(\mathbb{F}_q)$ with size at most

$$m + \frac{q+1}{m}$$

isolated double point case: $Y(X^2 - \beta) = 1$

- G cyclic of order $q + 1$

(Anbar-Bartoli-G.-Platoni, 2013)

if m is a divisor of $q + 1$ with $m < \sqrt[4]{q/36}$, and in addition $(m, \frac{q+1}{m}) = 1$, then there exists a complete cap in $\mathbb{A}^2(\mathbb{F}_q)$ with size at most

$$m + \frac{q+1}{m} \qquad \sim q^{3/4}$$

elliptic case: $Y^2 = X^3 + AX + B$

if

$$n \in [q + 1 - 2\sqrt{q}, q + 1 + 2\sqrt{q}] \qquad n \not\equiv q + 1 \pmod{p}$$

there exists an elliptic cubic curve $\mathcal{X}$ over $\mathbb{F}_q$ with $\#G = n$

elliptic case: $Y^2 = X^3 + AX + B$

if

$$n \in [q + 1 - 2\sqrt{q}, q + 1 + 2\sqrt{q}] \qquad n \not\equiv q + 1 \pmod{p}$$

there exists an elliptic cubic curve $\mathcal{X}$ over $\mathbb{F}_q$ with $\#G = n$

(Voloch, 1988)

if p does not divide $\#G - 1$, then G can be assumed to be cyclic

elliptic case: $Y^2 = X^3 + AX + B$

if

$$n \in [q + 1 - 2\sqrt{q}, q + 1 + 2\sqrt{q}] \qquad n \not\equiv q + 1 \pmod{p}$$

there exists an elliptic cubic curve $\mathcal{X}$ over $\mathbb{F}_q$ with $\#G = n$

(Voloch, 1988)

if p does not divide $\#G - 1$, then G can be assumed to be cyclic

- **problem:** no polynomial or rational parametrization of the points of S is possible

elliptic case: $Y^2 = X^3 + AX + B$

if

$$n \in [q + 1 - 2\sqrt{q}, q + 1 + 2\sqrt{q}] \qquad n \not\equiv q + 1 \pmod{p}$$

there exists an elliptic cubic curve $\mathcal{X}$ over $\mathbb{F}_q$ with $\#G = n$

(Voloch, 1988)

if p does not divide $\#G - 1$, then G can be assumed to be cyclic

- **problem:** no polynomial or rational parametrization of the points of S is possible
- Voloch's solution (1990): implicit description of $\mathcal{C}_P$

elliptic case: $Y^2 = X^3 + AX + B$

if

$$n \in [q + 1 - 2\sqrt{q}, q + 1 + 2\sqrt{q}] \qquad n \not\equiv q + 1 \pmod{p}$$

there exists an elliptic cubic curve $\mathcal{X}$ over $\mathbb{F}_q$ with $\#G = n$

(Voloch, 1988)

if p does not divide $\#G - 1$, then G can be assumed to be cyclic

- **problem:** no polynomial or rational parametrization of the points of S is possible
- Voloch's solution (1990): implicit description of $\mathcal{C}_P$
- Voloch's result would provide complete caps of size $\sim q^{3/4}$ for every q large enough

elliptic case: $Y^2 = X^3 + AX + B$

if

$$n \in [q + 1 - 2\sqrt{q}, q + 1 + 2\sqrt{q}] \qquad n \not\equiv q + 1 \pmod{p}$$

there exists an elliptic cubic curve $\mathcal{X}$ over $\mathbb{F}_q$ with $\#G = n$

(Voloch, 1988)

if p does not divide $\#G - 1$, then G can be assumed to be cyclic

- **problem:** no polynomial or rational parametrization of the points of S is possible
- Voloch's solution (1990): implicit description of $\mathcal{C}_P$
- Voloch's result would provide complete caps of size $\sim q^{3/4}$ for every q large enough

?

G cyclic $\qquad m \mid q-1 \qquad m$ prime

G cyclic $\qquad m \mid q-1 \qquad m$ prime

- Tate-Lichtenbaum Pairing

$$< \cdot, \cdot >: G[m] \times G/K \quad \rightarrow \quad \mathbb{F}_q^*/(\mathbb{F}_q^*)^m$$

$$G \text{ cyclic} \qquad m \mid q-1 \qquad m \text{ prime}$$

- Tate-Lichtenbaum Pairing

$$< \cdot, \cdot >: G[m] \times G/K \quad \rightarrow \quad \mathbb{F}_q^*/(\mathbb{F}_q^*)^m$$

- if m^2 does not divide $\#G$, then for some T in $G[m]$

$$< T, \cdot >: G/K \quad \rightarrow \quad \mathbb{F}_q^*/(\mathbb{F}_q^*)^m$$

is an isomorphism such that

$$K \oplus Q \mapsto [\alpha_T(Q)]$$

where α_T is a rational function on $\mathcal{X}$

$$G \text{ cyclic} \qquad m \mid q-1 \qquad m \text{ prime}$$

- Tate-Lichtenbaum Pairing

$$< \cdot, \cdot >: G[m] \times G/K \quad \rightarrow \quad \mathbb{F}_q^*/(\mathbb{F}_q^*)^m$$

- if m^2 does not divide $\#G$, then for some T in $G[m]$

$$< T, \cdot >: G/K \quad \rightarrow \quad \mathbb{F}_q^*/(\mathbb{F}_q^*)^m$$

is an isomorphism such that

$$K \oplus Q \mapsto [\alpha_T(Q)]$$

where α_T is a rational function on $\mathcal{X}$

$$S = \{R \in G \mid \alpha_T(R) = dt^m \text{ for some } t \in \mathbb{F}_q\}$$

$$S = \{R \in \mathcal{X} \mid \alpha(R) = dt^m \text{ for some } t \in \mathbb{F}_q\}$$

$$S = \{R \in \mathcal{X} \mid \alpha(R) = dt^m \text{ for some } t \in \mathbb{F}_q\}$$

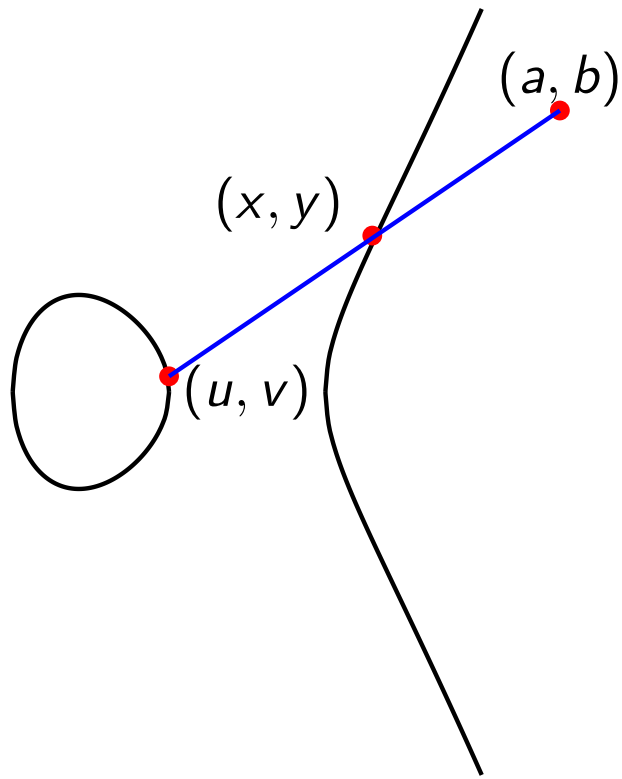
$$S = \{R \in \mathcal{X} \mid \alpha(R) = dt^m \text{ for some } t \in \mathbb{F}_q\}$$

$P = (a, b)$ collinear with two points $(x, y), (u, v) \in S$ if there exist $x, y, u, v, t, z \in \mathbb{F}_q$ with

$$S = \{R \in \mathcal{X} \mid \alpha(R) = dt^m \text{ for some } t \in \mathbb{F}_q\}$$

$P = (a, b)$ collinear with two points $(x, y), (u, v) \in S$ if there exist $x, y, u, v, t, z \in \mathbb{F}_q$ with

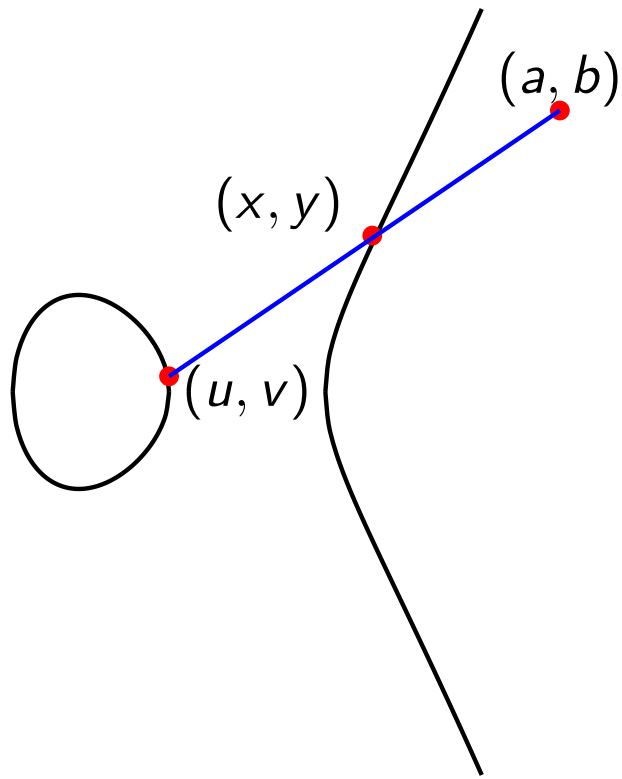
$$\begin{cases} y^2 = x^3 + Ax + B \\ v^2 = u^3 + Au + B \end{cases}$$



$$S = \{R \in \mathcal{X} \mid \alpha(R) = dt^m \text{ for some } t \in \mathbb{F}_q\}$$

$P = (a, b)$ collinear with two points $(x, y), (u, v) \in S$ if there exist $x, y, u, v, t, z \in \mathbb{F}_q$ with

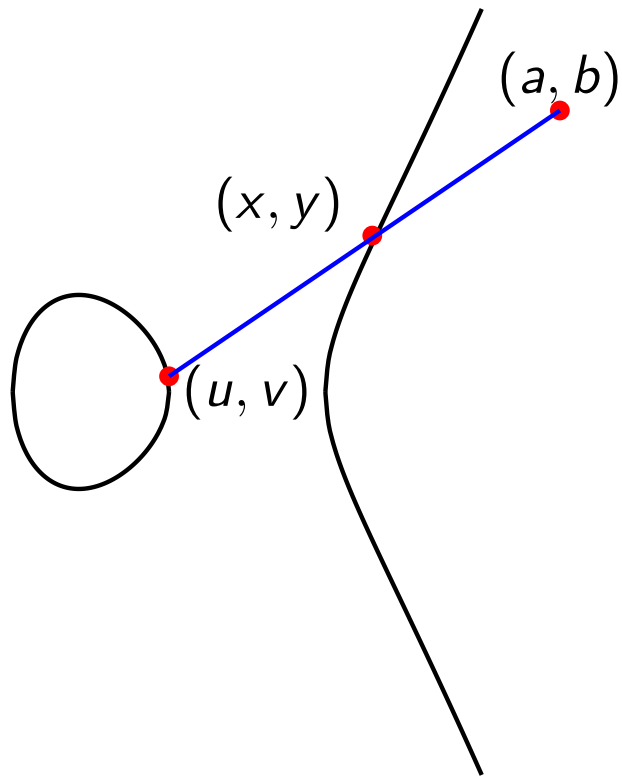
$$\begin{cases} y^2 = x^3 + Ax + B \\ v^2 = u^3 + Au + B \\ \alpha(x, y) = dt^m \\ \alpha(u, v) = dz^m \end{cases}$$



$$S = \{R \in \mathcal{X} \mid \alpha(R) = dt^m \text{ for some } t \in \mathbb{F}_q\}$$

$P = (a, b)$ collinear with two points $(x, y), (u, v) \in S$ if there exist $x, y, u, v, t, z \in \mathbb{F}_q$ with

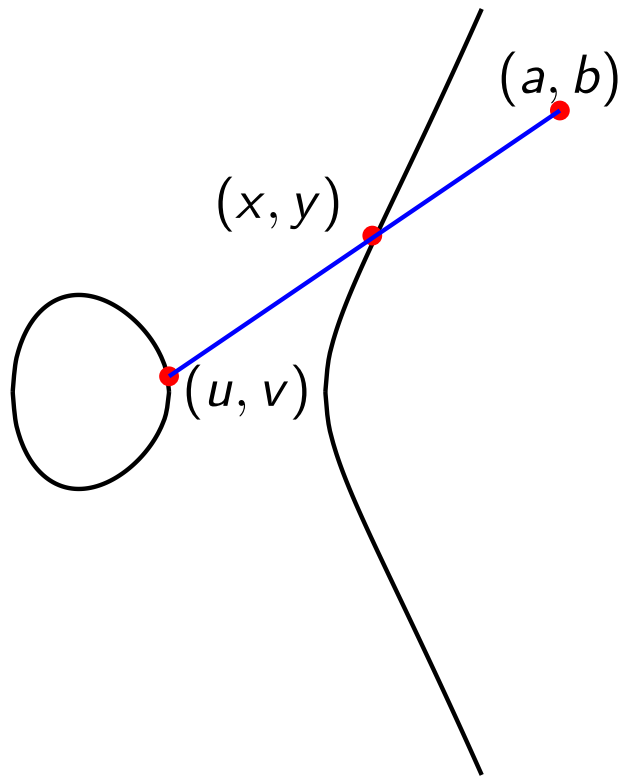
$$\begin{cases} y^2 = x^3 + Ax + B \\ v^2 = u^3 + Au + B \\ \alpha(x, y) = dt^m \\ \alpha(u, v) = dz^m \\ \det \begin{pmatrix} a & b & 1 \\ x & y & 1 \\ u & v & 1 \end{pmatrix} = 0 \end{cases}$$



$$S = \{R \in \mathcal{X} \mid \alpha(R) = dt^m \text{ for some } t \in \mathbb{F}_q\}$$

$P = (a, b)$ collinear with two points $(x, y), (u, v) \in S$ if there exist $x, y, u, v, t, z \in \mathbb{F}_q$ with

$$\mathcal{C}_P : \begin{cases} y^2 = x^3 + Ax + B \\ v^2 = u^3 + Au + B \\ \alpha(x, y) = dt^m \\ \alpha(u, v) = dz^m \\ \det \begin{pmatrix} a & b & 1 \\ x & y & 1 \\ u & v & 1 \end{pmatrix} = 0 \end{cases}$$



(Anbar-G., 2012)

if $A \neq 0$, then $\mathcal{C}_P$ is irreducible or admits an irreducible $\mathbb{F}_q$-rational component

(Anbar-G., 2012)

if $A \neq 0$, then $\mathcal{C}_P$ is irreducible or admits an irreducible $\mathbb{F}_q$-rational component

if m is a prime divisor of $q-1$ with $m < \sqrt[4]{q/64}$, then there exists a complete cap in $\mathbb{A}^2(\mathbb{F}_q)$ with size at most

$$m + \lfloor \frac{q - 2\sqrt{q} + 1}{m} \rfloor + 31$$

(Anbar-G., 2012)

if $A \neq 0$, then $\mathcal{C}_P$ is irreducible or admits an irreducible $\mathbb{F}_q$-rational component

if m is a prime divisor of $q-1$ with $m < \sqrt[4]{q/64}$, then there exists a complete cap in $\mathbb{A}^2(\mathbb{F}_q)$ with size at most

$$m + \lfloor \frac{q - 2\sqrt{q} + 1}{m} \rfloor + 31 \qquad \sim q^{3/4}$$

$$\ell(r, q)_{2,4}$$

in geometrical terms...

proposition

$$\ell(r,q)_{2,4} = \textit{minimum size of a complete cap in } \mathbb{P}^{r-1}(\mathbb{F}_q)$$

in geometrical terms...

proposition

$$\ell(r,q)_{2,4} = \textit{minimum size of a complete cap in } \mathbb{P}^{r-1}(\mathbb{F}_q)$$

trivial lower bound

$$\#S \geq \sqrt{2}q^{(N-1)/2} \textit{ in } \mathbb{P}^N(\mathbb{F}_q)$$

in geometrical terms...

proposition

$$\ell(r,q)_{2,4} = \textit{minimum size of a complete cap in } \mathbb{P}^{r-1}(\mathbb{F}_q)$$

trivial lower bound

$$\#S \geq \sqrt{2}q^{(N-1)/2} \textit{ in } \mathbb{P}^N(\mathbb{F}_q)$$

$N = 3$

- TLB:

$$\sqrt{2} \cdot q$$

$N = 3$

- TLB:

$$\sqrt{2} \cdot q$$

(Pellegrino, 1999)

$$\frac{1}{2} q \sqrt{q} + 2$$

$N = 3$

- TLB:

$$\sqrt{2} \cdot q$$

(Pellegrino, 1999)

$$\frac{1}{2} q \sqrt{q} + 2$$

(Faina, Faina-Pambianco, Hadnagy 1988-1999)

$$q^2/3$$

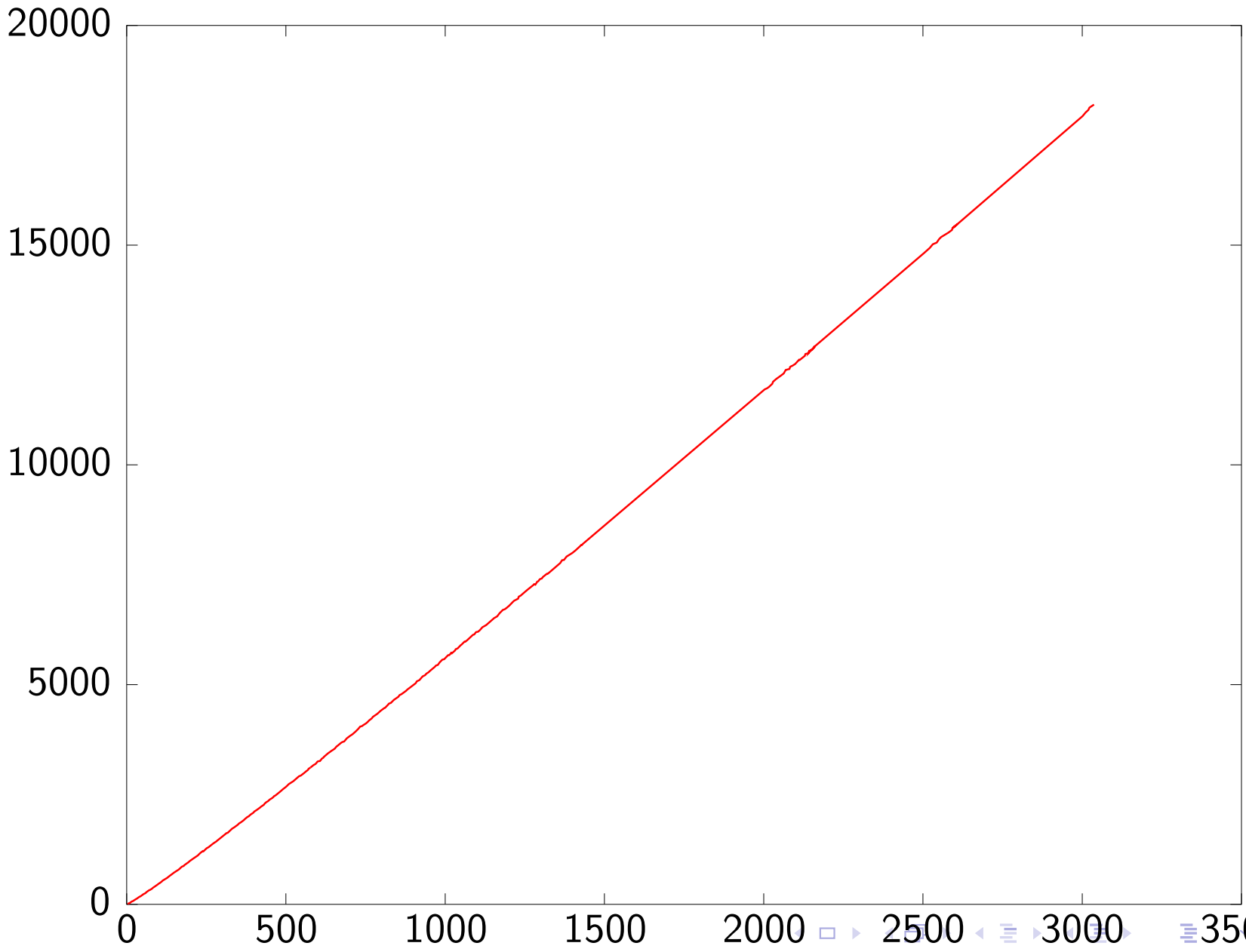
$N = 3$

- TLB:

$$\sqrt{2} \cdot q$$

(Pellegrino, 1999)

computational results



recursive constructions of complete caps

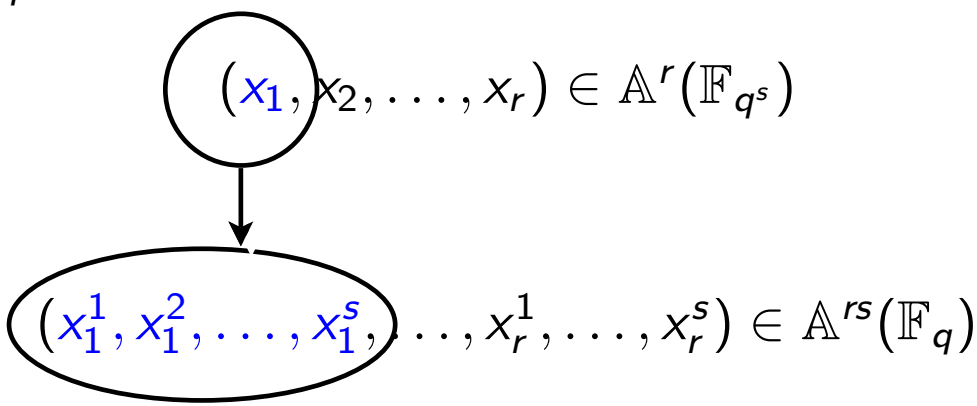
blow-up

- S cap in $\mathbb{A}^r(\mathbb{F}_{q^s})$

recursive constructions of complete caps

blow-up

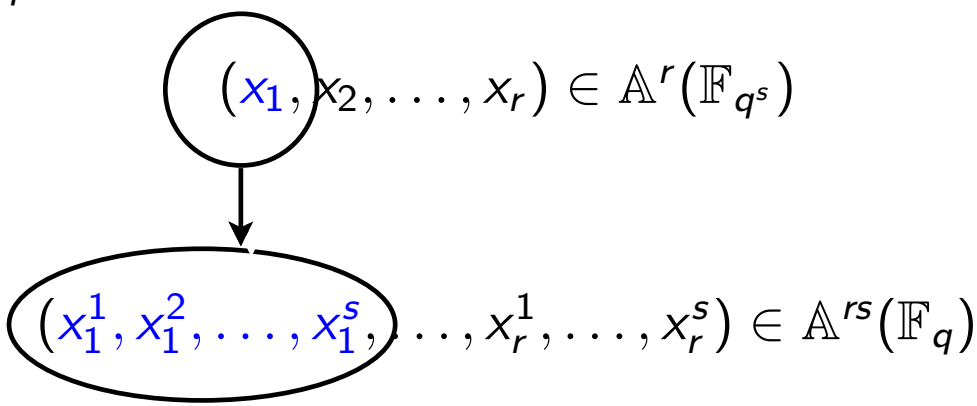
- S cap in $\mathbb{A}^r(\mathbb{F}_{q^s})$
- for each P in S, substitute each coordinate in $\mathbb{F}_{q^s}$ with its expansion over $\mathbb{F}_q$



recursive constructions of complete caps

blow-up

- S cap in $\mathbb{A}^r(\mathbb{F}_{q^s})$
- for each P in S, substitute each coordinate in $\mathbb{F}_{q^s}$ with its expansion over $\mathbb{F}_q$

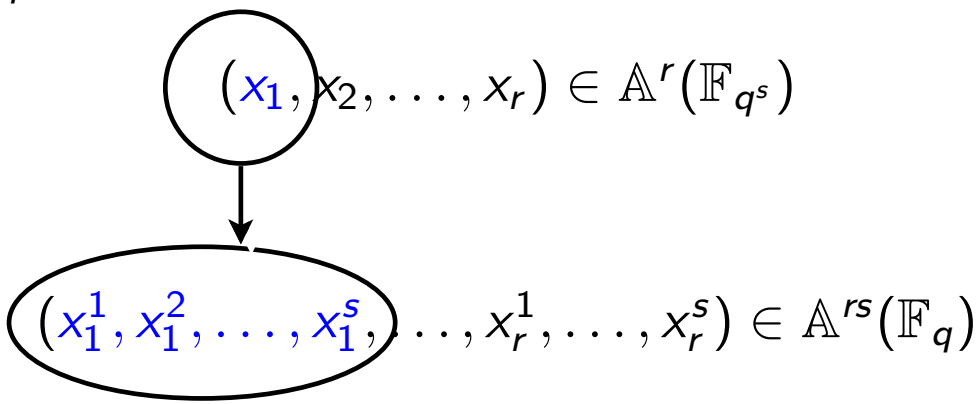


- the resulting subset of $\mathbb{A}^{rs}(\mathbb{F}_q)$ is a cap

recursive constructions of complete caps

blow-up

- S cap in $\mathbb{A}^r(\mathbb{F}_{q^s})$
- for each P in S, substitute each coordinate in $\mathbb{F}_{q^s}$ with its expansion over $\mathbb{F}_q$



- the resulting subset of $\mathbb{A}^{rs}(\mathbb{F}_q)$ is a cap

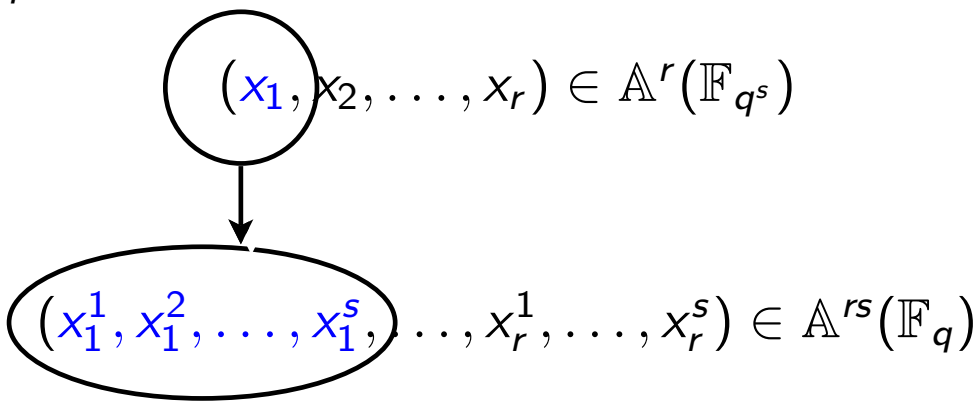
product

- S_1 cap in $\mathbb{A}^r(\mathbb{F}_q)$, S_2 cap in $\mathbb{A}^s(\mathbb{F}_q)$

recursive constructions of complete caps

blow-up

- S cap in $\mathbb{A}^r(\mathbb{F}_{q^s})$
- for each P in S, substitute each coordinate in $\mathbb{F}_{q^s}$ with its expansion over $\mathbb{F}_q$



- the resulting subset of $\mathbb{A}^{rs}(\mathbb{F}_q)$ is a cap

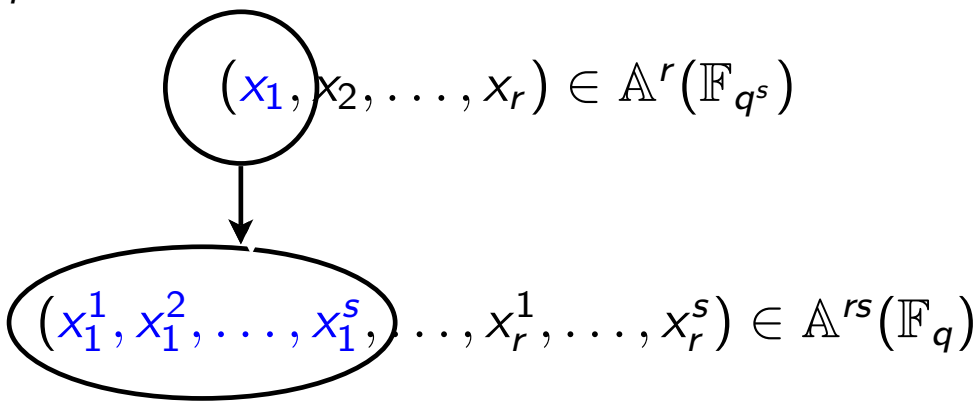
product

- S_1 cap in $\mathbb{A}^r(\mathbb{F}_q)$, S_2 cap in $\mathbb{A}^s(\mathbb{F}_q)$
- $S_1 \times S_2$ is a cap in $\mathbb{A}^{r+s}(\mathbb{F}_q)$

recursive constructions of complete caps

blow-up

- S cap in $\mathbb{A}^r(\mathbb{F}_{q^s})$
- for each P in S, substitute each coordinate in $\mathbb{F}_{q^s}$ with its expansion over $\mathbb{F}_q$



- the resulting subset of $\mathbb{A}^{rs}(\mathbb{F}_q)$ is a cap

product

- S_1 cap in $\mathbb{A}^r(\mathbb{F}_q)$, S_2 cap in $\mathbb{A}^s(\mathbb{F}_q)$
- $S_1 \times S_2$ is a cap in $\mathbb{A}^{r+s}(\mathbb{F}_q)$

- do such constructions preserve completeness?

recursive constructions of complete caps

T_N blow-up of a parabola of $\mathbb{A}^2(\mathbb{F}_{q^{N/2}})$

recursive constructions of complete caps

T_N blow-up of a parabola of $\mathbb{A}^2(\mathbb{F}_{q^{N/2}})$

(Davydov-Östergàrd, 2001)

T_N is complete in $\mathbb{A}^N(\mathbb{F}_q) \Leftrightarrow N/2$ is odd.

recursive constructions of complete caps

T_N blow-up of a parabola of $\mathbb{A}^2(\mathbb{F}_{q^{N/2}})$

(Davydov-Östergàrd, 2001)

T_N is complete in $\mathbb{A}^N(\mathbb{F}_q) \Leftrightarrow N/2$ is odd.

Problem: When $T_N \times S$ is complete?

external/internal points to a segment

external/internal points to a segment

(Segre, 1973)

P, P_1, P_2 distinct collinear points in $\mathbb{A}^2(\mathbb{F}_q)$

external/internal points to a segment

(Segre, 1973)

P, P_1, P_2 distinct collinear points in $\mathbb{A}^2(\mathbb{F}_q)$

the point P is internal or external to the segment P_1P_2 if

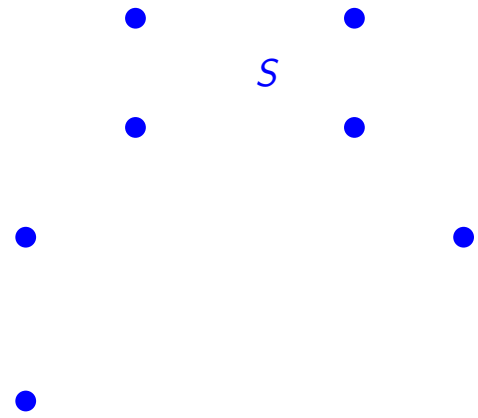
$$(x - x_1)(x - x_2) \text{ is a non-square in } \mathbb{F}_q \text{ or not,}$$

x, x_1, x_2 coordinates of P, P_1, P_2 w.r.t. any affine frame of ℓ.

bicovering and almost bicovering caps

bicovering and almost bicovering caps

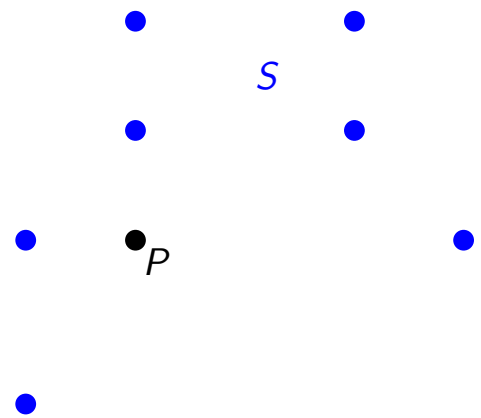
let S be a complete cap in $\mathbb{A}^2(\mathbb{F}_q)$.



bicovering and almost bicovering caps

let S be a complete cap in $\mathbb{A}^2(\mathbb{F}_q)$.

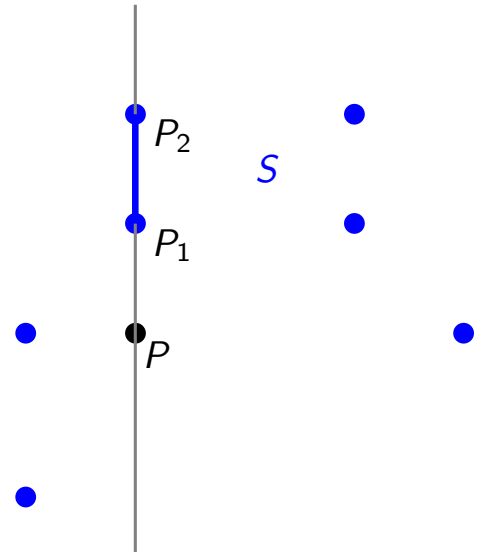
a point $P \notin S$ is bicovered by S if it is external to a segment P_1P_2, with $P_1, P_2 \in S$ and internal to another segment P_3P_4, with $P_3, P_4 \in S$



bicovering and almost bicovering caps

let S be a complete cap in $\mathbb{A}^2(\mathbb{F}_q)$.

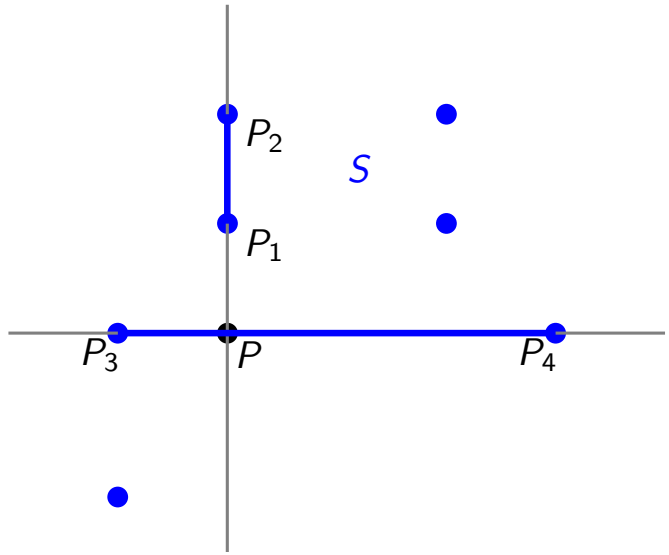
a point $P \notin S$ is bicovered by S if it is external to a segment P_1P_2, with $P_1, P_2 \in S$ and internal to another segment P_3P_4, with $P_3, P_4 \in S$



bicovering and almost bicovering caps

let S be a complete cap in $\mathbb{A}^2(\mathbb{F}_q)$.

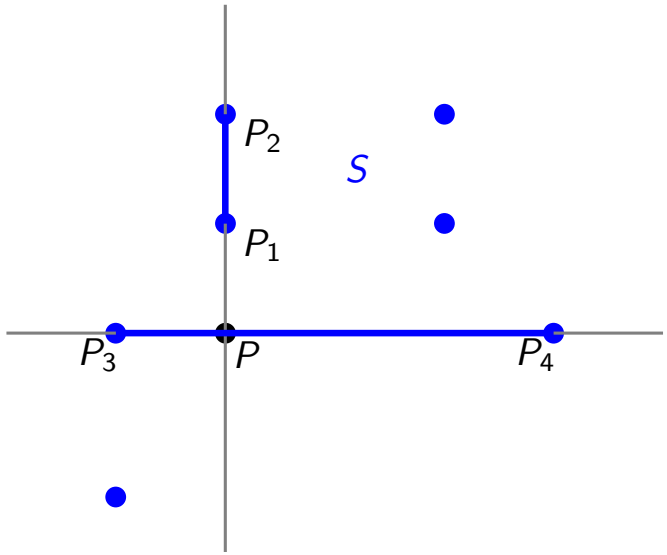
a point $P \notin S$ is bicovered by S if it is external to a segment P_1P_2, with $P_1, P_2 \in S$ and internal to another segment P_3P_4, with $P_3, P_4 \in S$



bicovering and almost bicovering caps

let S be a complete cap in $\mathbb{A}^2(\mathbb{F}_q)$.

a point $P \notin S$ is bicovered by S if it is external to a segment P_1P_2, with $P_1, P_2 \in S$ and internal to another segment P_3P_4, with $P_3, P_4 \in S$



definition

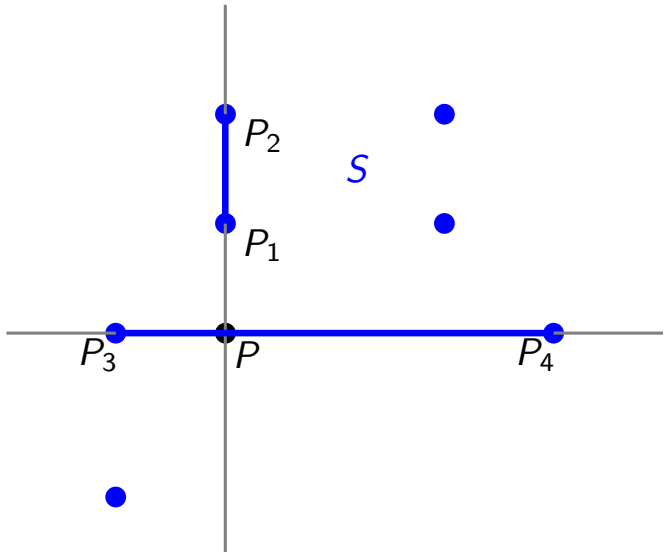
S is said to be

- *bicovering if for every $P \notin S$ is bicovered by S*

bicovering and almost bicovering caps

let S be a complete cap in $\mathbb{A}^2(\mathbb{F}_q)$.

a point $P \notin S$ is bicovered by S if it is external to a segment P_1P_2, with $P_1, P_2 \in S$ and internal to another segment P_3P_4, with $P_3, P_4 \in S$



definition

S is said to be

- *bicovering if for every $P \notin S$ is bicovered by S*
- *almost bicovering if there exists precisely one point not bicovered by S*

recursive constructions of complete caps

- T_N blow-up of a parabola in $\mathbb{A}^N(\mathbb{F}_q)$, $N \equiv 2 \pmod 4$
- S complete cap in $\mathbb{A}^2(\mathbb{F}_q)$

recursive constructions of complete caps

- T_N blow-up of a parabola in $\mathbb{A}^N(\mathbb{F}_q)$, $N \equiv 2 \pmod 4$
- S complete cap in $\mathbb{A}^2(\mathbb{F}_q)$

(G., 2007)

(i) $K_S = T_N \times S$ is complete if and only if S is bicovering

recursive constructions of complete caps

- T_N blow-up of a parabola in $\mathbb{A}^N(\mathbb{F}_q)$, $N \equiv 2 \pmod 4$
- S complete cap in $\mathbb{A}^2(\mathbb{F}_q)$

(G., 2007)

(i) $K_S = T_N \times S$ is complete if and only if S is bicovering

(ii) if S is almost bicovering, then

$$K_S \cup \{(a, a^2 - z_0, x_0, y_0) \mid a \in \mathbb{F}_{q^{N/2}}\}$$

is complete for some $x_0, y_0, z_0 \in \mathbb{F}_q$

bicovering caps in $\mathbb{A}^2(\mathbb{F}_q)$

remarks:

- no probabilistic result is known

bicovering caps in $\mathbb{A}^2(\mathbb{F}_q)$

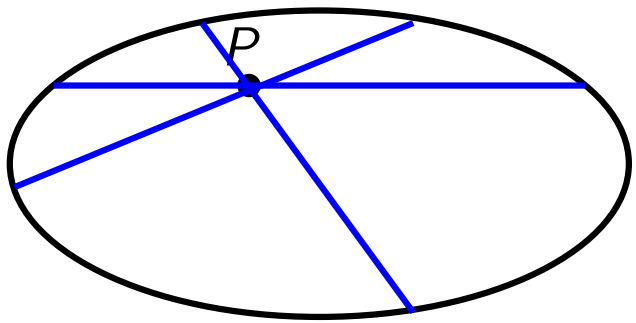
remarks:

- no probabilistic result is known
- no computational constructive method is known

bicovering caps in $\mathbb{A}^2(\mathbb{F}_q)$

remarks:

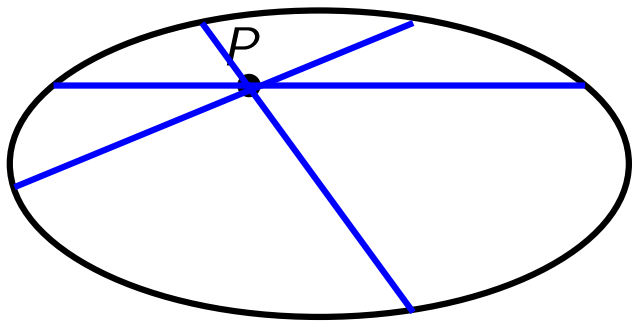
- no probabilistic result is known
- no computational constructive method is known
- in the Euclidean plane, no conic is bicovering or almost bicovering



bicovering caps in $\mathbb{A}^2(\mathbb{F}_q)$

remarks:

- no probabilistic result is known
- no computational constructive method is known
- in the Euclidean plane, no conic is bicovering or almost bicovering



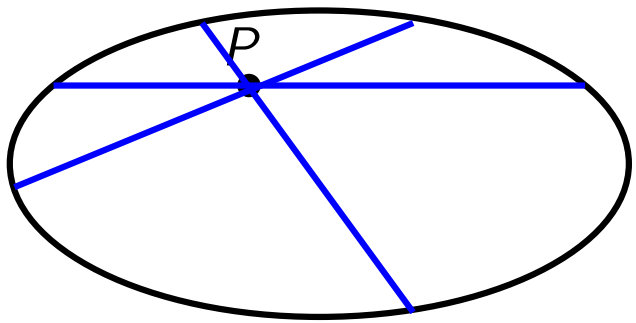
(Segre, 1973)

if $q > 13$, ellipses and hyperbolas are almost bicovering caps

bicovering caps in $\mathbb{A}^2(\mathbb{F}_q)$

remarks:

- no probabilistic result is known
- no computational constructive method is known
- in the Euclidean plane, no conic is bicovering or almost bicovering



(Segre, 1973)

if $q > 13$, ellipses and hyperbolas are almost bicovering caps

let $N \equiv 0 \pmod 4$; if $q > 13$, then there exists a complete cap of size $\#T_{N-2} \cdot [(q-1)+1] = q^{\frac{N}{2}}$

how to prove that an algebraic cap is bicovering

$$S = \{\underbrace{(f(t), g(t))}_{P_t} \mid t \in \mathbb{F}_q\}$$

how to prove that an algebraic cap is bicovering

$$S = \{\underbrace{(f(t), g(t))}_{P_t} \mid t \in \mathbb{F}_q\}$$

$P = (a, b) \in \mathbb{A}^2(\mathbb{F}_q)$

how to prove that an algebraic cap is bicovering

$$S = \{\underbrace{(f(t), g(t))}_{P_t} \mid t \in \mathbb{F}_q\}$$

$P = (a, b) \in \mathbb{A}^2(\mathbb{F}_q)$

(1) consider the space curve

$$\mathcal{Y}_P : \begin{cases} F_P(X, Y) = 0 \\ (a - f(X))(a - f(Y)) = Z^2 \end{cases}$$

how to prove that an algebraic cap is bicovering

$$S = \{\underbrace{(f(t), g(t))}_{P_t} \mid t \in \mathbb{F}_q\}$$

$P = (a, b) \in \mathbb{A}^2(\mathbb{F}_q)$

(1) consider the space curve

$$\mathcal{Y}_P : \begin{cases} F_P(X, Y) = 0 \\ (a - f(X))(a - f(Y)) = Z^2 \end{cases}$$

(2) apply Hasse-Weil to $\mathcal{Y}_P$ (if possible) and find a suitable point $(x, y, z) \in \mathcal{Y}_P(\mathbb{F}_q)$

how to prove that an algebraic cap is bicovering

$$S = \{\underbrace{(f(t), g(t))}_{P_t} \mid t \in \mathbb{F}_q\}$$

$P = (a, b) \in \mathbb{A}^2(\mathbb{F}_q)$

(1) consider the space curve

$$\mathcal{Y}_P : \begin{cases} F_P(X, Y) = 0 \\ (a - f(X))(a - f(Y)) = Z^2 \end{cases}$$

(2) apply Hasse-Weil to $\mathcal{Y}_P$ (if possible) and find a suitable point $(x, y, z) \in \mathcal{Y}_P(\mathbb{F}_q)$

the point P is external to the segment joining P_x and P_y

how to prove that an algebraic cap is bicovering

$$S = \{\underbrace{(f(t), g(t))}_{P_t} \mid t \in \mathbb{F}_q\}$$

$P = (a, b) \in \mathbb{A}^2(\mathbb{F}_q)$

(1) consider the space curve

$$\mathcal{Y}_{P,c} : \begin{cases} F_P(X, Y) = 0 \\ (a - f(X))(a - f(Y)) = cZ^2 \end{cases}$$

(2) apply Hasse-Weil to $\mathcal{Y}_P$ (if possible) and find a suitable point $(x, y, z) \in \mathcal{Y}_P(\mathbb{F}_q)$

the point P is external to the segment joining P_x and P_y

(3) fix a non-square c in $\mathbb{F}_q^*$ and repeat for $\mathcal{Y}_{P,c}$

bicovering caps from cubic curves

- the method works well for S a coset of a cubic $\mathcal{X}$, and P a point off the cubic.

bicovering caps from cubic curves

- the method works well for S a coset of a cubic $\mathcal{X}$, and P a point off the cubic.
- in order to bicover the points on the cubics, more cosets of the same subgroup are needed: the cosets corresponding to a maximal 3-independent subset in the factor group G/K

bicovering caps from cubic curves

- the method works well for S a coset of a cubic $\mathcal{X}$, and P a point off the cubic.
- in order to bicover the points on the cubics, more cosets of the same subgroup are needed: the cosets corresponding to a maximal 3-independent subset in the factor group G/K
- in the best case bicovering caps of size approximately $q^{7/8}$ are obtained

bicovering caps from cubic curves

- the method works well for S a coset of a cubic $\mathcal{X}$, and P a point off the cubic.
- in order to bicover the points on the cubics, more cosets of the same subgroup are needed: the cosets corresponding to a maximal 3-independent subset in the factor group G/K
- in the best case bicovering caps of size approximately $q^{7/8}$ are obtained
- for $N \equiv 0 \pmod 4$ complete caps of size approximately $q^{\frac{N}{2}-\frac{1}{8}}$ are obtained, provided that suitable divisors of $q, q-1, q+1$ exist

bicovering caps from cubic curves

- the method works well for S a coset of a cubic $\mathcal{X}$, and P a point off the cubic.
- in order to bicover the points on the cubics, more cosets of the same subgroup are needed: the cosets corresponding to a maximal 3-independent subset in the factor group G/K
- in the best case bicovering caps of size approximately $q^{7/8}$ are obtained
- for $N \equiv 0 \pmod 4$ complete caps of size approximately $q^{\frac{N}{2}-\frac{1}{8}}$ are obtained, provided that suitable divisors of $q, q-1, q+1$ exist
- if Voloch's gap is filled, we will have bicovering caps with roughly $q^{7/8}$ points for any odd q

$$\mathcal{X} : Y - X^3 = 0$$

(Anbar-Bartoli-G.-Platoni, 2013)

let

- $q = p^h$, with $p > 3$ a prime
- $m = p^{h'}$, with $h' < h$ and $m \leq \frac{\sqrt[4]{q}}{4}$

then there exists an almost bicovering cap contained in $\mathcal{X}$, of size

$$n = \begin{cases} (2\sqrt{m} - 3)\dfrac{q}{m}, & \text{if } h' \text{ is even} \\ \left(\sqrt{\dfrac{m}{p}} + \sqrt{mp} - 3\right)\dfrac{q}{m}, & \text{if } h' \text{ is odd} \end{cases} \sim q^{7/8}$$

$$\mathcal{X} : XY - (X-1)^3 = 0$$

(Anbar-Bartoli-G.-Platoni, 2013)

assume that

- $q = p^h$, with $p > 3$ a prime
- m is an odd divisor of $q-1$, with $(3, m) = 1$ and $m \leq \frac{\sqrt[4]{q}}{3.5}$
- $m = m_1 m_2$ s.t. $(m_1, m_2) = 1$ and $m_1, m_2 \geq 4$

then there exists a bicovering cap contained in $\mathcal{X}$ of size

$$n \leq \frac{m_1 + m_2}{m}(q-1) \sim q^{7/8}$$

$$\mathcal{X} : Y(X^2 - \beta) = 1$$

(Anbar-Bartoli-G.-Platoni, 2013)

assume that

- $q = p^h$, with $p > 3$ a prime
- m is a proper divisor of $q + 1$ such that $(m, 6) = 1$ and $m \leq \frac{\sqrt[4]{q}}{4}$
- $m = m_1 m_2$ with $(m_1, m_2) = 1$

then there exists an almost bicovering cap contained in $\mathcal{X}$ of size less than or equal to

$$(m_1 + m_2 - 3) \cdot \frac{q+1}{m} + 3 \sim q^{7/8}$$

$$\mathcal{X} : Y^2 - X^3 - AX - B = 0$$

(Anbar-G., 2012)

assume that

- $q = p^h$, with $p > 3$ a prime
- m is a prime divisor of $q - 1$, with $7 < m < \frac{1}{8}\sqrt[4]{q}$

then there exists a bicovering cap contained in $\mathcal{X}$ of size

$$n \leq 2\sqrt{m}\left(\left\lfloor \frac{q - 2\sqrt{q} + 1}{m} \right\rfloor + 31\right) \sim q^{7/8}$$

$$\ell(r, q)_{r-1,r+1}$$

$$\ell(r, q)_{r-1,r+1}$$

Reed-Solomon codes: $\ell(r, q)_{r-1,r+1} \leq q + 1$

AG codes from elliptic curves

- $\mathcal{X} : Y^2 = X^3 + AX + B \qquad 4A^3 + 27B^2 \neq 0$
- O common pole of x and y
- $P_1, \ldots, P_n$ rational points of $\mathcal{X}$ (distinct from O)

AG codes from elliptic curves

- $\mathcal{X} : Y^2 = X^3 + AX + B \qquad 4A^3 + 27B^2 \neq 0$
- O common pole of x and y
- $P_1, \ldots, P_n$ rational points of $\mathcal{X}$ (distinct from O)

$$C_r = C(D, G)^{\perp}, \qquad \text{where } G = rO, D = P_1 + \ldots + P_n, \qquad n > r$$

AG codes from elliptic curves

- $\mathcal{X} : Y^2 = X^3 + AX + B \qquad 4A^3 + 27B^2 \neq 0$
- O common pole of x and y
- $P_1, \ldots, P_n$ rational points of $\mathcal{X}$ (distinct from O)

$$C_r = C(D, G)^{\perp}, \qquad \text{where } G = rO, D = P_1 + \ldots + P_n, \qquad n > r$$

- C_r is an $[n, n-r, r+1]_q$-MDS-code if and only if for every $P_{i_1}, \ldots, P_{i_r}$

$$P_{i_1} \oplus \ldots \oplus P_{i_r} \neq O$$

AG codes from elliptic curves

- $\mathcal{X} : Y^2 = X^3 + AX + B \qquad 4A^3 + 27B^2 \neq 0$
- O common pole of x and y
- $P_1, \ldots, P_n$ rational points of $\mathcal{X}$ (distinct from O)

$$C_r = C(D, G)^{\perp}, \qquad \text{where } G = rO, D = P_1 + \ldots + P_n, \qquad n > r$$

- C_r is an $[n, n-r, r+1]_q$-MDS-code if and only if for every $P_{i_1}, \ldots, P_{i_r}$

$$P_{i_1} \oplus \ldots \oplus P_{i_r} \neq O$$

(Munuera, 1993)

If C_r is MDS then, for $n > r + 2$,

$$n \leq \frac{1}{2}(\#\mathcal{X}(\mathbb{F}_q) - 3 + 2r)$$

covering radius of elliptic MDS codes

a subset T of an abelian group H is *r-independent* if for each $a_1, \ldots, a_r \in T$,

$$a_1 + a_2 + \ldots + a_r \neq 0$$

covering radius of elliptic MDS codes

a subset T of an abelian group H is *r-independent* if for each $a_1, \ldots, a_r \in T$,

$$a_1 + a_2 + \ldots + a_r \neq 0$$

$\{P_1, \ldots, P_n\}$ maximal r-independent subset of $\mathcal{X}(\mathbb{F}_q)$

covering radius of elliptic MDS codes

a subset T of an abelian group H is *r-independent* if for each $a_1, \ldots, a_r \in T$,

$$a_1 + a_2 + \ldots + a_r \neq 0$$

$\{P_1, \ldots, P_n\}$ maximal r-independent subset of $\mathcal{X}(\mathbb{F}_q)$

- let

$$\phi_r : \mathcal{X} \to \mathbb{P}^{r-1} \qquad \phi_r = (1 : f_1 : \ldots : f_{r-1})$$

with

$$1, f_1, \ldots, f_{r-1} \qquad \text{basis of } L(rO)$$

covering radius of elliptic MDS codes

a subset T of an abelian group H is *r-independent* if for each $a_1, \ldots, a_r \in T$,

$$a_1 + a_2 + \ldots + a_r \neq 0$$

$\{P_1, \ldots, P_n\}$ maximal r-independent subset of $\mathcal{X}(\mathbb{F}_q)$

- let

$$\phi_r : \mathcal{X} \to \mathbb{P}^{r-1} \qquad \phi_r = (1 : f_1 : \ldots : f_{r-1})$$

with

$$1, f_1, \ldots, f_{r-1} \qquad \text{basis of } L(rO)$$

- $R(C_r) = r - 1$ if and only if each point in $\mathbb{P}^{r-1}(\mathbb{F}_q)$ belongs to the hyperplane generated by some

$$\phi_r(P_{i_1}), \phi_r(P_{i_2}), \ldots, \phi_r(P_{i_{r-1}})$$

(Bartoli-G.-Platoni, 2013)

if

- $(\mathcal{X}(\mathbb{F}_q), \oplus) \cong \mathbb{Z}_m \times K$ cyclic for $m > 3$ a prime
- $S = K \oplus P$ covers all the points in $\mathbb{A}^2(\mathbb{F}_q)$ off $\mathcal{X}$
- $T \supset S$ is a maximal r-independent subset of $\mathcal{X}(\mathbb{F}_q)$

(Bartoli-G.-Platoni, 2013)

if

- $(\mathcal{X}(\mathbb{F}_q), \oplus) \cong \mathbb{Z}_m \times K$ cyclic for $m > 3$ a prime
- $S = K \oplus P$ covers all the points in $\mathbb{A}^2(\mathbb{F}_q)$ off $\mathcal{X}$
- $T \supset S$ is a maximal r-independent subset of $\mathcal{X}(\mathbb{F}_q)$

then almost every point in $\mathbb{P}^{r-1}(\mathbb{F}_q)$ belongs to some hyperplane generated by $r-1$ points of $\phi_r(T)$

(Bartoli-G.-Platoni, 2013)

if

- $(\mathcal{X}(\mathbb{F}_q), \oplus) \cong \mathbb{Z}_m \times K$ cyclic for $m > 3$ a prime
- $S = K \oplus P$ covers all the points in $\mathbb{A}^2(\mathbb{F}_q)$ off $\mathcal{X}$
- $T \supset S$ is a maximal r-independent subset of $\mathcal{X}(\mathbb{F}_q)$

then almost every point in $\mathbb{P}^{r-1}(\mathbb{F}_q)$ belongs to some hyperplane generated by $r-1$ points of $\phi_r(T)$

- if m is a prime divisor of $q-1$ with $m < \sqrt[4]{q/64}$, then

$$\ell(r,q)_{r-1,r+1} \leq (\lceil r/2 \rceil - 1)(|S|-1) + 2\frac{m+1}{r-2} + 2r$$

(Bartoli-G.-Platoni, 2013)

if

- $(\mathcal{X}(\mathbb{F}_q), \oplus) \cong \mathbb{Z}_m \times K$ cyclic for $m > 3$ a prime
- $S = K \oplus P$ covers all the points in $\mathbb{A}^2(\mathbb{F}_q)$ off $\mathcal{X}$
- $T \supset S$ is a maximal r-independent subset of $\mathcal{X}(\mathbb{F}_q)$

then almost every point in $\mathbb{P}^{r-1}(\mathbb{F}_q)$ belongs to some hyperplane generated by $r-1$ points of $\phi_r(T)$

- if m is a prime divisor of $q-1$ with $m < \sqrt[4]{q/64}$, then

$$\ell(r,q)_{r-1,r+1} \leq (\lceil r/2 \rceil - 1)(|S|-1) + 2\frac{m+1}{r-2} + 2r \sim (\lceil r/2 \rceil - 1)q^{3/4}$$

(Bartoli-G.-Platoni, 2013)

if

- $(\mathcal{X}(\mathbb{F}_q), \oplus) \cong \mathbb{Z}_m \times K$ cyclic for $m > 3$ a prime
- $S = K \oplus P$ covers all the points in $\mathbb{A}^2(\mathbb{F}_q)$ off $\mathcal{X}$
- $T \supset S$ is a maximal r-independent subset of $\mathcal{X}(\mathbb{F}_q)$

then almost every point in $\mathbb{P}^{r-1}(\mathbb{F}_q)$ belongs to some hyperplane generated by $r-1$ points of $\phi_r(T)$

- if m is a prime divisor of $q-1$ with $m < \sqrt[4]{q/64}$, then

$$\ell(r,q)_{r-1,r+1} \leq (\lceil r/2 \rceil - 1)(|S|-1) + 2\frac{m+1}{r-2} + 2r \sim (\lceil r/2 \rceil - 1) q^{3/4}$$

problems

- explanation for experimental results

problems

- explanation for experimental results
- Voloch's proof for plane caps in elliptic cubics

problems

- explanation for experimental results
- Voloch's proof for plane caps in elliptic cubics
- bicovering caps in dimensions different from 2

problems

- explanation for experimental results
- Voloch's proof for plane caps in elliptic cubics
- bicovering caps in dimensions different from 2
- non-recursive constructions in higher dimensions

problems

- explanation for experimental results
- Voloch's proof for plane caps in elliptic cubics
- bicovering caps in dimensions different from 2
- non-recursive constructions in higher dimensions
- probabilistic results intrinsic to higher dimensions