## Weierstrass semigroups at several points, total inflection points on curves and coding theory

Cícero Carvalho<br>Faculdade de Matemática<br>Universidade Federal de Uberlândia<br>

Algebraic curves over finite fields RICAM - November 11-15, 2013

Partially supported by CNPq, FAPEMIG and CAPES

## Weierstrass semigroup at several points

Let $X$ be a smooth complete irreducible curve of genus $g \geq 1$ defined over a field $\mathbb{F}$, assumed to be the full field of constants of $\mathbb{F}(X)$.

Let $P_{1}, \ldots, D_{m}$ be distinet rational points of $X$
Definition The Weierstrass semigroup at $P_{1}, \ldots, P_{m}$ is defined as
$H=H\left(P_{1}, \ldots, P_{m}\right):=\left\{\left(\alpha_{1}, \ldots, \alpha_{m}\right) \in \mathbb{N}_{0}^{m} \mid \exists f \in \mathbb{F}(X)\right.$ with

$$
\left.\operatorname{div}_{\infty}(f)=\alpha_{1} P_{1}+\cdots+\alpha_{m} P_{m}\right\}
$$

Its systematic study was initiated by S. J. Kim and M. Homma in mid 90 's. They studied specially the case $m=2$; investigated properties of $H$ and its relationship with the theory of algebraic geometry (Goppa) codes. In a joint work with $F$. Torres we extended their results for any value of $m$, and also applied the results to obtain better lower bounds for the minimum distance of certain algebraic geometry codes.
A similar application of these semigroups was recently done by Korchmáros and Nagy, which improved such bounds for certain codes previously studied by Matthews and Michel.

## Weierstrass semigroup at several points

Let $X$ be a smooth complete irreducible curve of genus $g \geq 1$ defined over a field $\mathbb{F}$, assumed to be the full field of constants of $\mathbb{F}(X)$.


Definition The Weierstrass semigroup at $P_{1}, \ldots, P_{m}$ is defined as
$H=H\left(P_{1}, \ldots, P_{m}\right):=\left\{\left(\alpha_{1}, \ldots, \alpha_{m}\right) \in \mathbb{N}_{0} \mid \exists f \in \mathbb{F}(X)\right.$ with
Its systematic study was initiated by S. J. Kim and M. Homma in mid 90 's. They studied specially the case $m=2$; investigated properties of $H$ and its relationship with the theory of algebraic geometry (Goppa) codes. In a joint work with F. Torres we extended their results for any value of $m$, and also applied the results to obtain better lower bounds for the minimum distance of certain algebraic geometry codes.
A similar application of these semigroups was recently done by
Korchmáros and Nagy, which improved such bounds for certain codes previously studied by Matthews and Michel

## Weierstrass semigroup at several points

Let $X$ be a smooth complete irreducible curve of genus $g \geq 1$ defined over a field $\mathbb{F}$, assumed to be the full field of constants of $\mathbb{F}(X)$.

Let $P_{1}, \ldots, P_{m}$ be distinct rational points of $X$.
Definition The Weierstrass semigroup at $P_{1}, \ldots, P_{m}$ is defined as
$H=H\left(P_{1}, \ldots, P_{m}\right):=\left\{\left(\alpha_{1}, \ldots, \alpha_{m}\right) \in \mathbb{N}_{0}^{m} \mid \exists f \in \mathbb{F}(X)\right.$ with
Its systematic study was initiated by S. J. Kim and M. Homma in mid 90 's. They studied specially the case $m=2$; investigated properties of $H$ and its relationship with the theory of algebraic geometry (Goppa) codes. In a joint work with F. Torres we extended their results for any value of $m$, and also applied the results to obtain better lower bounds for the minimum distance of certain algebraic geometry codes.
A similar application of these semigroups was recently done by
Korchmáros and Nagy, which improved such bounds for certain codes previously studied by Matthews and Michel

## Weierstrass semigroup at several points

Let $X$ be a smooth complete irreducible curve of genus $g \geq 1$ defined over a field $\mathbb{F}$, assumed to be the full field of constants of $\mathbb{F}(X)$.

Let $P_{1}, \ldots, P_{m}$ be distinct rational points of $X$.
Definition The Weierstrass semigroup at $P_{1}, \ldots, P_{m}$ is defined as


## Weierstrass semigroup at several points

Let $X$ be a smooth complete irreducible curve of genus $g \geq 1$ defined over a field $\mathbb{F}$, assumed to be the full field of constants of $\mathbb{F}(X)$.

Let $P_{1}, \ldots, P_{m}$ be distinct rational points of $X$.
Definition The Weierstrass semigroup at $P_{1}, \ldots, P_{m}$ is defined as

$$
\begin{aligned}
H=H\left(P_{1}, \ldots, P_{m}\right):=\left\{\left(\alpha_{1}, \ldots, \alpha_{m}\right)\right. & \in \mathbb{N}_{0}^{m} \mid \exists f \in \mathbb{F}(X) \text { with } \\
& \left.\operatorname{div}_{\infty}(f)=\alpha_{1} P_{1}+\cdots+\alpha_{m} P_{m}\right\}
\end{aligned}
$$

Its systematic study was initiated by S. J. Kim and M. Homma in mid
90 's. They studied specially the case $m=2$; investigated properties of $H$ and its relationship with the theory of algebraic geometry (Goppa) codes. In a joint work with F. Torres we extended their results for any value of $m$, and also applied the results to obtain better lower bounds for the minimum distance of certain algebraic geometry codes.
A similar application of these semigroups was recently done by
Korchmáros and Nagy, which improved such bounds for certain codes previously studied by Matthews and Michel.

## Weierstrass semigroup at several points

Let $X$ be a smooth complete irreducible curve of genus $g \geq 1$ defined over a field $\mathbb{F}$, assumed to be the full field of constants of $\mathbb{F}(X)$.

Let $P_{1}, \ldots, P_{m}$ be distinct rational points of $X$.
Definition The Weierstrass semigroup at $P_{1}, \ldots, P_{m}$ is defined as

$$
\begin{aligned}
H=H\left(P_{1}, \ldots, P_{m}\right):=\left\{\left(\alpha_{1}, \ldots, \alpha_{m}\right)\right. & \in \mathbb{N}_{0}^{m} \mid \exists f \in \mathbb{F}(X) \text { with } \\
& \left.\operatorname{div}_{\infty}(f)=\alpha_{1} P_{1}+\cdots+\alpha_{m} P_{m}\right\}
\end{aligned}
$$

Its systematic study was initiated by S. J. Kim and M. Homma in mid 90 's. They studied specially the case $m=2$; investigated properties of $H$ and its relationship with the theory of algebraic geometry (Goppa) codes.
In a joint work with F. Torres we extended their results for any value of $m$ and also applied the results to obtain better lower bounds for the minimum distance of certain algehraic oeometry codes
A similar application of these semigroups was recently done by
Korchmáros and Nagy, which improved such bounds for certain codes previously studied by Matthews and Michel.

## Weierstrass semigroup at several points

Let $X$ be a smooth complete irreducible curve of genus $g \geq 1$ defined over a field $\mathbb{F}$, assumed to be the full field of constants of $\mathbb{F}(X)$.
Let $P_{1}, \ldots, P_{m}$ be distinct rational points of $X$.
Definition The Weierstrass semigroup at $P_{1}, \ldots, P_{m}$ is defined as
$H=H\left(P_{1}, \ldots, P_{m}\right):=\left\{\left(\alpha_{1}, \ldots, \alpha_{m}\right) \in \mathbb{N}_{0}^{m} \mid \exists f \in \mathbb{F}(X)\right.$ with

$$
\left.\operatorname{div}_{\infty}(f)=\alpha_{1} P_{1}+\cdots+\alpha_{m} P_{m}\right\}
$$

Its systematic study was initiated by S. J. Kim and M. Homma in mid 90 's. They studied specially the case $m=2$; investigated properties of $H$ and its relationship with the theory of algebraic geometry (Goppa) codes. In a joint work with F . Torres we extended their results for any value of $m$, and also applied the results to obtain better lower bounds for the minimum distance of certain algebraic geometry codes.
A similar application of these semigroups was recently done by
Korchmáros and Nagy, which improved such bounds for certain codes
previously studied hy Matthews and Michel

## Weierstrass semigroup at several points

Let $X$ be a smooth complete irreducible curve of genus $g \geq 1$ defined over a field $\mathbb{F}$, assumed to be the full field of constants of $\mathbb{F}(X)$.
Let $P_{1}, \ldots, P_{m}$ be distinct rational points of $X$.
Definition The Weierstrass semigroup at $P_{1}, \ldots, P_{m}$ is defined as
$H=H\left(P_{1}, \ldots, P_{m}\right):=\left\{\left(\alpha_{1}, \ldots, \alpha_{m}\right) \in \mathbb{N}_{0}^{m} \mid \exists f \in \mathbb{F}(X)\right.$ with

$$
\left.\operatorname{div}_{\infty}(f)=\alpha_{1} P_{1}+\cdots+\alpha_{m} P_{m}\right\}
$$

Its systematic study was initiated by S. J. Kim and M. Homma in mid 90 's. They studied specially the case $m=2$; investigated properties of $H$ and its relationship with the theory of algebraic geometry (Goppa) codes. In a joint work with F. Torres we extended their results for any value of $m$, and also applied the results to obtain better lower bounds for the minimum distance of certain algebraic geometry codes.
A similar application of these semigroups was recently done by Korchmáros and Nagy, which improved such bounds for certain codes previously studied by Matthews and Michel.

## Weierstrass semigroup at several points

We will write $\mathbf{n}:=\left(n_{1}, \ldots, n_{m}\right) \in \mathbb{N}_{0}^{m}, \mathbf{e}_{i} \in \mathbb{N}_{0}^{m}$ for the $m$-tuple that has 1 in the $i$-th position and 0 in the others, $L(\mathbf{n}):=L\left(n_{1} P_{1}+\cdots+n_{m} P_{m}\right)$ and $\ell(\mathbf{n}):=\operatorname{dim} L(\mathbf{n})$.

Lemma. The following are equivalent:
(i) $\mathbf{n} \in H$;
(ii) $\ell(\mathbf{n})=\ell\left(\mathbf{n}-\mathrm{e}_{i}\right)+1$ for all $i=1, \ldots, m$;
(iii) The linear system $\left|n_{1} P_{1}+\cdots+n_{m} P_{m}\right|$ is base-point free.

We call $\mathbb{N}_{0}^{m} \backslash H$ the set of gaps of $H$, it is a finite set whose cardinality may vary with $P_{1}, \ldots, P_{m}$. For example, if $X$ is a hyperelliptic curve of genus 4 , and $m=2$ we get:


[^0]
## Weierstrass semigroup at several points

We will write $\mathbf{n}:=\left(n_{1}, \ldots, n_{m}\right) \in \mathbb{N}_{0}^{m}, e_{i} \in \mathbb{N}_{0}^{m}$ for the $m$-tuple that has 1 in the $i$-th position and 0 in the others, $L(\mathbf{n}):=L\left(n_{1} P_{1}+\cdots+n_{m} P_{m}\right)$ and $\ell(\mathbf{n}):=\operatorname{dim} L(\mathbf{n})$.

Lemma. The following are equivalent:
(i) $n \in H$;
(ii) $\ell(\mathbf{n})=\ell\left(\mathbf{n}-\mathbf{e}_{i}\right)+1$ for all $i=1, \ldots, m$;
(iii) The linear system $\left|n_{1} P_{1}+\cdots+n_{m} P_{m}\right|$ is base-point free.

We call $\mathbb{N}_{0}^{m} \backslash H$ the set of gaps of $H$, it is a finite set whose cardinality may vary with $P_{1}, \ldots, P_{m}$. For example, if $X$ is a hyperelliptic curve of genus 4 , and $m=2$ we get:


## Weierstrass semigroup at several points

We will write $\mathbf{n}:=\left(n_{1}, \ldots, n_{m}\right) \in \mathbb{N}_{0}^{m}, \mathbf{e}_{i} \in \mathbb{N}_{0}^{m}$ for the $m$-tuple that has 1 in the $i$-th position and 0 in the others, $L(\mathbf{n}):=L\left(n_{1} P_{1}+\cdots+n_{m} P_{m}\right)$

Lemma. The following are equivalent:
(i) $\mathbf{n} \in H$;
(ii) $\ell(\mathbf{n})=\ell\left(\mathbf{n}-\mathbf{e}_{i}\right)+1$ for all $i=1, \ldots, m$;
(iii) The linear system $\left|n_{1} P_{1}+\cdots+n_{m} P_{m}\right|$ is base-point free.

We call $\mathbb{N}_{0} m \backslash H$ the set of gaps of $H$, it is a finite set whose carclinality may vary with $P_{1}, \ldots, P_{m}$. For example, if $X$ is a hyperelliptic curve of genus 4 , and $m=2$ we get:


## Weierstrass semigroup at several points

We will write $\mathbf{n}:=\left(n_{1}, \ldots, n_{m}\right) \in \mathbb{N}_{0}^{m}, \mathbf{e}_{i} \in \mathbb{N}_{0}^{m}$ for the $m$-tuple that has 1 in the $i$-th position and 0 in the others, $L(\mathbf{n}):=L\left(n_{1} P_{1}+\cdots+n_{m} P_{m}\right)$ and $\ell(\mathbf{n}):=\operatorname{dim} L(\mathbf{n})$.

Lemma. The following are equivalent:
(i) $\mathbf{n} \in H$;
(ii) $\ell(\mathbf{n})=\ell\left(\mathbf{n}-\mathbf{e}_{i}\right)+1$ for all $i=1, \ldots, m$;
(iii) The linear system $\left|n_{1} P_{1}+\cdots+n_{m} P_{m}\right|$ is base-point free.

We call $\mathbb{N}_{0} m \backslash H$ the set of gaps of $H$, it is a finite set whose cardinality may vary with $P_{1}, \ldots, P_{m}$. For example, if $X$ is a hyperelliptic curve of genus 4 , and $m=2$ we get:


## Weierstrass semigroup at several points

```
We will write n := (n},\ldots,\mp@subsup{n}{m}{})\in\mp@subsup{\mathbb{N}}{0}{m},\mp@subsup{\mathbf{e}}{i}{}\in\mp@subsup{\mathbb{N}}{0}{m}\mathrm{ for the m-tuple that has 1
in the i-th position and 0 in the others, L(\mathbf{n}):=L(\mp@subsup{n}{1}{}\mp@subsup{P}{1}{}+\cdots+\mp@subsup{n}{m}{}\mp@subsup{P}{m}{})
and \ell(\mathbf{n}):=\operatorname{dim}L(\mathbf{n}).
```

Lemma. The following are equivalent:
(i) $\mathbf{n} \in H$;
(ii) $\ell(\mathbf{n})=\ell\left(\mathrm{n}-\mathrm{e}_{\mathrm{i}}\right)+1$ for all $i=1, \ldots, m_{\text {; }}$
(iii) The linear system $\left|n_{1} P_{1}+\cdots+n_{m} P_{m}\right|$ is base-point free.
We call $\mathbb{N}_{0}^{m} \backslash H$ the set of gaps of $H$, it is a finite set whose cardinality
may vary with $P_{1}, \ldots, P_{m}$. For example, if $X$ is a hyperelliptic curve of
genus 4 , and $m=2$ we get:


## Weierstrass semigroup at several points

$$
\begin{aligned}
& \text { We will write } \mathbf{n}:=\left(n_{1}, \ldots, n_{m}\right) \in \mathbb{N}_{0}^{m}, \quad \mathbf{e}_{i} \in \mathbb{N}_{0}^{m} \text { for the } m \text {-tuple that has } 1 \\
& \text { in the } i \text {-th position and } 0 \text { in the others, } L(\mathbf{n}):=L\left(n_{1} P_{1}+\cdots+n_{m} P_{m}\right) \\
& \text { and } \ell(\mathbf{n}):=\operatorname{dim} L(\mathbf{n}) \text {. } \\
& \text { Lemma. The following are equivalent: }
\end{aligned}
$$

(ii) $\ell(\mathbf{n})=\ell\left(\mathbf{n}-\mathbf{e}_{i}\right)+1$ for all $i=1, \ldots, m$;
(iii) The linear system $\left|n_{1} P_{1}+\cdots+n_{m} P_{m}\right|$ is base-point free.

We call $\mathbb{N}_{0}^{m} \backslash H$ the set of gaps of $H$, it is a finite set whose cardinality may vary with $P_{1}, \ldots, P_{m}$. For example, if $X$ is a hyperelliptic curve of genus 4 , and $m=2$ we get:


## Weierstrass semigroup at several points

We will write $\mathbf{n}:=\left(n_{1}, \ldots, n_{m}\right) \in \mathbb{N}_{0}^{m}, \mathbf{e}_{i} \in \mathbb{N}_{0}^{m}$ for the $m$-tuple that has 1 in the $i$-th position and 0 in the others, $L(\mathbf{n}):=L\left(n_{1} P_{1}+\cdots+n_{m} P_{m}\right)$ and $\ell(\mathbf{n}):=\operatorname{dim} L(\mathbf{n})$.

Lemma. The following are equivalent:
(i) $\mathbf{n} \in H$;
(ii) $\ell(\mathbf{n})=\ell\left(\mathbf{n}-\mathbf{e}_{i}\right)+1$ for all $i=1, \ldots, m$;
(iii) The linear system $\left|n_{1} P_{1}+\cdots+n_{m} P_{m}\right|$ is base-point free.

We call $\mathbb{N}_{0}^{m} \backslash H$ the set of gaps of $H$, it is a finite set whose cardinality may vary with $P_{1}, \ldots, P_{m}$. For example, if $X$ is a hyperelliptic curve of genus 4 , and $m=2$ we get:


## Weierstrass semigroup at several points

We will write $\mathbf{n}:=\left(n_{1}, \ldots, n_{m}\right) \in \mathbb{N}_{0}^{m}, \mathbf{e}_{i} \in \mathbb{N}_{0}^{m}$ for the $m$-tuple that has 1 in the $i$-th position and 0 in the others, $L(\mathbf{n}):=L\left(n_{1} P_{1}+\cdots+n_{m} P_{m}\right)$ and $\ell(\mathbf{n}):=\operatorname{dim} L(\mathbf{n})$.

Lemma. The following are equivalent:
(i) $\mathbf{n} \in H$;
(ii) $\ell(\mathbf{n})=\ell\left(\mathbf{n}-\mathbf{e}_{i}\right)+1$ for all $i=1, \ldots, m$;
(iii) The linear system $\left|n_{1} P_{1}+\cdots+n_{m} P_{m}\right|$ is base-point free.

We call $\mathbb{N}_{0}^{m} \backslash H$ the set of gaps of $H$, it is a finite set whose cardinality may vary with $P_{1}, \ldots, P_{m}$. For example, if $X$ is a hyperelliptic curve of genus 4 , and $m=2$ we get:

## Weierstrass semigroup at several points

We will write $\mathbf{n}:=\left(n_{1}, \ldots, n_{m}\right) \in \mathbb{N}_{0}^{m}, \mathbf{e}_{i} \in \mathbb{N}_{0}^{m}$ for the $m$-tuple that has 1 in the $i$-th position and 0 in the others, $L(\mathbf{n}):=L\left(n_{1} P_{1}+\cdots+n_{m} P_{m}\right)$ and $\ell(\mathbf{n}):=\operatorname{dim} L(\mathbf{n})$.

Lemma. The following are equivalent:
(i) $\mathbf{n} \in H$;
(ii) $\ell(\mathbf{n})=\ell\left(\mathbf{n}-\mathbf{e}_{i}\right)+1$ for all $i=1, \ldots, m$;
(iii) The linear system $\left|n_{1} P_{1}+\cdots+n_{m} P_{m}\right|$ is base-point free.

We call $\mathbb{N}_{0}^{m} \backslash H$ the set of gaps of $H$, it is a finite set whose cardinality may vary with $P_{1}, \ldots, P_{m}$. For example, if $X$ is a hyperelliptic curve of genus 4 , and $m=2$ we get:

## Weierstrass semigroup at several points

We will write $\mathbf{n}:=\left(n_{1}, \ldots, n_{m}\right) \in \mathbb{N}_{0}^{m}, \mathbf{e}_{i} \in \mathbb{N}_{0}^{m}$ for the $m$-tuple that has 1 in the $i$-th position and 0 in the others, $L(\mathbf{n}):=L\left(n_{1} P_{1}+\cdots+n_{m} P_{m}\right)$ and $\ell(\mathbf{n}):=\operatorname{dim} L(\mathbf{n})$.

Lemma. The following are equivalent:
(i) $\mathbf{n} \in H$;
(ii) $\ell(\mathbf{n})=\ell\left(\mathbf{n}-\mathbf{e}_{i}\right)+1$ for all $i=1, \ldots, m$;
(iii) The linear system $\left|n_{1} P_{1}+\cdots+n_{m} P_{m}\right|$ is base-point free.

We call $\mathbb{N}_{0}^{m} \backslash H$ the set of gaps of $H$, it is a finite set whose cardinality

genus 4 , and $m=2$ we get:

## Weierstrass semigroup at several points

We will write $\mathbf{n}:=\left(n_{1}, \ldots, n_{m}\right) \in \mathbb{N}_{0}^{m}, \mathbf{e}_{i} \in \mathbb{N}_{0}^{m}$ for the $m$-tuple that has 1 in the $i$-th position and 0 in the others, $L(\mathbf{n}):=L\left(n_{1} P_{1}+\cdots+n_{m} P_{m}\right)$ and $\ell(\mathbf{n}):=\operatorname{dim} L(\mathbf{n})$.

Lemma. The following are equivalent:
(i) $\mathbf{n} \in H$;
(ii) $\ell(\mathbf{n})=\ell\left(\mathbf{n}-\mathbf{e}_{i}\right)+1$ for all $i=1, \ldots, m$;
(iii) The linear system $\left|n_{1} P_{1}+\cdots+n_{m} P_{m}\right|$ is base-point free.

We call $\mathbb{N}_{0}^{m} \backslash H$ the set of gaps of $H$, it is a finite set whose cardinality may vary with $P_{1}, \ldots, P_{m}$.
genus 4 , and $m=2$ we get:

## Weierstrass semigroup at several points

We will write $\mathbf{n}:=\left(n_{1}, \ldots, n_{m}\right) \in \mathbb{N}_{0}^{m}, \mathbf{e}_{i} \in \mathbb{N}_{0}^{m}$ for the $m$-tuple that has 1 in the $i$-th position and 0 in the others, $L(\mathbf{n}):=L\left(n_{1} P_{1}+\cdots+n_{m} P_{m}\right)$ and $\ell(\mathbf{n}):=\operatorname{dim} L(\mathbf{n})$.

Lemma. The following are equivalent:
(i) $\mathbf{n} \in H$;
(ii) $\ell(\mathbf{n})=\ell\left(\mathbf{n}-\mathbf{e}_{i}\right)+1$ for all $i=1, \ldots, m$;
(iii) The linear system $\left|n_{1} P_{1}+\cdots+n_{m} P_{m}\right|$ is base-point free.

We call $\mathbb{N}_{0}^{m} \backslash H$ the set of gaps of $H$, it is a finite set whose cardinality may vary with $P_{1}, \ldots, P_{m}$. For example, if $X$ is a hyperelliptic curve of genus 4 , and $m=2$ we get:
$\square$

## Weierstrass semigroup at several points

We will write $\mathbf{n}:=\left(n_{1}, \ldots, n_{m}\right) \in \mathbb{N}_{0}^{m}, \mathbf{e}_{i} \in \mathbb{N}_{0}^{m}$ for the $m$-tuple that has 1 in the $i$-th position and 0 in the others, $L(\mathbf{n}):=L\left(n_{1} P_{1}+\cdots+n_{m} P_{m}\right)$ and $\ell(\mathbf{n}):=\operatorname{dim} L(\mathbf{n})$.

Lemma. The following are equivalent:
(i) $\mathbf{n} \in H$;
(ii) $\ell(\mathbf{n})=\ell\left(\mathbf{n}-\mathbf{e}_{i}\right)+1$ for all $i=1, \ldots, m$;
(iii) The linear system $\left|n_{1} P_{1}+\cdots+n_{m} P_{m}\right|$ is base-point free.

We call $\mathbb{N}_{0}^{m} \backslash H$ the set of gaps of $H$, it is a finite set whose cardinality may vary with $P_{1}, \ldots, P_{m}$. For example, if $X$ is a hyperelliptic curve of genus 4 , and $m=2$ we get:


[^1]
## Weierstrass semigroup at several points

We will write $\mathbf{n}:=\left(n_{1}, \ldots, n_{m}\right) \in \mathbb{N}_{0}^{m}, \mathbf{e}_{i} \in \mathbb{N}_{0}^{m}$ for the $m$-tuple that has 1 in the $i$-th position and 0 in the others, $L(\mathbf{n}):=L\left(n_{1} P_{1}+\cdots+n_{m} P_{m}\right)$ and $\ell(\mathbf{n}):=\operatorname{dim} L(\mathbf{n})$.

Lemma. The following are equivalent:
(i) $\mathbf{n} \in H$;
(ii) $\ell(\mathbf{n})=\ell\left(\mathbf{n}-\mathbf{e}_{i}\right)+1$ for all $i=1, \ldots, m$;
(iii) The linear system $\left|n_{1} P_{1}+\cdots+n_{m} P_{m}\right|$ is base-point free.

We call $\mathbb{N}_{0}^{m} \backslash H$ the set of gaps of $H$, it is a finite set whose cardinality may vary with $P_{1}, \ldots, P_{m}$. For example, if $X$ is a hyperelliptic curve of genus 4 , and $m=2$ we get:


[^2]

## Weierstrass semigroup at several points

```
From now on we assume that #(\mathbb{F})\geqm
Properties of H
- For all i=1,m}\mathrm{ we get that }a\inH(\mp@subsup{P}{i}{})\mathrm{ if and only if a.e ii }\inH\mathrm{ .
- Let }(\mp@subsup{n}{1}{},\ldots,\mp@subsup{n}{m}{}),(\mp@subsup{p}{1}{},\ldots,\mp@subsup{p}{m}{})\inH\mathrm{ and set }\mp@subsup{q}{i}{}:=m\operatorname{max}{\mp@subsup{n}{i}{},\mp@subsup{p}{i}{}}\mathrm{ ,
i=1,\ldots,m}\mathrm{ . Then }(\mp@subsup{q}{1}{},\ldots,\mp@subsup{q}{m}{})\inH\mathrm{ .
Define (n},\ldots,\ldots,\mp@subsup{n}{m}{})\preceq(\mp@subsup{n}{1}{},\ldots,\mp@subsup{n}{m}{})\mathrm{ if }\mp@subsup{n}{:}{\prime}\leq\mp@subsup{p}{i}{}\foralli=1,\ldots.,m.Then \preceq叟 a
partial order in N}\mp@subsup{\mathbb{N}}{0}{m}\mathrm{ .
Let i}\in{1,\ldots,m}, let \mp@subsup{n}{i}{}\in\mp@subsup{\mathbb{N}}{0}{}\mathrm{ and let n =
( }\mp@subsup{n}{1}{},\ldots,\mp@subsup{n}{m}{})\mathrm{ be a minimal element (w.r.t. }\preceq
of the set {(\mp@subsup{p}{1}{},\ldots,\mp@subsup{p}{m}{})\inH| \mp@subsup{p}{i}{}=\mp@subsup{n}{i}{}}\mathrm{ . If}
n
j}=i\mathrm{ , then:
(i) }\mp@subsup{n}{i}{}\mp@subsup{\textrm{e}}{i}{}\not\inH(\mathrm{ hence }\mp@subsup{n}{i}{}\not\inH(\mp@subsup{P}{i}{}))\mathrm{ ;
(ii) }\mathbf{n}\mathrm{ is a minimal element of the set
{(\mp@subsup{p}{1}{},\ldots,\mp@subsup{p}{m}{})\inH|\mp@subsup{p}{i}{}=\mp@subsup{n}{j}{}},\mathrm{ so }\mp@subsup{n}{j}{}\not\inH(\mp@subsup{P}{j}{})\mathrm{ .}
```


## Weierstrass semigroup at several points

From now on we assume that $\#(\mathbb{F}) \geq m$.
Properties of H :

- For all $i=1, \ldots, m$ we get that $a \in H\left(P_{i}\right)$ if and only if $a . e_{i} \in H$.
- Let $\left(n_{1}, \ldots, n_{m}\right),\left(p_{1}, \ldots, p_{m}\right) \in H$ and set $q_{i}:=\max \left\{n_{i}, p_{i}\right\}$,
$i=1, \ldots, m$. Then $\left(q_{1}, \ldots, q_{m}\right) \in H$.
Define $\left(n_{1}, \ldots, n_{m}\right) \preceq\left(p_{1}, \ldots, p_{m}\right)$ if $n_{i} \leq p_{i} \forall i=1, \ldots, m$. Then $\preceq$ is a partial order in $\mathbb{N}_{0}^{m}$.
Let $i \in\{1, \ldots, m\}$, let $n_{i} \in \mathbb{N}_{0}$ and let $\mathbf{n}=$
$\left(n_{1}, \ldots, n_{m}\right)$ be a minimal element (w.r.t. $\preceq$ )
of the set $\left\{\left(p_{1}, \ldots, p_{m}\right) \in H \mid p_{i}=n_{i}\right\}$. If
$n_{i}>0$ and $n_{j}>0$ for some $j \in\{1, \ldots, m\}$,
$j \neq i$, then:
(i) $n_{i} \mathbf{e}_{i} \notin H$ (hence $\left.n_{i} \notin H\left(P_{i}\right)\right)$;
(ii) n is a minimal element of the set
$\left\{\left(p_{1}, \ldots, p_{m}\right) \in H \mid p_{j}=n_{j}\right\}$, so $n_{j} \notin H\left(P_{j}\right)$.


## Weierstrass semigroup at several points

From now on we assume that $\#(\mathbb{F}) \geq m$.
Properties of $H$ :


## Weierstrass semigroup at several points

From now on we assume that $\#(\mathbb{F}) \geq m$.

## Properties of $H$ :

- For all $i=1, \ldots, m$ we get that $a \in H\left(P_{i}\right)$ if and only if $a . \mathbf{e}_{i} \in H$.



## Weierstrass semigroup at several points

From now on we assume that $\#(\mathbb{F}) \geq m$.
Properties of $H$ :

- For all $i=1, \ldots, m$ we get that $a \in H\left(P_{i}\right)$ if and only if $a . \mathbf{e}_{i} \in H$.



## Weierstrass semigroup at several points

From now on we assume that $\#(\mathbb{F}) \geq m$.
Properties of $H$ :

- For all $i=1, \ldots, m$ we get that $a \in H\left(P_{i}\right)$ if and only if $a . \mathbf{e}_{i} \in H$.
- Let $\left(n_{1}, \ldots, n_{m}\right),\left(p_{1}, \ldots, p_{m}\right) \in H$ and set $q_{i}:=\max \left\{n_{i}, p_{i}\right\}$, $i=1, \ldots, m$. Then $\left(q_{1}, \ldots, q_{m}\right) \in H$.



## Weierstrass semigroup at several points

From now on we assume that $\#(\mathbb{F}) \geq m$.
Properties of $H$ :

- For all $i=1, \ldots, m$ we get that $a \in H\left(P_{i}\right)$ if and only if $a . \mathbf{e}_{i} \in H$.
- Let $\left(n_{1}, \ldots, n_{m}\right),\left(p_{1}, \ldots, p_{m}\right) \in H$ and set $q_{i}:=\max \left\{n_{i}, p_{i}\right\}$, $i=1, \ldots, m$. Then $\left(q_{1}, \ldots, q_{m}\right) \in H$.
Define $\left(n_{1}, \ldots, n_{m}\right) \preceq\left(p_{1}, \ldots, p_{m}\right)$ if $n_{i} \leq p_{i} \forall i=1, \ldots, m$. Then $\preceq$ is a partial order in $\mathbb{N}_{0}^{m}$.



## Weierstrass semigroup at several points

From now on we assume that $\#(\mathbb{F}) \geq m$.
Properties of $H$ :

- For all $i=1, \ldots, m$ we get that $a \in H\left(P_{i}\right)$ if and only if $a . \mathbf{e}_{i} \in H$.
- Let $\left(n_{1}, \ldots, n_{m}\right),\left(p_{1}, \ldots, p_{m}\right) \in H$ and set $q_{i}:=\max \left\{n_{i}, p_{i}\right\}$, $i=1, \ldots, m$. Then $\left(q_{1}, \ldots, q_{m}\right) \in H$.
Define $\left(n_{1}, \ldots, n_{m}\right) \preceq\left(p_{1}, \ldots, p_{m}\right)$ if $n_{i} \leq p_{i} \forall i=1, \ldots, m$. Then $\preceq$ is a partial order in $\mathbb{N}_{0}^{m}$.
Let $i \in\{1, \ldots, m\}$, let $n_{i} \in \mathbb{N}_{0}$ and let $\mathbf{n}=$ $\left(n_{1}, \ldots, n_{m}\right)$ be a minimal element (w.r.t. $\preceq$ ) of the set $\left\{\left(p_{1}, \ldots, p_{m}\right) \in H \mid p_{i}=n_{i}\right\}$.




## Weierstrass semigroup at several points

From now on we assume that $\#(\mathbb{F}) \geq m$.
Properties of $H$ :

- For all $i=1, \ldots, m$ we get that $a \in H\left(P_{i}\right)$ if and only if $a . \mathbf{e}_{i} \in H$.
- Let $\left(n_{1}, \ldots, n_{m}\right),\left(p_{1}, \ldots, p_{m}\right) \in H$ and set $q_{i}:=\max \left\{n_{i}, p_{i}\right\}$, $i=1, \ldots, m$. Then $\left(q_{1}, \ldots, q_{m}\right) \in H$.
Define $\left(n_{1}, \ldots, n_{m}\right) \preceq\left(p_{1}, \ldots, p_{m}\right)$ if $n_{i} \leq p_{i} \forall i=1, \ldots, m$. Then $\preceq$ is a partial order in $\mathbb{N}_{0}^{m}$.
Let $i \in\{1, \ldots, m\}$, let $n_{i} \in \mathbb{N}_{0}$ and let $\mathbf{n}=$ $\left(n_{1}, \ldots, n_{m}\right)$ be a minimal element (w.r.t. $\preceq$ ) of the set $\left\{\left(p_{1}, \ldots, p_{m}\right) \in H \mid p_{i}=n_{i}\right\}$. $n_{i}>0$ and $n_{j}>0$ for some $j \in\{1, \ldots, m\}$, $j \neq i$, then:




## Weierstrass semigroup at several points

From now on we assume that $\#(\mathbb{F}) \geq m$.
Properties of $H$ :

- For all $i=1, \ldots, m$ we get that $a \in H\left(P_{i}\right)$ if and only if $a . \mathbf{e}_{i} \in H$.
- Let $\left(n_{1}, \ldots, n_{m}\right),\left(p_{1}, \ldots, p_{m}\right) \in H$ and set $q_{i}:=\max \left\{n_{i}, p_{i}\right\}$, $i=1, \ldots, m$. Then $\left(q_{1}, \ldots, q_{m}\right) \in H$.
Define $\left(n_{1}, \ldots, n_{m}\right) \preceq\left(p_{1}, \ldots, p_{m}\right)$ if $n_{i} \leq p_{i} \forall i=1, \ldots, m$. Then $\preceq$ is a partial order in $\mathbb{N}_{0}^{m}$.
Let $i \in\{1, \ldots, m\}$, let $n_{i} \in \mathbb{N}_{0}$ and let $\mathbf{n}=$ $\left(n_{1}, \ldots, n_{m}\right)$ be a minimal element (w.r.t. $\preceq$ ) of the set $\left\{\left(p_{1}, \ldots, p_{m}\right) \in H \mid p_{i}=n_{i}\right\}$. $n_{i}>0$ and $n_{j}>0$ for some $j \in\{1, \ldots, m\}$, $j \neq i$, then:
(i) $n_{i} \mathbf{e}_{i} \notin H$ (hence $\left.n_{i} \notin H\left(P_{i}\right)\right)$;



## Weierstrass semigroup at several points

From now on we assume that $\#(\mathbb{F}) \geq m$.
Properties of $H$ :

- For all $i=1, \ldots, m$ we get that $a \in H\left(P_{i}\right)$ if and only if $a . \mathbf{e}_{i} \in H$.
- Let $\left(n_{1}, \ldots, n_{m}\right),\left(p_{1}, \ldots, p_{m}\right) \in H$ and set $q_{i}:=\max \left\{n_{i}, p_{i}\right\}$, $i=1, \ldots, m$. Then $\left(q_{1}, \ldots, q_{m}\right) \in H$.
Define $\left(n_{1}, \ldots, n_{m}\right) \preceq\left(p_{1}, \ldots, p_{m}\right)$ if $n_{i} \leq p_{i} \forall i=1, \ldots, m$. Then $\preceq$ is a partial order in $\mathbb{N}_{0}^{m}$.
Let $i \in\{1, \ldots, m\}$, let $n_{i} \in \mathbb{N}_{0}$ and let $\mathbf{n}=$ $\left(n_{1}, \ldots, n_{m}\right)$ be a minimal element (w.r.t. $\preceq$ ) of the set $\left\{\left(p_{1}, \ldots, p_{m}\right) \in H \mid p_{i}=n_{i}\right\}$. $n_{i}>0$ and $n_{j}>0$ for some $j \in\{1, \ldots, m\}$, $j \neq i$, then:
(i) $n_{i} \mathbf{e}_{i} \notin H$ (hence $n_{i} \notin H\left(P_{i}\right)$ );
(ii) $\mathbf{n}$ is a minimal element of the set $\left\{\left(p_{1}, \ldots, p_{m}\right) \in H \mid p_{j}=n_{j}\right\}$, so $n_{j} \notin H\left(P_{j}\right)$.



## Weierstrass semigroup and AG codes

```
Given \mathbf{n}=(\mp@subsup{n}{1}{},\ldots,\mp@subsup{n}{m}{})\in\mp@subsup{\mathbb{N}}{0}{m}\mathrm{ , define}
\nabla
Lemma. Let }\mathbf{n}\in\mp@subsup{\mathbb{N}}{0}{m}\mathrm{ . The following are equivalent:
(i) n}\not\inH\mathrm{ ;
(ii)}\mp@subsup{\nabla}{i}{}(\mathbf{n})=\emptyset\mathrm{ for some }i\in{1,\ldots.,m}\mathrm{ .
We say that \mathbf{n}\in\mp@subsup{\mathbb{N}}{0}{m}}\mathrm{ is a pure gap if }\mp@subsup{\nabla}{i}{}(\mathbf{n})=\emptyset\mathrm{ for all }i\in{1,\ldots,m
Denote the set of pure gaps by G}\mp@subsup{\mathbf{G}}{0}{
Lemma: Let }\mathbf{n}=(\mp@subsup{n}{1}{}\ldots..\mp@subsup{n}{m}{})\in\mp@subsup{\mathbb{N}}{0}{m}
(i) If n }\in\mp@subsup{\textrm{G}}{0}{}\mathrm{ then }\mp@subsup{n}{i}{}\not\inH(\mp@subsup{P}{i}{})\mathrm{ for all }i
1,\ldots,m
(ii) If 1+ \sum i ni\leq\gamma, where \gamma is the gonality of
X}\mathrm{ , then }\textrm{n}\in\mp@subsup{\textrm{G}}{0}{}\mathrm{ .
```


## Weierstrass semigroup and AG codes

$$
\begin{aligned}
& \text { Given } \mathbf{n}=\left(n_{1}, \ldots, n_{m}\right) \in \mathbb{N}_{0}^{m}, \text { define } \\
& \nabla_{i}(\mathbf{n}):=\left\{\left(p_{1}, \ldots, p_{m}\right) \in H \mid p_{i}=n_{i} \text { and } p_{j} \leq n_{j} \forall j \neq i\right\}
\end{aligned}
$$

$$
\text { Lemma. Let } \mathbf{n} \in \mathbb{N}_{0}^{m} \text {. The following are equivalent: }
$$

$$
\text { (i) } \mathbf{n} \notin H \text {; }
$$

$$
\text { (ii) } \nabla_{i}(\mathbf{n})=\emptyset \text { for some } i \in\{1, \ldots, m\} \text {. }
$$

$$
\text { We say that } \mathbf{n} \in \mathbb{N}_{0}^{m} \text { is a pure gap if } \nabla_{i}(\mathbf{n})=\emptyset \text { for all } i \in\{1, \ldots, m\}
$$

$$
\text { Denote the set of pure gaps by } \mathbf{G}_{0} \text {. }
$$

$$
\text { Lemma: Let } \mathbf{n}=\left(n_{1}, \ldots, n_{m}\right) \in \mathbb{N}_{0}^{m}
$$

$$
\text { (i) If } \mathbf{n} \in \mathbf{G}_{0} \text { then } n_{i} \notin H\left(P_{i}\right) \text { for all } i=
$$(ii) If $1+\sum_{i} n_{i} \leq \gamma$, where $\gamma$ is the gonality ofthen $\mathbf{n} \in \mathbf{G}_{0}$

## Weierstrass semigroup and AG codes

$$
\begin{aligned}
& \text { Given } \mathbf{n}=\left(n_{1}, \ldots, n_{m}\right) \in \mathbb{N}_{0}^{m} \text {, define } \\
& \nabla_{i}(\mathbf{n}):=\left\{\left(p_{1}, \ldots, p_{m}\right) \in H \mid p_{i}=n_{i} \text { and } p_{j} \leq n_{j} \forall j \neq i\right\}
\end{aligned}
$$

$$
\text { Lemma. Let } \mathbf{n} \in \mathbb{N}_{0}^{m} \text {. The following are equivalent: }
$$

$$
\text { (i) } \mathbf{n} \notin H \text {; }
$$

$$
\text { (ii) } \nabla_{i}(\mathbf{n})=\emptyset \text { for some } i \in\{1, \ldots, m\} \text {. }
$$

$$
\text { We say that } \mathbf{n} \in \mathbb{N}_{0}^{m} \text { is a pure gap if } \nabla_{i}(\mathbf{n})=\emptyset \text { for all } i \in\{1, \ldots, m\}
$$

$$
\text { Denote the set of pure gaps by } \mathbf{G}_{0} \text {. }
$$




## Weierstrass semigroup and AG codes

$$
\begin{aligned}
& \text { Given } \mathbf{n}=\left(n_{1}, \ldots, n_{m}\right) \in \mathbb{N}_{0}^{m} \text {, define } \\
& \nabla_{i}(\mathbf{n}):=\left\{\left(p_{1}, \ldots, p_{m}\right) \in H \mid p_{i}=n_{i} \text { and } p_{j} \leq n_{j} \forall j \neq i\right\}
\end{aligned}
$$

$$
\text { Lemma. Let } \mathbf{n} \in \mathbb{N}_{0}^{m} \text {. The following are equivalent: }
$$

$$
\text { (i) } \mathbf{n} \notin H \text {; }
$$

$$
\text { (ii) } \nabla_{i}(\mathbf{n})=\emptyset \text { for some } i \in\{1, \ldots, m\} \text {. }
$$

$$
\text { We say that } \mathbf{n} \in \mathbb{N}_{0}^{m} \text { is a pure gap if } \nabla_{i}(\mathbf{n})=\emptyset \text { for all } i \in\{1, \ldots, m\}
$$

$$
\text { Denote the set of pure gaps by } \mathbf{G}_{0} \text {. }
$$




## Weierstrass semigroup and AG codes

Given $\mathbf{n}=\left(n_{1}, \ldots, n_{m}\right) \in \mathbb{N}_{0}^{m}$, define $\nabla_{i}(\mathbf{n}):=\left\{\left(p_{1}, \ldots, p_{m}\right) \in H \mid p_{i}=n_{i}\right.$ and $\left.p_{j} \leq n_{j} \forall j \neq i\right\}$

Lemma. Let $\mathbf{n} \in \mathbb{N}_{0}^{m}$. The following are equivalent:
(i) $\mathbf{n} \notin H$;
(ii) $\nabla_{i}(\mathbf{n})=\emptyset$ for some $i \in\{1, \ldots, m\}$

We say that $\mathbf{n} \in \mathbb{N}_{0}^{m}$ is a pure gap if $\nabla_{i}(\mathbf{n})=\emptyset$ for all $i \in\{1, \ldots, m\}$
Denote the set of pure gaps by $\mathbf{G}_{0}$.



## Weierstrass semigroup and AG codes

$$
\begin{aligned}
& \text { Given } \mathbf{n}=\left(n_{1}, \ldots, n_{m}\right) \in \mathbb{N}_{0}^{m} \text {, define } \\
& \nabla_{i}(\mathbf{n}):=\left\{\left(p_{1}, \ldots, p_{m}\right) \in H \mid p_{i}=n_{i} \text { and } p_{j} \leq n_{j} \forall j \neq i\right\}
\end{aligned}
$$

$$
\text { Lemma. Let } \mathbf{n} \in \mathbb{N}_{0}^{m} \text {. The following are equivalent: }
$$

$$
\text { (i) } \mathbf{n} \notin H \text {; }
$$

$$
\text { (ii) } \nabla_{i}(\mathbf{n})=\emptyset \text { for some } i \in\{1, \ldots, m\} \text {. }
$$

$$
\text { We say that } \mathbf{n} \in \mathbb{N}_{0}^{m} \text { is a pure gap if } \nabla_{i}(\mathbf{n})=\emptyset \text { for all } i \in\{1, \ldots, m\}
$$

$$
\text { Denote the set of pure gaps by } \mathbf{G}_{0} \text {. }
$$




## Weierstrass semigroup and AG codes

$$
\begin{aligned}
& \text { Given } \mathbf{n}=\left(n_{1}, \ldots, n_{m}\right) \in \mathbb{N}_{0}^{m} \text {, define } \\
& \nabla_{i}(\mathbf{n}):=\left\{\left(p_{1}, \ldots, p_{m}\right) \in H \mid p_{i}=n_{i} \text { and } p_{j} \leq n_{j} \forall j \neq i\right\}
\end{aligned}
$$

$$
\text { Lemma. Let } \mathbf{n} \in \mathbb{N}_{0}^{m} \text {. The following are equivalent: }
$$

$$
\text { (i) } \mathbf{n} \notin H \text {; }
$$

$$
\text { (ii) } \nabla_{i}(\mathbf{n})=\emptyset \text { for some } i \in\{1, \ldots, m\} \text {. }
$$

$$
\text { We say that } \mathbf{n} \in \mathbb{N}_{0}^{m} \text { is a pure gap if } \nabla_{i}(\mathbf{n})=\emptyset \text { for all } i \in\{1, \ldots, m\}
$$

$$
\text { Denote the set of pure gaps by } \mathbf{G}_{0} \text {. }
$$




## Weierstrass semigroup and AG codes

Given $\mathbf{n}=\left(n_{1}, \ldots, n_{m}\right) \in \mathbb{N}_{0}^{m}$, define
$\nabla_{i}(\mathbf{n}):=\left\{\left(p_{1}, \ldots, p_{m}\right) \in H \mid p_{i}=n_{i}\right.$ and $\left.p_{j} \leq n_{j} \forall j \neq i\right\}$
Lemma. Let $\mathbf{n} \in \mathbb{N}_{0}^{m}$. The following are equivalent:
(ii) $\nabla_{i}(\mathbf{n})=\emptyset$ for some $i \in\{1, \ldots, m\}$

We say that $\mathbf{n} \in \mathbb{N}_{0}^{m}$ is a pure gap if $\nabla_{i}(\mathbf{n})=\emptyset$ for all $i \in\{1, \ldots, m\}$
Denote the set of nure gans by $\mathbf{G}_{0}$.



## Weierstrass semigroup and AG codes

Given $\mathbf{n}=\left(n_{1}, \ldots, n_{m}\right) \in \mathbb{N}_{0}^{m}$, define
$\nabla_{i}(\mathbf{n}):=\left\{\left(p_{1}, \ldots, p_{m}\right) \in H \mid p_{i}=n_{i}\right.$ and $\left.p_{j} \leq n_{j} \forall j \neq i\right\}$
Lemma. Let $\mathbf{n} \in \mathbb{N}_{0}^{m}$. The following are equivalent:
(ii) $\nabla_{i}(\mathbf{n})=\emptyset$ for some $i \in\{1, \ldots, m\}$

We say that $\mathbf{n} \in \mathbb{N}_{0}^{m}$ is a pure gap if $\nabla_{i}(\mathbf{n})=\emptyset$ for all $i \in\{1, \ldots, m\}$
Denote the set of nure gans by $\mathbf{G}_{0}$.



## Weierstrass semigroup and AG codes

Given $\mathbf{n}=\left(n_{1}, \ldots, n_{m}\right) \in \mathbb{N}_{0}^{m}$, define
$\nabla_{i}(\mathbf{n}):=\left\{\left(p_{1}, \ldots, p_{m}\right) \in H \mid p_{i}=n_{i}\right.$ and $\left.p_{j} \leq n_{j} \forall j \neq i\right\}$
Lemma. Let $\mathbf{n} \in \mathbb{N}_{0}^{m}$. The following are equivalent:
(i) $\mathbf{n} \notin H$;
(ii) $\nabla_{i}(\mathrm{n})=\emptyset$ for some $i \in\{1, \ldots, m\}$

We say that $\mathbf{n} \in \mathbb{N}_{0}^{m}$ is a pure gap if $\nabla_{i}(\mathbf{n})=\emptyset$ for all $i \in\{1, \ldots, m\}$
Denote the set of nure gans by $\mathbf{G}_{0}$.



## Weierstrass semigroup and AG codes

Given $\mathbf{n}=\left(n_{1}, \ldots, n_{m}\right) \in \mathbb{N}_{0}^{m}$, define
$\nabla_{i}(\mathbf{n}):=\left\{\left(p_{1}, \ldots, p_{m}\right) \in H \mid p_{i}=n_{i}\right.$ and $\left.p_{j} \leq n_{j} \forall j \neq i\right\}$
Lemma. Let $\mathbf{n} \in \mathbb{N}_{0}^{m}$. The following are equivalent:
(i) $\mathbf{n} \notin H$;
(ii) $\nabla_{i}(\mathbf{n})=\emptyset$ for some $i \in\{1, \ldots, m\}$.

We say that $\mathbf{n} \in \mathbb{N}_{0}^{m}$ is a pure gap if $\nabla_{i}(\mathbf{n})=\emptyset$ for all $i \in\{1, \ldots, m\}$
Denote the set of pure gaps by $\mathbf{G}_{0}$.



## Weierstrass semigroup and AG codes

Given $\mathbf{n}=\left(n_{1}, \ldots, n_{m}\right) \in \mathbb{N}_{0}^{m}$, define
$\nabla_{i}(\mathbf{n}):=\left\{\left(p_{1}, \ldots, p_{m}\right) \in H \mid p_{i}=n_{i}\right.$ and $\left.p_{j} \leq n_{j} \forall j \neq i\right\}$
Lemma. Let $\mathbf{n} \in \mathbb{N}_{0}^{m}$. The following are equivalent:
(i) $\mathbf{n} \notin H$;
(ii) $\nabla_{i}(\mathbf{n})=\emptyset$ for some $i \in\{1, \ldots, m\}$.

We say that $\mathbf{n} \in \mathbb{N}_{0}^{m}$ is a pure gap if $\nabla_{i}(\mathbf{n})=\emptyset$ for all $i \in\{1, \ldots, m\}$
Denote the set of pure gaps by $\mathbf{G}_{0}$.



## Weierstrass semigroup and AG codes

Given $\mathbf{n}=\left(n_{1}, \ldots, n_{m}\right) \in \mathbb{N}_{0}^{m}$, define
$\nabla_{i}(\mathbf{n}):=\left\{\left(p_{1}, \ldots, p_{m}\right) \in H \mid p_{i}=n_{i}\right.$ and $\left.p_{j} \leq n_{j} \forall j \neq i\right\}$
Lemma. Let $\mathbf{n} \in \mathbb{N}_{0}^{m}$. The following are equivalent:
(i) $\mathbf{n} \notin H$;
(ii) $\nabla_{i}(\mathbf{n})=\emptyset$ for some $i \in\{1, \ldots, m\}$.

We say that $\mathbf{n} \in \mathbb{N}_{0}^{m}$ is a pure gap if $\nabla_{i}(\mathbf{n})=\emptyset$ for all $i \in\{1, \ldots, m\}$.

## Denote the set of pure gaps by $\mathrm{G}_{0}$.




## Weierstrass semigroup and AG codes

Given $\mathbf{n}=\left(n_{1}, \ldots, n_{m}\right) \in \mathbb{N}_{0}^{m}$, define
$\nabla_{i}(\mathbf{n}):=\left\{\left(p_{1}, \ldots, p_{m}\right) \in H \mid p_{i}=n_{i}\right.$ and $\left.p_{j} \leq n_{j} \forall j \neq i\right\}$
Lemma. Let $\mathbf{n} \in \mathbb{N}_{0}^{m}$. The following are equivalent:
(i) $\mathbf{n} \notin H$;
(ii) $\nabla_{i}(\mathbf{n})=\emptyset$ for some $i \in\{1, \ldots, m\}$.

We say that $\mathbf{n} \in \mathbb{N}_{0}^{m}$ is a pure gap if $\nabla_{i}(\mathbf{n})=\emptyset$ for all $i \in\{1, \ldots, m\}$.
Denote the set of pure gaps by $\mathbf{G}_{0}$.



## Weierstrass semigroup and AG codes

Given $\mathbf{n}=\left(n_{1}, \ldots, n_{m}\right) \in \mathbb{N}_{0}^{m}$, define
$\nabla_{i}(\mathbf{n}):=\left\{\left(p_{1}, \ldots, p_{m}\right) \in H \mid p_{i}=n_{i}\right.$ and $\left.p_{j} \leq n_{j} \forall j \neq i\right\}$
Lemma. Let $\mathbf{n} \in \mathbb{N}_{0}^{m}$. The following are equivalent:
(i) $\mathbf{n} \notin H$;
(ii) $\nabla_{i}(\mathbf{n})=\emptyset$ for some $i \in\{1, \ldots, m\}$.

We say that $\mathbf{n} \in \mathbb{N}_{0}^{m}$ is a pure gap if $\nabla_{i}(\mathbf{n})=\emptyset$ for all $i \in\{1, \ldots, m\}$.
Denote the set of pure gaps by $\mathbf{G}_{0}$.
Lemma: Let $\mathbf{n}=\left(n_{1}, \ldots, n_{m}\right) \in \mathbb{N}_{0}^{m}$.



## Weierstrass semigroup and AG codes

Given $\mathbf{n}=\left(n_{1}, \ldots, n_{m}\right) \in \mathbb{N}_{0}^{m}$, define
$\nabla_{i}(\mathbf{n}):=\left\{\left(p_{1}, \ldots, p_{m}\right) \in H \mid p_{i}=n_{i}\right.$ and $\left.p_{j} \leq n_{j} \forall j \neq i\right\}$
Lemma. Let $\mathbf{n} \in \mathbb{N}_{0}^{m}$. The following are equivalent:
(i) $\mathbf{n} \notin H$;
(ii) $\nabla_{i}(\mathbf{n})=\emptyset$ for some $i \in\{1, \ldots, m\}$.

We say that $\mathbf{n} \in \mathbb{N}_{0}^{m}$ is a pure gap if $\nabla_{i}(\mathbf{n})=\emptyset$ for all $i \in\{1, \ldots, m\}$.
Denote the set of pure gaps by $\mathbf{G}_{0}$.
Lemma: Let $\mathbf{n}=\left(n_{1}, \ldots, n_{m}\right) \in \mathbb{N}_{0}^{m}$.
(i) If $\mathbf{n} \in \mathbf{G}_{0}$ then $n_{i} \notin H\left(P_{i}\right)$ for all $i=$ $1, \ldots, m$.



## Weierstrass semigroup and AG codes

Given $\mathbf{n}=\left(n_{1}, \ldots, n_{m}\right) \in \mathbb{N}_{0}^{m}$, define
$\nabla_{i}(\mathbf{n}):=\left\{\left(p_{1}, \ldots, p_{m}\right) \in H \mid p_{i}=n_{i}\right.$ and $\left.p_{j} \leq n_{j} \forall j \neq i\right\}$
Lemma. Let $\mathbf{n} \in \mathbb{N}_{0}^{m}$. The following are equivalent:
(i) $\mathbf{n} \notin H$;
(ii) $\nabla_{i}(\mathbf{n})=\emptyset$ for some $i \in\{1, \ldots, m\}$.

We say that $\mathbf{n} \in \mathbb{N}_{0}^{m}$ is a pure gap if $\nabla_{i}(\mathbf{n})=\emptyset$ for all $i \in\{1, \ldots, m\}$.
Denote the set of pure gaps by $\mathbf{G}_{0}$.
Lemma: Let $\mathbf{n}=\left(n_{1}, \ldots, n_{m}\right) \in \mathbb{N}_{0}^{m}$.
(i) If $\mathbf{n} \in \mathbf{G}_{0}$ then $n_{i} \notin H\left(P_{i}\right)$ for all $i=$ $1, \ldots, m$.
(ii) If $1+\sum_{i} n_{i} \leq \gamma$, where $\gamma$ is the gonality of $X$, then $\mathbf{n} \in \mathbf{G}_{0}$.


## Weierstrass semigroup and AG codes

Assume that $\mathbb{F}$ is a finite field, let $D:=Q_{1}+\cdots+Q_{n}$, where $Q_{1}, \ldots, Q_{n}$ are distinct rational points of $X$, all distinct from $P_{1}, \ldots, P_{m}$, and let $G$ be a divisor with support on $P_{1}, \ldots, P_{m}$.
Let $C_{\Omega}(D, G)$ be the algebraic geometry code which is the image of the $\operatorname{map} \varphi: \Omega(G-D) \rightarrow \mathbb{F}^{n}$ defined by $\varphi(\eta)=\left(\operatorname{res}_{Q_{1}}(\eta), \ldots, \operatorname{res}_{Q_{n}}(\eta)\right)$. We know that $C_{\Omega}(D, G)$ is an $[n, k, d]$-code, with $d \geq \operatorname{deg}(G)-(2 g-2)$. Building on previous work by Homma, Kim and Matthews, F. Torres and myself proved the following results.
Theorem. Assume that $\left(n_{1}, \ldots, n_{m}\right)$ and $\left(p_{1}, \ldots, p_{m}\right)$ are pure gaps of $H$, and let $G=\sum_{i=1}^{m}\left(n_{i}+p_{i}-1\right) P_{i}$. Then $C_{\Omega}(D, G)$ is an $[n, k, d]$-code, with $d \geq \operatorname{deg}(G)-(2 g-2)+m$.
Theorem. Assume that $\left(n_{1}, \ldots, n_{m}\right)$ and $\left(p_{1}, \ldots, p_{m}\right)$ are pure gaps of $H$, with $n_{i} \leq p_{i}$ for all $i=1, \ldots, m$, and that $\left(q_{1}, \ldots, q_{m}\right)$ is also a pure gap whenever $n_{i} \leq q_{i} \leq p_{i}$, for all $i=1, \ldots, m$. Let $G=\sum_{i=1}^{m}\left(n_{i}+p_{i}-1\right) P_{i}$, then $C_{\Omega}(D, G)$ is an $[n, k, d]$-code, with
$d \geq \operatorname{deg}(G)-(2 g-2)+m+\sum_{i=1}^{n}\left(p_{i}-n_{i}\right)$.

## Weierstrass semigroup and AG codes

Assume that $\mathbb{F}$ is a finite field, let $D:=Q_{1}+\cdots+Q_{n}$, where $Q_{1} \ldots, Q_{n}$ are distinct rational points of $X$, all distinct from $P_{1}, \ldots, P_{m}$, and let $G$ be a divisor with support on $P_{1}, \ldots, P_{m}$.
Let $C_{\Omega}(D, G)$ be the algebraic geometry code which is the image of the $\operatorname{map} \varphi: \Omega(G-D) \rightarrow \mathbb{F}^{n}$ defined by $\varphi(\eta)=\left(\operatorname{res}_{Q_{1}}(\eta), \ldots, \operatorname{res}_{Q_{n}}(\eta)\right)$ We know that $C_{\Omega}(D, G)$ is an $[n, k, d]$-code, with $d \geq \operatorname{deg}(G)-(2 g-2)$. Building on previous work by Homma, Kim and Matthews, F. Torres and myself proved the following results.
Theorem. Assume that $\left(n_{1}, \ldots, n_{m}\right)$ and $\left(p_{1}, \ldots, p_{m}\right)$ are pure gaps of $H$, and let $G=\sum_{i=1}^{m}\left(n_{i}+p_{i}-1\right) P_{i}$. Then $C_{\Omega}(D, G)$ is an $[n, k, d]$-code, with $d \geq \operatorname{deg}(G)-(2 g-2)+m$.
Theorem. Assume that $\left(n_{1}, \ldots, n_{m}\right)$ and $\left(p_{1}, \ldots, p_{m}\right)$ are pure gaps of $H$, with $n_{i} \leq p_{i}$ for all $i=1, \ldots, m$, and that $\left(q_{1}, \ldots, q_{m}\right)$ is also a pure gap whenever $n_{i} \leq q_{i} \leq p_{i}$, for all $i=1, \ldots, m$. Let $G=\sum_{i=1}^{m}\left(n_{i}+p_{i}-1\right) P_{i}$, then $C_{\Omega}(D, G)$ is an $[n, k, d]$-code, with
$d \geq \operatorname{deg}(G)-(2 g-2)+m+\sum_{i=1}^{n}\left(p_{i}-n_{i}\right)$.

## Weierstrass semigroup and AG codes

Assume that $\mathbb{F}$ is a finite field, let $D:=Q_{1}+\cdots+Q_{n}$, where $Q_{1}, \ldots, Q_{n}$ are distinct rational points of $X$, all distinct from $P_{1}, \ldots, P_{m}$, a divisor with support on $P_{1}, \ldots, P_{m}$ Let $C_{\Omega}(D, G)$ be the algebraic geometry code which is the image of the $\operatorname{map} \varphi: \Omega(G-D) \rightarrow \mathbb{F}^{n}$ defined by $\varphi(\eta)=\left(\operatorname{res}_{Q_{1}}(\eta), \ldots, \operatorname{res}_{Q_{n}}(\eta)\right)$ We know that $C_{\Omega}(D, G)$ is an $[n, k, d]$-code, with $d \geq \operatorname{deg}(G)-(2 g-2)$ Building on previous work by Homma, Kim and Matthews, F. Torres and myself proved the following results.
Theorem. Assume that $\left(n_{1}, \ldots, n_{m}\right)$ and $\left(p_{1}, \ldots, p_{m}\right)$ are pure gaps of $H$, and let $G=\sum_{i=1}^{m}\left(n_{i}+p_{i}-1\right) P_{i}$. Then $C_{\Omega}(D, G)$ is an $[n, k, d]$-code, with $d \geq \operatorname{deg}(G)-(2 g-2)+m$.
Theorem. Assume that $\left(n_{1}, \ldots, n_{m}\right)$ and $\left(p_{1}, \ldots, p_{m}\right)$ are pure gaps of $H$, with $n_{i} \leq p_{i}$ for all $i=1, \ldots, m$, and that $\left(q_{1}, \ldots, q_{m}\right)$ is also a pure gap whenever $n_{i} \leq q_{i} \leq p_{i}$, for all $i=1, \ldots, m$. Let $G=\sum_{i=1}^{m}\left(n_{i}+p_{i}-1\right) P_{i}$ then $C_{\Omega}(D, G)$ is an $[n, k, d]$-code, with
$\square$

## Weierstrass semigroup and AG codes

Assume that $\mathbb{F}$ is a finite field, let $D:=Q_{1}+\cdots+Q_{n}$, where $Q_{1}, \ldots, Q_{n}$ are distinct rational points of $X$, all distinct from $P_{1}, \ldots, P_{m}$, and let $G$ be a divisor with support on $P_{1}, \ldots, P_{m}$.
Let $C_{\Omega}(D, G)$ be the algebraic geometry code which is the image of the $\operatorname{map} \varphi: \Omega(G-D) \rightarrow \mathbb{F}^{n}$ defined by $\varphi(\eta)=\left(\operatorname{res}_{Q_{1}}(\eta), \ldots, \operatorname{res}_{Q_{n}}(\eta)\right)$ We know that $C_{\Omega}(D, G)$ is an $[n, k, d]$-code, with $d \geq \operatorname{deg}(G)-(2 g-2)$ Building on previous work by Homma, Kim and Matthews, F. Torres and myself proved the following results.
Theorem. Assume that $\left(n_{1}, \ldots, n_{m}\right)$ and $\left(p_{1}, \ldots, p_{m}\right)$ are pure gaps of $H$ and let $G=\sum_{i=1}^{m}\left(n_{i}+p_{i}-1\right) P_{i}$. Then $C_{\Omega}(D, G)$ is an $[n, k, d]$-code, with $d \geq \operatorname{deg}(G)-(2 g-2)+m$.

then $C_{\Omega}(D, G)$ is an $[n, k, d]$-code, with
$d \geq \operatorname{deg}(G)-(2 g-2)+m+\sum_{i=1}^{n}\left(p_{i}-n_{i}\right)$

## Weierstrass semigroup and AG codes

Assume that $\mathbb{F}$ is a finite field, let $D:=Q_{1}+\cdots+Q_{n}$, where $Q_{1}, \ldots, Q_{n}$ are distinct rational points of $X$, all distinct from $P_{1}, \ldots, P_{m}$, and let $G$ be a divisor with support on $P_{1}, \ldots, P_{m}$.
Let $C_{\Omega}(D, G)$ be the algebraic geometry code which is the image of the $\operatorname{map} \varphi: \Omega(G-D) \rightarrow \mathbb{F}^{n}$ defined by $\varphi(\eta)=\left(\operatorname{res}_{Q_{1}}(\eta), \ldots, \operatorname{res}_{Q_{n}}(\eta)\right)$.

Building on previous work by Homma, Kim and Matthews, F. Torres and myself proved the following results.

then $C_{\Omega}(D, G)$ is an $[n, k, d]$-code, with
$\square$

## Weierstrass semigroup and AG codes

Assume that $\mathbb{F}$ is a finite field, let $D:=Q_{1}+\cdots+Q_{n}$, where $Q_{1}, \ldots, Q_{n}$ are distinct rational points of $X$, all distinct from $P_{1}, \ldots, P_{m}$, and let $G$ be a divisor with support on $P_{1}, \ldots, P_{m}$.
Let $C_{\Omega}(D, G)$ be the algebraic geometry code which is the image of the $\operatorname{map} \varphi: \Omega(G-D) \rightarrow \mathbb{F}^{n}$ defined by $\varphi(\eta)=\left(\operatorname{res}_{Q_{1}}(\eta), \ldots, \operatorname{res}_{Q_{n}}(\eta)\right)$. We know that $C_{\Omega}(D, G)$ is an $[n, k, d]$-code, with $d \geq \operatorname{deg}(G)-(2 g-2)$.


## Weierstrass semigroup and AG codes

Assume that $\mathbb{F}$ is a finite field, let $D:=Q_{1}+\cdots+Q_{n}$, where $Q_{1}, \ldots, Q_{n}$ are distinct rational points of $X$, all distinct from $P_{1}, \ldots, P_{m}$, and let $G$ be a divisor with support on $P_{1}, \ldots, P_{m}$.
Let $C_{\Omega}(D, G)$ be the algebraic geometry code which is the image of the $\operatorname{map} \varphi: \Omega(G-D) \rightarrow \mathbb{F}^{n}$ defined by $\varphi(\eta)=\left(\operatorname{res}_{Q_{1}}(\eta), \ldots, \operatorname{res}_{Q_{n}}(\eta)\right)$. We know that $C_{\Omega}(D, G)$ is an $[n, k, d]$-code, with $d \geq \operatorname{deg}(G)-(2 g-2)$. Building on previous work by Homma, Kim and Matthews, F. Torres and myself proved the following results.


## Weierstrass semigroup and AG codes

Assume that $\mathbb{F}$ is a finite field, let $D:=Q_{1}+\cdots+Q_{n}$, where $Q_{1}, \ldots, Q_{n}$ are distinct rational points of $X$, all distinct from $P_{1}, \ldots, P_{m}$, and let $G$ be a divisor with support on $P_{1}, \ldots, P_{m}$.
Let $C_{\Omega}(D, G)$ be the algebraic geometry code which is the image of the $\operatorname{map} \varphi: \Omega(G-D) \rightarrow \mathbb{F}^{n}$ defined by $\varphi(\eta)=\left(\operatorname{res}_{Q_{1}}(\eta), \ldots, \operatorname{res}_{Q_{n}}(\eta)\right)$. We know that $C_{\Omega}(D, G)$ is an $[n, k, d]$-code, with $d \geq \operatorname{deg}(G)-(2 g-2)$. Building on previous work by Homma, Kim and Matthews, F. Torres and myself proved the following results.
Theorem. Assume that $\left(n_{1}, \ldots, n_{m}\right)$ and $\left(p_{1}, \ldots, p_{m}\right)$ are pure gaps of $H$, and let $G=\sum_{i=1}^{m}\left(n_{i}+p_{i}-1\right) P_{i}$.


## Weierstrass semigroup and AG codes

Assume that $\mathbb{F}$ is a finite field, let $D:=Q_{1}+\cdots+Q_{n}$, where $Q_{1}, \ldots, Q_{n}$ are distinct rational points of $X$, all distinct from $P_{1}, \ldots, P_{m}$, and let $G$ be a divisor with support on $P_{1}, \ldots, P_{m}$.
Let $C_{\Omega}(D, G)$ be the algebraic geometry code which is the image of the $\operatorname{map} \varphi: \Omega(G-D) \rightarrow \mathbb{F}^{n}$ defined by $\varphi(\eta)=\left(\operatorname{res}_{Q_{1}}(\eta), \ldots, \operatorname{res}_{Q_{n}}(\eta)\right)$. We know that $C_{\Omega}(D, G)$ is an $[n, k, d]$-code, with $d \geq \operatorname{deg}(G)-(2 g-2)$. Building on previous work by Homma, Kim and Matthews, F. Torres and myself proved the following results.
Theorem. Assume that $\left(n_{1}, \ldots, n_{m}\right)$ and $\left(p_{1}, \ldots, p_{m}\right)$ are pure gaps of $H$, and let $G=\sum_{i=1}^{m}\left(n_{i}+p_{i}-1\right) P_{i}$. Then $C_{\Omega}(D, G)$ is an $[n, k, d]$-code, with $d \geq \operatorname{deg}(G)-(2 g-2)+m$.


## Weierstrass semigroup and AG codes

Assume that $\mathbb{F}$ is a finite field, let $D:=Q_{1}+\cdots+Q_{n}$, where $Q_{1}, \ldots, Q_{n}$ are distinct rational points of $X$, all distinct from $P_{1}, \ldots, P_{m}$, and let $G$ be a divisor with support on $P_{1}, \ldots, P_{m}$.
Let $C_{\Omega}(D, G)$ be the algebraic geometry code which is the image of the $\operatorname{map} \varphi: \Omega(G-D) \rightarrow \mathbb{F}^{n}$ defined by $\varphi(\eta)=\left(\operatorname{res}_{Q_{1}}(\eta), \ldots, \operatorname{res}_{Q_{n}}(\eta)\right)$. We know that $C_{\Omega}(D, G)$ is an $[n, k, d]$-code, with $d \geq \operatorname{deg}(G)-(2 g-2)$. Building on previous work by Homma, Kim and Matthews, F. Torres and myself proved the following results.
Theorem. Assume that $\left(n_{1}, \ldots, n_{m}\right)$ and $\left(p_{1}, \ldots, p_{m}\right)$ are pure gaps of $H$, and let $G=\sum_{i=1}^{m}\left(n_{i}+p_{i}-1\right) P_{i}$. Then $C_{\Omega}(D, G)$ is an $[n, k, d]$-code, with $d \geq \operatorname{deg}(G)-(2 g-2)+m$.
Theorem. Assume that $\left(n_{1}, \ldots, n_{m}\right)$ and $\left(p_{1}, \ldots, p_{m}\right)$ are pure gaps of $H$,


## Weierstrass semigroup and AG codes

Assume that $\mathbb{F}$ is a finite field, let $D:=Q_{1}+\cdots+Q_{n}$, where $Q_{1}, \ldots, Q_{n}$ are distinct rational points of $X$, all distinct from $P_{1}, \ldots, P_{m}$, and let $G$ be a divisor with support on $P_{1}, \ldots, P_{m}$.
Let $C_{\Omega}(D, G)$ be the algebraic geometry code which is the image of the $\operatorname{map} \varphi: \Omega(G-D) \rightarrow \mathbb{F}^{n}$ defined by $\varphi(\eta)=\left(\operatorname{res}_{Q_{1}}(\eta), \ldots, \operatorname{res}_{Q_{n}}(\eta)\right)$.
We know that $C_{\Omega}(D, G)$ is an $[n, k, d]$-code, with $d \geq \operatorname{deg}(G)-(2 g-2)$. Building on previous work by Homma, Kim and Matthews, F. Torres and myself proved the following results.
Theorem. Assume that $\left(n_{1}, \ldots, n_{m}\right)$ and $\left(p_{1}, \ldots, p_{m}\right)$ are pure gaps of $H$, and let $G=\sum_{i=1}^{m}\left(n_{i}+p_{i}-1\right) P_{i}$. Then $C_{\Omega}(D, G)$ is an $[n, k, d]$-code, with $d \geq \operatorname{deg}(G)-(2 g-2)+m$.
Theorem. Assume that $\left(n_{1}, \ldots, n_{m}\right)$ and $\left(p_{1}, \ldots, p_{m}\right)$ are pure gaps of $H$, with $n_{i} \leq p_{i}$ for all $i=1, \ldots, m$,


## Weierstrass semigroup and AG codes

Assume that $\mathbb{F}$ is a finite field, let $D:=Q_{1}+\cdots+Q_{n}$, where $Q_{1}, \ldots, Q_{n}$ are distinct rational points of $X$, all distinct from $P_{1}, \ldots, P_{m}$, and let $G$ be a divisor with support on $P_{1}, \ldots, P_{m}$.
Let $C_{\Omega}(D, G)$ be the algebraic geometry code which is the image of the $\operatorname{map} \varphi: \Omega(G-D) \rightarrow \mathbb{F}^{n}$ defined by $\varphi(\eta)=\left(\operatorname{res}_{Q_{1}}(\eta), \ldots, \operatorname{res}_{Q_{n}}(\eta)\right)$.
We know that $C_{\Omega}(D, G)$ is an $[n, k, d]$-code, with $d \geq \operatorname{deg}(G)-(2 g-2)$. Building on previous work by Homma, Kim and Matthews, F. Torres and myself proved the following results.
Theorem. Assume that $\left(n_{1}, \ldots, n_{m}\right)$ and $\left(p_{1}, \ldots, p_{m}\right)$ are pure gaps of $H$, and let $G=\sum_{i=1}^{m}\left(n_{i}+p_{i}-1\right) P_{i}$. Then $C_{\Omega}(D, G)$ is an $[n, k, d]$-code, with $d \geq \operatorname{deg}(G)-(2 g-2)+m$.
Theorem. Assume that $\left(n_{1}, \ldots, n_{m}\right)$ and $\left(p_{1}, \ldots, p_{m}\right)$ are pure gaps of $H$, with $n_{i} \leq p_{i}$ for all $i=1, \ldots, m$, and that $\left(q_{1}, \ldots, q_{m}\right)$ is also a pure gap whenever $n_{i} \leq q_{i} \leq p_{i}$, for all $i=1, \ldots, m$.

## Weierstrass semigroup and AG codes

Assume that $\mathbb{F}$ is a finite field, let $D:=Q_{1}+\cdots+Q_{n}$, where $Q_{1}, \ldots, Q_{n}$ are distinct rational points of $X$, all distinct from $P_{1}, \ldots, P_{m}$, and let $G$ be a divisor with support on $P_{1}, \ldots, P_{m}$.
Let $C_{\Omega}(D, G)$ be the algebraic geometry code which is the image of the $\operatorname{map} \varphi: \Omega(G-D) \rightarrow \mathbb{F}^{n}$ defined by $\varphi(\eta)=\left(\operatorname{res}_{Q_{1}}(\eta), \ldots, \operatorname{res}_{Q_{n}}(\eta)\right)$.
We know that $C_{\Omega}(D, G)$ is an $[n, k, d]$-code, with $d \geq \operatorname{deg}(G)-(2 g-2)$. Building on previous work by Homma, Kim and Matthews, F. Torres and myself proved the following results.
Theorem. Assume that $\left(n_{1}, \ldots, n_{m}\right)$ and $\left(p_{1}, \ldots, p_{m}\right)$ are pure gaps of $H$, and let $G=\sum_{i=1}^{m}\left(n_{i}+p_{i}-1\right) P_{i}$. Then $C_{\Omega}(D, G)$ is an $[n, k, d]$-code, with $d \geq \operatorname{deg}(G)-(2 g-2)+m$.
Theorem. Assume that $\left(n_{1}, \ldots, n_{m}\right)$ and $\left(p_{1}, \ldots, p_{m}\right)$ are pure gaps of $H$, with $n_{i} \leq p_{i}$ for all $i=1, \ldots, m$, and that $\left(q_{1}, \ldots, q_{m}\right)$ is also a pure gap whenever $n_{i} \leq q_{i} \leq p_{i}$, for all $i=1, \ldots, m$. Let $G=\sum_{i=1}^{m}\left(n_{i}+p_{i}-1\right) P_{i}$,

## Weierstrass semigroup and AG codes

Assume that $\mathbb{F}$ is a finite field, let $D:=Q_{1}+\cdots+Q_{n}$, where $Q_{1}, \ldots, Q_{n}$ are distinct rational points of $X$, all distinct from $P_{1}, \ldots, P_{m}$, and let $G$ be a divisor with support on $P_{1}, \ldots, P_{m}$.
Let $C_{\Omega}(D, G)$ be the algebraic geometry code which is the image of the $\operatorname{map} \varphi: \Omega(G-D) \rightarrow \mathbb{F}^{n}$ defined by $\varphi(\eta)=\left(\operatorname{res}_{Q_{1}}(\eta), \ldots, \operatorname{res}_{Q_{n}}(\eta)\right)$.
We know that $C_{\Omega}(D, G)$ is an $[n, k, d]$-code, with $d \geq \operatorname{deg}(G)-(2 g-2)$. Building on previous work by Homma, Kim and Matthews, F. Torres and myself proved the following results.
Theorem. Assume that $\left(n_{1}, \ldots, n_{m}\right)$ and $\left(p_{1}, \ldots, p_{m}\right)$ are pure gaps of $H$, and let $G=\sum_{i=1}^{m}\left(n_{i}+p_{i}-1\right) P_{i}$. Then $C_{\Omega}(D, G)$ is an $[n, k, d]$-code, with $d \geq \operatorname{deg}(G)-(2 g-2)+m$.
Theorem. Assume that $\left(n_{1}, \ldots, n_{m}\right)$ and $\left(p_{1}, \ldots, p_{m}\right)$ are pure gaps of $H$, with $n_{i} \leq p_{i}$ for all $i=1, \ldots, m$, and that $\left(q_{1}, \ldots, q_{m}\right)$ is also a pure gap whenever $n_{i} \leq q_{i} \leq p_{i}$, for all $i=1, \ldots, m$. Let $G=\sum_{i=1}^{m}\left(n_{i}+p_{i}-1\right) P_{i}$, then $C_{\Omega}(D, G)$ is an $[n, k, d]$-code, with $d \geq \operatorname{deg}(G)-(2 g-2)+m+\sum_{i=1}^{n}\left(p_{i}-n_{i}\right)$.

## Weierstrass semigroup and AG codes



Weierstrass semigroup of two rational points in $Y^{8}+Y=X^{9}$ over $\mathbb{F}_{64}$.

Theorem. Assume that $\left(n_{1}, \ldots, n_{m}\right)$ and $\left(p_{1}, \ldots, p_{m}\right)$ are pure gaps of $H$, and let $G=\sum_{i=1}^{m}\left(n_{i}+p_{i}-1\right) P_{i}$. Then $C_{\Omega}(D, G)$ is an $[n, k, d]$-code, with $d \geq \operatorname{deg}(G)-(2 g-2)+m$.
Theorem. Assume that $\left(n_{1}, \ldots, n_{m}\right)$ and $\left(p_{1}, \ldots, p_{m}\right)$ are pure gaps of $H$, with $n_{i} \leq p_{i}$ for all $i=1, \ldots, m$, and that $\left(q_{1}, \ldots, q_{m}\right)$ is also a pure gap whenever $n_{i} \leq q_{i} \leq p_{i}$, for all $i=1, \ldots, m$. Let $G=\sum_{i=1}^{m}\left(n_{i}+p_{i}-1\right) P_{i}$, then $C_{\Omega}(D, G)$ is an $[n, k, d]$-code, with $d \geq \operatorname{deg}(G)-(2 g-2)+m+\sum_{i=1}^{n}\left(p_{i}-n_{i}\right)$.

## Weierstrass semigroup and AG codes

A similar improvement can be made to bounds for the generalized Hamming distance of AG codes.
Let $r$ be a positive integer, and $C \subset \mathbb{F}^{m}$ a linear code. Let $U$ be a subcode of $C$, the support of $U$ is defined as
$\operatorname{supp}(U):=\left\{i \mid c_{i} \neq 0\right.$ for some $\left.\left(c_{1}, \ldots, c_{m}\right) \in U\right\}$
The $r$-th generalized Hamming distance of $C$ is defined as

$$
d_{r}(C)=\min \{\#(\operatorname{supp}(U)) \mid U \text { is a subcode of } C, \operatorname{dim}(U)=r\}
$$

Let $s$ be a positive integer and set
$\gamma_{s}:=\{\min (\operatorname{deg}(A)) \mid A$ a divisor with $\operatorname{dim} L(A)=s\}$. The sequence $\gamma_{1}, \gamma_{2}, \ldots$ is the gonality sequence of $X ; \gamma_{1}=0$ and $\gamma_{2}$ is the gonality of $X$. This concept was introduced by Yang, Kummar and Stichtenoth in a paper where they proved the following result.
Theorem. The $r$-th generalized Hamming distance of an AG code of length $n$ defined over $X$ satisfies
i) $d_{r}\left(C_{L}(D, G) \geq n-\operatorname{deg}(G)+\gamma_{r}\right.$;
ii) $d_{r}\left(C_{\Omega}(D, G) \geq \operatorname{deg}(G)-(2 g-2)+\gamma_{r}\right.$.

## Weierstrass semigroup and AG codes

A similar improvement can be made to bounds for the generalized Hamming distance of AG codes.
Let $r$ be a positive integer, and $C \subset \mathbb{F}^{m}$ a linear code. Let $U$ be a subcode of $C$, the support of $U$ is defined as
$\operatorname{supp}(U):=\left\{i \mid c_{i} \neq 0\right.$ for some $\left.\left(c_{1}, \ldots, c_{m}\right) \in U\right\}$
The $r$-th generalized Hamming distance of $C$ is defined as

$$
d_{r}(C)=\min \{\#(\operatorname{supp}(U)) \mid U \text { is a subcode of } C, \operatorname{dim}(U)=r\} .
$$

Let $s$ be a positive integer and set $\gamma_{s}:=\{\min (\operatorname{deg}(A)) \mid A$ a divisor with $\operatorname{dim} L(A)=s\}$. The sequence $\gamma_{1}, \gamma_{2}, \ldots$ is the gonality sequence of $X ; \gamma_{1}=0$ and $\gamma_{2}$ is the gonality of $X$. This concept was introduced by Yang, Kummar and Stichtenoth in a paper where they proved the following result.
Theorem. The $r$-th generalized Hamming distance of an AG code of length $n$ defined over $X$ satisfies
i) $d_{r}\left(C_{L}(D, G) \geq n-\operatorname{deg}(G)\right.$
ii) $d_{r}\left(C_{\Omega}(D, G) \geq \operatorname{deg}(G)-(2 g-2)\right.$

## Weierstrass semigroup and AG codes

A similar improvement can be made to bounds for the generalized Hamming distance of AG codes.
Let $r$ be a positive integer, and $C \subset \mathbb{F}^{m}$ a linear code.
of $C$, the support of $U$ is defined as
$\operatorname{supp}(U):=\left\{i \mid c_{i} \neq 0\right.$ for some $\left.\left(c_{1}, \ldots, c_{m}\right) \in U\right\}$.
The $r$-th generalized Hamming distance of $C$ is defined as
$d_{t}(C)=\min \{\#(\operatorname{sinn}(I I)) \mid U$ is a subcode of $C \quad \operatorname{dim}(U)=r\}$.
Let $s$ be a positive integer and set $\gamma_{s}:=\{\min (\operatorname{deg}(A)) \mid A$ a divisor with $\operatorname{dim} L(A)=s\}$. The sequence $\gamma_{1}, \gamma_{2}, \ldots$ is the gonality sequence of $X ; \gamma_{1}=0$ and $\gamma_{2}$ is the gonality of $X$ This concept was introduced by Yang, Kummar and Stichtenoth in a paper where they proved the following result.
Theorem. The $r$-th generalized Hamming distance of an AG code of length $n$ defined over $X$ satisfies
i) $d_{r}\left(C_{L}(D, G) \geq n-\operatorname{deg}(G)\right.$
ii) $d_{r}\left(C_{\Omega}(D, G) \geq \operatorname{deg}(G)-(2 g-2)\right.$

## Weierstrass semigroup and AG codes

A similar improvement can be made to bounds for the generalized Hamming distance of AG codes.
Let $r$ be a positive integer, and $C \subset \mathbb{F}^{m}$ a linear code. Let $U$ be a subcode of $C$, the support of $U$ is defined as


$$
d_{r}(C)=\min \{\#(\operatorname{supp}(U)) \mid U \text { is a subcode of } C, \operatorname{dim}(U)=r\} .
$$

Let $s$ be a positive integer and set
$\gamma_{s}:=\{\min (\operatorname{deg}(A)) \mid A$ a divisor with $\operatorname{dim} L(A)=s\}$. The sequence $\gamma_{1}, \gamma_{2}, \ldots$ is the gonality sequence of $X ; \gamma_{1}=0$ and $\gamma_{2}$ is the gonality of $X$. This concept was introduced by Yang, Kummar and Stichtenoth in a paper where they proved the following result.
Theorem. The $r$-th generalized Hamming distance of an AG code of length $n$ defined over $X$ satisfies
i) $d_{r}\left(C_{L}(D, G) \geq n-\operatorname{deg}(G)\right.$
ii) $d_{r}\left(C_{\Omega}(D, G) \geq \operatorname{deg}(G)\right.$

## Weierstrass semigroup and AG codes

A similar improvement can be made to bounds for the generalized Hamming distance of AG codes.
Let $r$ be a positive integer, and $C \subset \mathbb{F}^{m}$ a linear code. Let $U$ be a subcode of $C$, the support of $U$ is defined as
$\operatorname{supp}(U):=\left\{i \mid c_{i} \neq 0\right.$ for some $\left.\left(c_{1}, \ldots, c_{m}\right) \in U\right\}$.
$d_{r}(C)=\min \{\#(\operatorname{supp}(U)) \mid U$ is a subcode of $C, \operatorname{dim}(U)=r\}$
Let $s$ be a nositive integer and set
$\gamma_{s}:=\{\min (\operatorname{deg}(A)) \mid A$ a divisor with $\operatorname{dim} L(A)=s\}$. The sequence
$\gamma_{1}, \gamma_{2}, \ldots$ is the gonality sequence of $X ; \gamma_{1}=0$ and $\gamma_{2}$ is the gonality of $X$
This concept was introduced by Yang, Kummar and Stichtenoth in a paper where they proved the following result.
Theorem. The $r$-th generalized Hamming distance of an AG code of length $n$ defined over $X$ satisfies
i) $d_{r}\left(C_{L}(D, G) \geq n-\operatorname{deg}(G)\right.$
ii) $d_{r}\left(C_{\Omega}(D, G) \geq \operatorname{deg}(G)-(2 g-2)\right.$

## Weierstrass semigroup and AG codes

A similar improvement can be made to bounds for the generalized Hamming distance of AG codes.
Let $r$ be a positive integer, and $C \subset \mathbb{F}^{m}$ a linear code. Let $U$ be a subcode of $C$, the support of $U$ is defined as
$\operatorname{supp}(U):=\left\{i \mid c_{i} \neq 0\right.$ for some $\left.\left(c_{1}, \ldots, c_{m}\right) \in U\right\}$.
The $r$-th generalized Hamming distance of $C$ is defined as


## Weierstrass semigroup and AG codes

A similar improvement can be made to bounds for the generalized Hamming distance of AG codes.
Let $r$ be a positive integer, and $C \subset \mathbb{F}^{m}$ a linear code. Let $U$ be a subcode of $C$, the support of $U$ is defined as
$\operatorname{supp}(U):=\left\{i \mid c_{i} \neq 0\right.$ for some $\left.\left(c_{1}, \ldots, c_{m}\right) \in U\right\}$.
The $r$-th generalized Hamming distance of $C$ is defined as

$$
d_{r}(C)=\min \{\#(\operatorname{supp}(U)) \mid U \text { is a subcode of } C, \operatorname{dim}(U)=r\}
$$


$\gamma_{s}:=\{\min (\operatorname{deg}(A)) \mid A$ a divisor with $\operatorname{dim} L(A)=s\}$. The sequence $\gamma_{1}, \gamma_{2}, \ldots$ is the gonality sequence of $X ; \gamma_{1}=0$ and $\gamma_{2}$ is the gonality of $X$ This concept was introduced by Yang, Kummar and Stichtenoth in a paper where they proved the following result.
Theorem. The $r$-th generalized Hamming distance of an AG code of length $n$ defined over $X$ satisfies
$\square$
ii) $d_{r}\left(C_{\Omega}(D, G) \geq \operatorname{deg}(G)-(2 \xi-2)\right.$

## Weierstrass semigroup and AG codes

A similar improvement can be made to bounds for the generalized Hamming distance of AG codes.
Let $r$ be a positive integer, and $C \subset \mathbb{F}^{m}$ a linear code. Let $U$ be a subcode of $C$, the support of $U$ is defined as
$\operatorname{supp}(U):=\left\{i \mid c_{i} \neq 0\right.$ for some $\left.\left(c_{1}, \ldots, c_{m}\right) \in U\right\}$.
The $r$-th generalized Hamming distance of $C$ is defined as

$$
d_{r}(C)=\min \{\#(\operatorname{supp}(U)) \mid U \text { is a subcode of } C, \operatorname{dim}(U)=r\}
$$

Let $s$ be a positive integer and set


## Weierstrass semigroup and AG codes

A similar improvement can be made to bounds for the generalized Hamming distance of AG codes.
Let $r$ be a positive integer, and $C \subset \mathbb{F}^{m}$ a linear code. Let $U$ be a subcode of $C$, the support of $U$ is defined as
$\operatorname{supp}(U):=\left\{i \mid c_{i} \neq 0\right.$ for some $\left.\left(c_{1}, \ldots, c_{m}\right) \in U\right\}$.
The $r$-th generalized Hamming distance of $C$ is defined as

$$
d_{r}(C)=\min \{\#(\operatorname{supp}(U)) \mid U \text { is a subcode of } C, \operatorname{dim}(U)=r\}
$$

Let $s$ be a positive integer and set
$\gamma_{s}:=\{\min (\operatorname{deg}(A)) \mid A$ divisor with $\operatorname{dim} L(A)=s\}$.

This concept was introduced by Yang, Kummar and Stichtenoth in a paper where they proved the following result.
Theorem. The $r$-th generalized Hamming distance of an AG code of length $n$ defined over $X$ satisfies
$\square$

## Weierstrass semigroup and AG codes

A similar improvement can be made to bounds for the generalized Hamming distance of AG codes.
Let $r$ be a positive integer, and $C \subset \mathbb{F}^{m}$ a linear code. Let $U$ be a subcode of $C$, the support of $U$ is defined as
$\operatorname{supp}(U):=\left\{i \mid c_{i} \neq 0\right.$ for some $\left.\left(c_{1}, \ldots, c_{m}\right) \in U\right\}$.
The $r$-th generalized Hamming distance of $C$ is defined as

$$
d_{r}(C)=\min \{\#(\operatorname{supp}(U)) \mid U \text { is a subcode of } C, \operatorname{dim}(U)=r\}
$$

Let $s$ be a positive integer and set
$\gamma_{s}:=\{\min (\operatorname{deg}(A)) \mid A$ a divisor with $\operatorname{dim} L(A)=s\}$. The sequence $\gamma_{1}, \gamma_{2}, \ldots$ is the gonality sequence of $X$;
This concept was introduced by Yang, Kummar and Stichtenoth in a paper where they proved the following result.
Theorem. The $r$-th generalized Hamming distance of an AG code of length $n$ defined over $X$ satisfies
$\square$

## Weierstrass semigroup and AG codes

A similar improvement can be made to bounds for the generalized Hamming distance of AG codes.
Let $r$ be a positive integer, and $C \subset \mathbb{F}^{m}$ a linear code. Let $U$ be a subcode of $C$, the support of $U$ is defined as
$\operatorname{supp}(U):=\left\{i \mid c_{i} \neq 0\right.$ for some $\left.\left(c_{1}, \ldots, c_{m}\right) \in U\right\}$.
The $r$-th generalized Hamming distance of $C$ is defined as

$$
d_{r}(C)=\min \{\#(\operatorname{supp}(U)) \mid U \text { is a subcode of } C, \operatorname{dim}(U)=r\} .
$$

Let $s$ be a positive integer and set
$\gamma_{s}:=\{\min (\operatorname{deg}(A)) \mid A$ a divisor with $\operatorname{dim} L(A)=s\}$. The sequence $\gamma_{1}, \gamma_{2}, \ldots$ is the gonality sequence of $X ; \gamma_{1}=0$ and $\gamma_{2}$ is the gonality of $X$.
where they proved the following result.
Theorem. The $r$-th generalized Hamming distance of an AG code of length $n$ defined over $X$ satisfies

## Weierstrass semigroup and AG codes

A similar improvement can be made to bounds for the generalized Hamming distance of AG codes.
Let $r$ be a positive integer, and $C \subset \mathbb{F}^{m}$ a linear code. Let $U$ be a subcode of $C$, the support of $U$ is defined as
$\operatorname{supp}(U):=\left\{i \mid c_{i} \neq 0\right.$ for some $\left.\left(c_{1}, \ldots, c_{m}\right) \in U\right\}$.
The $r$-th generalized Hamming distance of $C$ is defined as

$$
d_{r}(C)=\min \{\#(\operatorname{supp}(U)) \mid U \text { is a subcode of } C, \operatorname{dim}(U)=r\} .
$$

Let $s$ be a positive integer and set
$\gamma_{s}:=\{\min (\operatorname{deg}(A)) \mid A$ a divisor with $\operatorname{dim} L(A)=s\}$. The sequence $\gamma_{1}, \gamma_{2}, \ldots$ is the gonality sequence of $X ; \gamma_{1}=0$ and $\gamma_{2}$ is the gonality of $X$. This concept was introduced by Yang, Kummar and Stichtenoth in a paper where they proved the following result.
Theorem. The $r$-th generalized Hamming distance of an AG code of length $n$ defined over $X$ satisfies

## Weierstrass semigroup and AG codes

A similar improvement can be made to bounds for the generalized Hamming distance of AG codes.
Let $r$ be a positive integer, and $C \subset \mathbb{F}^{m}$ a linear code. Let $U$ be a subcode of $C$, the support of $U$ is defined as
$\operatorname{supp}(U):=\left\{i \mid c_{i} \neq 0\right.$ for some $\left.\left(c_{1}, \ldots, c_{m}\right) \in U\right\}$.
The $r$-th generalized Hamming distance of $C$ is defined as

$$
d_{r}(C)=\min \{\#(\operatorname{supp}(U)) \mid U \text { is a subcode of } C, \operatorname{dim}(U)=r\}
$$

Let $s$ be a positive integer and set
$\gamma_{s}:=\{\min (\operatorname{deg}(A)) \mid A$ a divisor with $\operatorname{dim} L(A)=s\}$. The sequence $\gamma_{1}, \gamma_{2}, \ldots$ is the gonality sequence of $X ; \gamma_{1}=0$ and $\gamma_{2}$ is the gonality of $X$. This concept was introduced by Yang, Kummar and Stichtenoth in a paper where they proved the following result.
Theorem. The $r$-th generalized Hamming distance of an AG code of length $n$ defined over $X$ satisfies

## Weierstrass semigroup and AG codes

A similar improvement can be made to bounds for the generalized Hamming distance of AG codes.
Let $r$ be a positive integer, and $C \subset \mathbb{F}^{m}$ a linear code. Let $U$ be a subcode of $C$, the support of $U$ is defined as
$\operatorname{supp}(U):=\left\{i \mid c_{i} \neq 0\right.$ for some $\left.\left(c_{1}, \ldots, c_{m}\right) \in U\right\}$.
The $r$-th generalized Hamming distance of $C$ is defined as

$$
d_{r}(C)=\min \{\#(\operatorname{supp}(U)) \mid U \text { is a subcode of } C, \operatorname{dim}(U)=r\}
$$

Let $s$ be a positive integer and set
$\gamma_{s}:=\{\min (\operatorname{deg}(A)) \mid A$ divisor with $\operatorname{dim} L(A)=s\}$. The sequence $\gamma_{1}, \gamma_{2}, \ldots$ is the gonality sequence of $X ; \gamma_{1}=0$ and $\gamma_{2}$ is the gonality of $X$. This concept was introduced by Yang, Kummar and Stichtenoth in a paper where they proved the following result.
Theorem. The $r$-th generalized Hamming distance of an AG code of length $n$ defined over $X$ satisfies
i) $d_{r}\left(C_{L}(D, G) \geq n-\operatorname{deg}(G)+\gamma_{r}\right.$;

## Weierstrass semigroup and AG codes

A similar improvement can be made to bounds for the generalized Hamming distance of AG codes.
Let $r$ be a positive integer, and $C \subset \mathbb{F}^{m}$ a linear code. Let $U$ be a subcode of $C$, the support of $U$ is defined as
$\operatorname{supp}(U):=\left\{i \mid c_{i} \neq 0\right.$ for some $\left.\left(c_{1}, \ldots, c_{m}\right) \in U\right\}$.
The $r$-th generalized Hamming distance of $C$ is defined as

$$
d_{r}(C)=\min \{\#(\operatorname{supp}(U)) \mid U \text { is a subcode of } C, \operatorname{dim}(U)=r\}
$$

Let $s$ be a positive integer and set
$\gamma_{s}:=\{\min (\operatorname{deg}(A)) \mid A$ divisor with $\operatorname{dim} L(A)=s\}$. The sequence $\gamma_{1}, \gamma_{2}, \ldots$ is the gonality sequence of $X ; \gamma_{1}=0$ and $\gamma_{2}$ is the gonality of $X$. This concept was introduced by Yang, Kummar and Stichtenoth in a paper where they proved the following result.
Theorem. The $r$-th generalized Hamming distance of an AG code of length $n$ defined over $X$ satisfies
i) $d_{r}\left(C_{L}(D, G) \geq n-\operatorname{deg}(G)+\gamma_{r}\right.$;
ii) $d_{r}\left(C_{\Omega}(D, G) \geq \operatorname{deg}(G)-(2 g-2)+\gamma_{r}\right.$.

## Weierstrass semigroup and AG codes

Using the concept of pure gaps we were able to prove the following result.
Theorem. Assume that $\left(n_{1}, \ldots, n_{m}\right)$ and $\left(p_{1}, \ldots, p_{m}\right)$ are pure gaps of $H$, with $n_{i} \leq p_{i}$ for all $i=1, \ldots, m$, and that $\left(q_{1}, \ldots, q_{m}\right)$ is also a pure gap whenever $n_{i} \leq q_{i} \leq p_{i}$, for all $i=1, \ldots, m$. Let $G:=\sum_{i=1}^{m} p_{i} p_{i}$. Then:
i) $d_{r}\left(C_{L}(D, G) \geq n-\operatorname{deg}(G)+\sum_{i=1}^{m}\left(p_{i}-n_{i}\right)+m+\gamma_{r}\right.$;
ii)
$d_{r}\left(C_{\Omega}(D, G) \geq \operatorname{deg}(G)-(2 g-2)+\gamma_{r+\sum_{i=1}^{m}\left(p_{i}-n_{i}\right)+m}-\left(\sum_{i=1}^{m}\left(p_{i}-n_{i}\right)+m\right)\right.$.
The last item can lead to an improvement because $\gamma_{r}<\gamma_{r+1}$ for all positive integers $r$.
Thus the existence of pure gaps in $H$ may lead to an improvement of the bounds for the generalized Hamming distances of AG codes. Now we will show that the existence of total inflection points in plane curves determine the existence of pure gaps in certain Weierstrass semigroups.

## Weierstrass semigroup and AG codes

Using the concept of pure gaps we were able to prove the following result.


The last item can lead to an improvement because $\gamma_{r}<\gamma_{r+1}$ for all positive integers $r$.
Thus the existence of pure gaps in $H$ may lead to an improvement of the bounds for the generalized Hamming distances of AG codes. Now we will show that the existence of total inflection points in plane curves determine the existence of pure gaps in certain Weierstrass semigroups.

## Weierstrass semigroup and AG codes

Using the concept of pure gaps we were able to prove the following result.
Theorem. Assume that $\left(n_{1}, \ldots, n_{m}\right)$ and $\left(p_{1}, \ldots, p_{m}\right)$ are pure gaps of $H$, with $n_{i} \leq p_{i}$ for all $i=1, \ldots, m$, and that $\left(q_{1}, \ldots, q_{m}\right)$ is also a pure gap
whenever $n_{i} \leq q_{i} \leq p_{i}$, for all $i=1, \ldots, m$. Let $G:=\sum_{i=1}^{m} p_{i} p_{i}$. Then:
i) $d_{r}\left(C_{L}(D, G) \geq n-\operatorname{deg}(G)+\sum_{i=1}^{m}\left(p_{i}-n_{i}\right)+m+\gamma_{r} ;\right.$
ii)
$d_{r}\left(C_{\Omega}(D, G) \geq \operatorname{deg}(G)-(2 g-2)+\gamma_{r+\sum_{i=1}^{m}\left(p_{i}-n_{i}\right)+m}-\left(\sum_{i=1}^{m}\left(p_{i}-n_{i}\right)+m\right)\right.$

The last item can lead to an improvement because $\gamma_{r}<\gamma_{r+1}$ for all positive integers $r$.
Thus the existence of pure gaps in $H$ may lead to an improvement of the bounds for the generalized Hamming distances of AG codes. Now we will show that the existence of total inflection points in plane curves determine the existence of pure gaps in certain Weierstrass semigroups.

## Weierstrass semigroup and AG codes

Using the concept of pure gaps we were able to prove the following result.
Theorem. Assume that $\left(n_{1}, \ldots, n_{m}\right)$ and $\left(p_{1}, \ldots, p_{m}\right)$ are pure gaps of $H$, with $n_{i} \leq p_{i}$ for all $i=1, \ldots, m$, whenever $n_{i} \leq q_{i} \leq p_{i}$, for all $i=1, \ldots, m$. Let $G:=\sum_{i=1}^{m} p_{i} P_{i}$. Then:
i) $d_{r}\left(C_{L}(D, G) \geq n-\operatorname{deg}(G)+\sum_{i=1}^{m}\left(p_{i}-n_{i}\right)+m+\gamma_{r} ;\right.$
ii)
$d_{r}\left(C_{\Omega}(D, G) \geq \operatorname{deg}(G)-(2 g-2)+\gamma_{r+\sum_{i=1}^{m}\left(p_{i}-n_{i}\right)+m}-\left(\sum_{i=1}^{m}\left(p_{i}-n_{i}\right)+m\right)\right.$.

The last item can lead to an improvement because $\gamma_{r}<\gamma_{r+1}$ for all positive integers $r$.
Thus the existence of pure gaps in $H$ may lead to an improvement of the bounds for the generalized Hamming distances of AG codes. Now we will show that the existence of total inflection points in plane curves determine the existence of pure gaps in certain Weierstrass semigroups.

## Weierstrass semigroup and AG codes

Using the concept of pure gaps we were able to prove the following result.
Theorem. Assume that $\left(n_{1}, \ldots, n_{m}\right)$ and $\left(p_{1}, \ldots, p_{m}\right)$ are pure gaps of $H$, with $n_{i} \leq p_{i}$ for all $i=1, \ldots, m$, and that $\left(q_{1}, \ldots, q_{m}\right)$ is also a pure gap whenever $n_{i} \leq q_{i} \leq p_{i}$, for all $i=1, \ldots, m$.


The last item can lead to an improvement because $\gamma_{r}<\gamma_{r+1}$ for all positive integers $r$.
Thus the existence of pure gaps in $H$ may lead to an improvement of the bounds for the generalized Hamming distances of AG codes. Now we will show that the existence of total inflection points in plane curves determine the existence of pure gaps in certain Weierstrass semigroups.

## Weierstrass semigroup and AG codes

Using the concept of pure gaps we were able to prove the following result.
Theorem. Assume that $\left(n_{1}, \ldots, n_{m}\right)$ and $\left(p_{1}, \ldots, p_{m}\right)$ are pure gaps of $H$, with $n_{i} \leq p_{i}$ for all $i=1, \ldots, m$, and that $\left(q_{1}, \ldots, q_{m}\right)$ is also a pure gap whenever $n_{i} \leq q_{i} \leq p_{i}$, for all $i=1, \ldots, m$. Let $G:=\sum_{i=1}^{m} p_{i} P_{i}$. Then:


The last item can lead to an improvement because $\gamma_{r}<\gamma_{r+1}$ for all positive integers $r$.
Thus the existence of pure gaps in $H$ may lead to an improvement of the bounds for the generalized Hamming distances of AG codes. Now we will show that the existence of total inflection points in plane curves determine the existence of pure gaps in certain Weierstrass semigroups.

## Weierstrass semigroup and AG codes

Using the concept of pure gaps we were able to prove the following result.
Theorem. Assume that $\left(n_{1}, \ldots, n_{m}\right)$ and $\left(p_{1}, \ldots, p_{m}\right)$ are pure gaps of $H$, with $n_{i} \leq p_{i}$ for all $i=1, \ldots, m$, and that $\left(q_{1}, \ldots, q_{m}\right)$ is also a pure gap whenever $n_{i} \leq q_{i} \leq p_{i}$, for all $i=1, \ldots, m$. Let $G:=\sum_{i=1}^{m} p_{i} P_{i}$. Then:
i) $d_{r}\left(C_{L}(D, G) \geq n-\operatorname{deg}(G)+\sum_{i=1}^{m}\left(p_{i}-n_{i}\right)+m+\gamma_{r}\right.$;
ii)
$d_{r}\left(C_{\Omega}(D, G) \geq \operatorname{deg}(G)-(2 g-2)+\gamma_{r+\sum_{i=1}^{m}\left(p_{i}-n_{i}\right)+m}-\left(\sum_{i=1}^{m}\left(p_{i}-n_{i}\right)+m\right)\right.$
The last item can lead to an improvement because $\gamma_{r}<\gamma_{r+1}$ for all positive integers $r$.
Thus the existence of pure gaps in $H$ may lead to an improvement of the bounds for the generalized Hamming distances of AG codes. Now we will show that the existence of total inflection points in plane curves determine the existence of pure gaps in certain Weierstrass semigroups.

## Weierstrass semigroup and AG codes

Using the concept of pure gaps we were able to prove the following result.
Theorem. Assume that $\left(n_{1}, \ldots, n_{m}\right)$ and $\left(p_{1}, \ldots, p_{m}\right)$ are pure gaps of $H$, with $n_{i} \leq p_{i}$ for all $i=1, \ldots, m$, and that $\left(q_{1}, \ldots, q_{m}\right)$ is also a pure gap whenever $n_{i} \leq q_{i} \leq p_{i}$, for all $i=1, \ldots, m$. Let $G:=\sum_{i=1}^{m} p_{i} P_{i}$. Then:
i) $d_{r}\left(C_{L}(D, G) \geq n-\operatorname{deg}(G)+\sum_{i=1}^{m}\left(p_{i}-n_{i}\right)+m+\gamma_{r}\right.$;
ii)

$$
d_{r}\left(C_{\Omega}(D, G) \geq \operatorname{deg}(G)-(2 g-2)+\gamma_{r+\sum_{i=1}^{m}\left(p_{i}-n_{i}\right)+m}-\left(\sum_{i=1}^{m}\left(p_{i}-n_{i}\right)+m\right)\right.
$$

The last item can lead to an improvement because $\gamma_{r}<\gamma_{r+1}$ for all positive integers $r$.
Thus the existence of pure gaps in $H$ may lead to an improvement of the bounds for the generalized Hamming distances of AG codes. Now we will show that the existence of total inflection points in plane curves determine the existence of pure gaps in certain Weierstrass semigroups.

## Weierstrass semigroup and AG codes

Using the concept of pure gaps we were able to prove the following result.
Theorem. Assume that $\left(n_{1}, \ldots, n_{m}\right)$ and $\left(p_{1}, \ldots, p_{m}\right)$ are pure gaps of $H$, with $n_{i} \leq p_{i}$ for all $i=1, \ldots, m$, and that $\left(q_{1}, \ldots, q_{m}\right)$ is also a pure gap whenever $n_{i} \leq q_{i} \leq p_{i}$, for all $i=1, \ldots, m$. Let $G:=\sum_{i=1}^{m} p_{i} P_{i}$. Then:
i) $d_{r}\left(C_{L}(D, G) \geq n-\operatorname{deg}(G)+\sum_{i=1}^{m}\left(p_{i}-n_{i}\right)+m+\gamma_{r}\right.$;
ii)
$d_{r}\left(C_{\Omega}(D, G) \geq \operatorname{deg}(G)-(2 g-2)+\gamma_{r+\sum_{i=1}^{m}\left(p_{i}-n_{i}\right)+m}-\left(\sum_{i=1}^{m}\left(p_{i}-n_{i}\right)+m\right)\right.$.
The last item can lead to an improvement because $\gamma_{r}<\gamma_{r+1}$ for all positive integers $r$.
Thus the existence of pure gaps in $H$ may lead to an improvement of the bounds for the generalized Hamming distances of AG codes. Now we will show that the existence of total inflection points in plane curves determine the existence of pure gaps in certain Weierstrass semigroups.

## Weierstrass semigroup and AG codes

Using the concept of pure gaps we were able to prove the following result.
Theorem. Assume that $\left(n_{1}, \ldots, n_{m}\right)$ and $\left(p_{1}, \ldots, p_{m}\right)$ are pure gaps of $H$, with $n_{i} \leq p_{i}$ for all $i=1, \ldots, m$, and that $\left(q_{1}, \ldots, q_{m}\right)$ is also a pure gap whenever $n_{i} \leq q_{i} \leq p_{i}$, for all $i=1, \ldots, m$. Let $G:=\sum_{i=1}^{m} p_{i} P_{i}$. Then:
i) $d_{r}\left(C_{L}(D, G) \geq n-\operatorname{deg}(G)+\sum_{i=1}^{m}\left(p_{i}-n_{i}\right)+m+\gamma_{r}\right.$;
ii)

$$
d_{r}\left(C_{\Omega}(D, G) \geq \operatorname{deg}(G)-(2 g-2)+\gamma_{r+\sum_{i=1}^{m}\left(p_{i}-n_{i}\right)+m}-\left(\sum_{i=1}^{m}\left(p_{i}-n_{i}\right)+m\right)\right.
$$

The last item can lead to an improvement because $\gamma_{r}<\gamma_{r+1}$ for all positive integers $r$.
Thus the existence of pure gaps in $H$ may lead to an improvement of the bounds for the generalized Hamming distances of AG codes.
show that the existence of total inflection points in plane curves determine the existence of pure gaps in certain Weierstrass semigroups.

## Weierstrass semigroup and AG codes

Using the concept of pure gaps we were able to prove the following result.
Theorem. Assume that $\left(n_{1}, \ldots, n_{m}\right)$ and $\left(p_{1}, \ldots, p_{m}\right)$ are pure gaps of $H$, with $n_{i} \leq p_{i}$ for all $i=1, \ldots, m$, and that $\left(q_{1}, \ldots, q_{m}\right)$ is also a pure gap whenever $n_{i} \leq q_{i} \leq p_{i}$, for all $i=1, \ldots, m$. Let $G:=\sum_{i=1}^{m} p_{i} P_{i}$. Then:
i) $d_{r}\left(C_{L}(D, G) \geq n-\operatorname{deg}(G)+\sum_{i=1}^{m}\left(p_{i}-n_{i}\right)+m+\gamma_{r}\right.$;
ii)
$d_{r}\left(C_{\Omega}(D, G) \geq \operatorname{deg}(G)-(2 g-2)+\gamma_{r+\sum_{i=1}^{m}\left(p_{i}-n_{i}\right)+m}-\left(\sum_{i=1}^{m}\left(p_{i}-n_{i}\right)+m\right)\right.$.
The last item can lead to an improvement because $\gamma_{r}<\gamma_{r+1}$ for all positive integers $r$.
Thus the existence of pure gaps in $H$ may lead to an improvement of the bounds for the generalized Hamming distances of AG codes. Now we will show that the existence of total inflection points in plane curves determine the existence of pure gaps in certain Weierstrass semigroups.

## Total inflection points and pure gaps

Assume that $X$ is a smooth plane, projective curve, of degree $r>4$. We say that $P \in X$ is a total inflection point if the tangent line at $P$ intersects $X$ only at $P$. In a work with T. Kato, we proved the following.
Theorem. Let $P_{1}, P_{2}$ and $P_{3}$ be rational, total inflection points of $X$ which do not lie in a line. Then $((r-4) r, 1,1),(1,(r-4) r, 1)$ and $(1,1,(r-4) r)$ are pure gaps of $H\left(P_{1}, P_{2}, P_{3}\right)$.

Theorem. Let $P_{1}, \ldots, P_{m}$ be total inflection points on $X$. Then $\left(s_{1} r+\alpha_{1}, \ldots, s_{m} r+\alpha_{m}\right)$ is a pure gap of $H\left(P_{1}, \ldots, P_{m}\right)$, whenever $s_{i}, \alpha_{i}$ are integers such that $s_{i} \geq 0,1 \leq \alpha_{i} \leq r-1-i-\sum_{j=1}^{m} s_{j}$, for all $i=1, \ldots, m$, and $\sum_{j=1}^{m} s_{j} \leq r-2-m$.
Theorem. Let $P, P_{1}, \ldots, P_{m} \in X$ be rational points, with $P$ a total inflection point. Let $0 \leq i<r-3$ and $\alpha_{1}, \ldots, \alpha_{m}$ be positive integers such that $\sum_{j=1}^{m} \alpha_{j} \leq r-i-3$. Then (ir $+1, \alpha_{1}, \ldots, \alpha_{m}$ ) is a pure gap of $H\left(P, P_{1}, \ldots, P_{m}\right)$.

## Total inflection points and pure gaps

Assume that $X$ is a smooth plane, projective curve, of degree $r>4$. We say that $P \in X$ is a total inflection point if the tangent line at $P$ intersects $X$ only at $P$. In a work with T. Kato, we proved the following.

Theorem. Let $P_{1}, P_{2}$ and $P_{3}$ be rational, total inflection points of $X$ which do not lie in a line. Then $((r-4) r, 1,1),(1,(r-4) r, 1)$ and $(1,1,(r-4) r)$ are pure gaps of $H\left(P_{1}, P_{2}, P_{3}\right)$

Theorem. Let $P_{1} \ldots \ldots P_{m}$ be total inflection noints on $X$. Then $\left(s_{1} r+\alpha_{1}, \ldots, s_{m} r+\alpha_{m}\right)$ is a pure gap of $H\left(P_{1}, \ldots, P_{m}\right)$, whenever $s_{i}, \alpha_{i}$ are integers such that $s_{i} \geq 0,1 \leq \alpha_{i} \leq r-1-i-\sum_{j=1}^{m} s_{j}$, for all $i=1, \ldots, m$, and $\sum_{j=1}^{m} s_{j} \leq r-2-m$


 $H\left(P, P_{1}, \ldots, P_{m}\right)$

## Total inflection points and pure gaps

Assume that $X$ is a smooth plane, projective curve, of degree $r>4$. We say that $P \in X$ is a total inflection point if the tangent line at $P$ intersects $X$ only at $P$.


## Total inflection points and pure gaps

Assume that $X$ is a smooth plane, projective curve, of degree $r>4$. We say that $P \in X$ is a total inflection point if the tangent line at $P$ intersects $X$ only at $P$. In a work with T. Kato, we proved the following.


## Total inflection points and pure gaps

Assume that $X$ is a smooth plane, projective curve, of degree $r>4$. We say that $P \in X$ is a total inflection point if the tangent line at $P$ intersects $X$ only at $P$. In a work with T. Kato, we proved the following.
Theorem. Let $P_{1}, P_{2}$ and $P_{3}$ be rational, total inflection points of $X$ which do not lie in a line. Then $((r-4) r, 1,1),(1,(r-4) r, 1)$ and $(1,1,(r-4) r)$ are pure gaps of $H\left(P_{1}, P_{2}, P_{3}\right)$.


## Total inflection points and pure gaps

Assume that $X$ is a smooth plane, projective curve, of degree $r>4$. We say that $P \in X$ is a total inflection point if the tangent line at $P$ intersects $X$ only at $P$. In a work with T. Kato, we proved the following.
Theorem. Let $P_{1}, P_{2}$ and $P_{3}$ be rational, total inflection points of $X$ which do not lie in a line. Then $((r-4) r, 1,1),(1,(r-4) r, 1)$ and $(1,1,(r-4) r)$ are pure gaps of $H\left(P_{1}, P_{2}, P_{3}\right)$.
Theorem. Let $P_{1}, \ldots, P_{m}$ be total inflection points on $X$. Then $\left(s_{1} r+\alpha_{1}, \ldots, s_{m} r+\alpha_{m}\right)$ is a pure gap of $H\left(P_{1}, \ldots, P_{m}\right)$, whenever $s_{i}, \alpha_{i}$ are integers such that $s_{i} \geq 0,1 \leq \alpha_{i} \leq r-1-i-\sum_{j=1}^{m} s_{j}$, for all $i=1, \ldots, m$, and $\sum_{j=1}^{m} s_{j} \leq r-2-m$.


## Total inflection points and pure gaps

Assume that $X$ is a smooth plane, projective curve, of degree $r>4$. We say that $P \in X$ is a total inflection point if the tangent line at $P$ intersects $X$ only at $P$. In a work with T. Kato, we proved the following.
Theorem. Let $P_{1}, P_{2}$ and $P_{3}$ be rational, total inflection points of $X$ which do not lie in a line. Then $((r-4) r, 1,1),(1,(r-4) r, 1)$ and $(1,1,(r-4) r)$ are pure gaps of $H\left(P_{1}, P_{2}, P_{3}\right)$.

Theorem. Let $P_{1}, \ldots, P_{m}$ be total inflection points on $X$. Then $\left(s_{1} r+\alpha_{1}, \ldots, s_{m} r+\alpha_{m}\right)$ is a pure gap of $H\left(P_{1}, \ldots, P_{m}\right)$, whenever $s_{i}, \alpha_{i}$ are integers such that $s_{i} \geq 0,1 \leq \alpha_{i} \leq r-1-i-\sum_{j=1}^{m} s_{j}$, for all $i=1, \ldots, m$, and $\sum_{j=1}^{m} s_{j} \leq r-2-m$.
Theorem. Let $P, P_{1}, \ldots, P_{m} \in X$ be rational points, with $P$ a total inflection point. Let $0 \leq i<r-3$ and $\alpha_{1}, \ldots, \alpha_{m}$ be positive integers such that $\sum_{j=1}^{m} \alpha_{j} \leq r-i-3$. Then (ir $+1, \alpha_{1}, \ldots, \alpha_{m}$ ) is a pure gap of $H\left(P, P_{1}, \ldots, P_{m}\right)$.

## Total inflection points and pure gaps

Theorem. Let $P_{1}, \ldots, P_{m}$ be total inflection points on $X$. Then $\left(s_{1} r+\alpha_{1}, \ldots, s_{m} r+\alpha_{m}\right)$ is a pure gap of $H\left(P_{1}, \ldots, P_{m}\right)$, whenever $s_{i}, \alpha_{i}$ are integers such that $s_{i} \geq 0,1 \leq \alpha_{i} \leq r-1-i-\sum_{i=1}^{m} s_{i}$, for all $i=1, \ldots, m$, and $\sum_{i=1}^{m} s_{i} \leq r-2-m$.
Application. Take $X \subset \mathbb{P}^{2}(K)$ the Hermitian curve of degree $q+1$ defined over $\mathbb{F}=G F\left(q^{2}\right)$. Let $s$ and $m$ be positive integers such that $s+m \leq q-1$; let $P_{1}, \ldots, P_{m}$ be distinct rational points of $X$. Take $s_{1}=s, s_{2}=\cdots=s_{m}=0$, from the above theorem we get that $\left(s r+\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m}\right)$ is a pure gap at $H\left(P_{1}, \ldots, P_{m}\right)$ whenever $1 \leq \alpha_{i} \leq q-i-s(i=1, \ldots, m)$ (i.e. only have pure gaps "between" $(s r+1,1, \ldots, 1)$ and $(s r+q-1-s, q-2-s, \ldots, q-m-s))$. Let $G=(2 s r+q-1-s) P_{1}+\sum_{i=2}^{m}(q-i-s) P_{i}$ and let $D$ be the sum of the other $q^{3}+1-m$ rational points of $X$. From the work together with $F$. Torres we know that $C_{\Omega}(D, G)$ is an $\left[q^{3}+1-m, k, d\right]$ code with $d \geq 2 s(q+1)+m(2 q-2 s-1)-m^{2}-(q+1)(q-2)$.

## Total inflection points and pure gaps

Theorem. Let $P_{1}, \ldots, P_{m}$ be total inflection points on $X$. Then $\left(s_{1} r+\alpha_{1}, \ldots, s_{m} r+\alpha_{m}\right)$ is a pure gap of $H\left(P_{1}, \ldots, P_{m}\right)$, whenever $s_{i}, \alpha_{i}$ are integers such that $s_{i} \geq 0,1 \leq \alpha_{i} \leq r-1-i-\sum_{i=1}^{m} s_{i}$, for all $i=1, \ldots, m$, and $\sum_{i=1}^{m} s_{i} \leq r-2-m$.


## Total inflection points and pure gaps

Theorem. Let $P_{1}, \ldots, P_{m}$ be total inflection points on $X$. Then $\left(s_{1} r+\alpha_{1}, \ldots, s_{m} r+\alpha_{m}\right)$ is a pure gap of $H\left(P_{1}, \ldots, P_{m}\right)$, whenever $s_{i}, \alpha_{i}$ are integers such that $s_{i} \geq 0,1 \leq \alpha_{i} \leq r-1-i-\sum_{i=1}^{m} s_{i}$, for all $i=1, \ldots, m$, and $\sum_{i=1}^{m} s_{i} \leq r-2-m$.
Application. Take $X \subset \mathbb{P}^{2}(K)$ the Hermitian curve of degree $q+1$ defined over $\mathbb{F}=G F\left(q^{2}\right)$.

$\square$

## Total inflection points and pure gaps

Theorem. Let $P_{1}, \ldots, P_{m}$ be total inflection points on $X$. Then $\left(s_{1} r+\alpha_{1}, \ldots, s_{m} r+\alpha_{m}\right)$ is a pure gap of $H\left(P_{1}, \ldots, P_{m}\right)$, whenever $s_{i}, \alpha_{i}$ are integers such that $s_{i} \geq 0,1 \leq \alpha_{i} \leq r-1-i-\sum_{i=1}^{m} s_{i}$, for all $i=1, \ldots, m$, and $\sum_{i=1}^{m} s_{i} \leq r-2-m$.
Application. Take $X \subset \mathbb{P}^{2}(K)$ the Hermitian curve of degree $q+1$ defined over $\mathbb{F}=G F\left(q^{2}\right)$. Let $s$ and $m$ be positive integers such that $s+m \leq q-1$; let $P_{1}, \ldots, P_{m}$ be distinct rational points of $X$.


## Total inflection points and pure gaps

Theorem. Let $P_{1}, \ldots, P_{m}$ be total inflection points on $X$. Then $\left(s_{1} r+\alpha_{1}, \ldots, s_{m} r+\alpha_{m}\right)$ is a pure gap of $H\left(P_{1}, \ldots, P_{m}\right)$, whenever $s_{i}, \alpha_{i}$ are integers such that $s_{i} \geq 0,1 \leq \alpha_{i} \leq r-1-i-\sum_{i=1}^{m} s_{i}$, for all $i=1, \ldots, m$, and $\sum_{i=1}^{m} s_{i} \leq r-2-m$.
Application. Take $X \subset \mathbb{P}^{2}(K)$ the Hermitian curve of degree $q+1$ defined over $\mathbb{F}=G F\left(q^{2}\right)$. Let $s$ and $m$ be positive integers such that $s+m \leq q-1$; let $P_{1}, \ldots, P_{m}$ be distinct rational points of $X$.
Take $s_{1}=s, s_{2}=\cdots=s_{m}=0$, from the above theorem we get that $\left(s r+\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m}\right)$ is a pure gap at $H\left(P_{1}, \ldots, P_{m}\right)$ whenever $1 \leq \alpha_{i} \leq q-i-s(i=1, \ldots, m)$


## Total inflection points and pure gaps

Theorem. Let $P_{1}, \ldots, P_{m}$ be total inflection points on $X$. Then $\left(s_{1} r+\alpha_{1}, \ldots, s_{m} r+\alpha_{m}\right)$ is a pure gap of $H\left(P_{1}, \ldots, P_{m}\right)$, whenever $s_{i}, \alpha_{i}$ are integers such that $s_{i} \geq 0,1 \leq \alpha_{i} \leq r-1-i-\sum_{i=1}^{m} s_{i}$, for all $i=1, \ldots, m$, and $\sum_{i=1}^{m} s_{i} \leq r-2-m$.
Application. Take $X \subset \mathbb{P}^{2}(K)$ the Hermitian curve of degree $q+1$ defined over $\mathbb{F}=G F\left(q^{2}\right)$. Let $s$ and $m$ be positive integers such that $s+m \leq q-1$; let $P_{1}, \ldots, P_{m}$ be distinct rational points of $X$.

Take $s_{1}=s, s_{2}=\cdots=s_{m}=0$, from the above theorem we get that $\left(s r+\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m}\right)$ is a pure gap at $H\left(P_{1}, \ldots, P_{m}\right)$ whenever $1 \leq \alpha_{i} \leq q-i-s(i=1, \ldots, m)$ (i.e. only have pure gaps "between" $(s r+1,1, \ldots, 1)$ and $(s r+q-1-s, q-2-s, \ldots, q-m-s))$.


Torres we know that $C_{\Omega}(D, G)$ is an $\left[q^{3}+1-m, k, d\right]$ code with

## Total inflection points and pure gaps

Theorem. Let $P_{1}, \ldots, P_{m}$ be total inflection points on $X$. Then $\left(s_{1} r+\alpha_{1}, \ldots, s_{m} r+\alpha_{m}\right)$ is a pure gap of $H\left(P_{1}, \ldots, P_{m}\right)$, whenever $s_{i}, \alpha_{i}$ are integers such that $s_{i} \geq 0,1 \leq \alpha_{i} \leq r-1-i-\sum_{i=1}^{m} s_{i}$, for all $i=1, \ldots, m$, and $\sum_{i=1}^{m} s_{i} \leq r-2-m$.
Application. Take $X \subset \mathbb{P}^{2}(K)$ the Hermitian curve of degree $q+1$ defined over $\mathbb{F}=G F\left(q^{2}\right)$. Let $s$ and $m$ be positive integers such that $s+m \leq q-1$; let $P_{1}, \ldots, P_{m}$ be distinct rational points of $X$.
Take $s_{1}=s, s_{2}=\cdots=s_{m}=0$, from the above theorem we get that $\left(s r+\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m}\right)$ is a pure gap at $H\left(P_{1}, \ldots, P_{m}\right)$ whenever $1 \leq \alpha_{i} \leq q-i-s(i=1, \ldots, m)$ (i.e. only have pure gaps "between" $(s r+1,1, \ldots, 1)$ and $(s r+q-1-s, q-2-s, \ldots, q-m-s))$. Let $G=(2 s r+q-1-s) P_{1}+\sum_{i=2}^{m}(q-i-s) P_{i}$ and let $D$ be the sum of the other $q^{3}+1-m$ rational points of $X$.
Torres we know that $C_{\Omega}(D, G)$ is an $\left[q^{3}+1-m, k, d\right]$ code with

## Total inflection points and pure gaps

Theorem. Let $P_{1}, \ldots, P_{m}$ be total inflection points on $X$. Then $\left(s_{1} r+\alpha_{1}, \ldots, s_{m} r+\alpha_{m}\right)$ is a pure gap of $H\left(P_{1}, \ldots, P_{m}\right)$, whenever $s_{i}, \alpha_{i}$ are integers such that $s_{i} \geq 0,1 \leq \alpha_{i} \leq r-1-i-\sum_{i=1}^{m} s_{i}$, for all $i=1, \ldots, m$, and $\sum_{i=1}^{m} s_{i} \leq r-2-m$.
Application. Take $X \subset \mathbb{P}^{2}(K)$ the Hermitian curve of degree $q+1$ defined over $\mathbb{F}=G F\left(q^{2}\right)$. Let $s$ and $m$ be positive integers such that $s+m \leq q-1$; let $P_{1}, \ldots, P_{m}$ be distinct rational points of $X$.
Take $s_{1}=s, s_{2}=\cdots=s_{m}=0$, from the above theorem we get that $\left(s r+\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m}\right)$ is a pure gap at $H\left(P_{1}, \ldots, P_{m}\right)$ whenever $1 \leq \alpha_{i} \leq q-i-s(i=1, \ldots, m)$ (i.e. only have pure gaps "between" $(s r+1,1, \ldots, 1)$ and $(s r+q-1-s, q-2-s, \ldots, q-m-s))$.
Let $G=(2 s r+q-1-s) P_{1}+\sum_{i=2}^{m}(q-i-s) P_{i}$ and let $D$ be the sum of the other $q^{3}+1-m$ rational points of $X$. From the work together with $F$. Torres we know that $C_{\Omega}(D, G)$ is an $\left[q^{3}+1-m, k, d\right]$ code with

## Total inflection points and pure gaps

Theorem. Let $P_{1}, \ldots, P_{m}$ be total inflection points on $X$. Then $\left(s_{1} r+\alpha_{1}, \ldots, s_{m} r+\alpha_{m}\right)$ is a pure gap of $H\left(P_{1}, \ldots, P_{m}\right)$, whenever $s_{i}, \alpha_{i}$ are integers such that $s_{i} \geq 0,1 \leq \alpha_{i} \leq r-1-i-\sum_{i=1}^{m} s_{i}$, for all $i=1, \ldots, m$, and $\sum_{i=1}^{m} s_{i} \leq r-2-m$.
Application. Take $X \subset \mathbb{P}^{2}(K)$ the Hermitian curve of degree $q+1$ defined over $\mathbb{F}=G F\left(q^{2}\right)$. Let $s$ and $m$ be positive integers such that $s+m \leq q-1$; let $P_{1}, \ldots, P_{m}$ be distinct rational points of $X$.
Take $s_{1}=s, s_{2}=\cdots=s_{m}=0$, from the above theorem we get that $\left(s r+\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m}\right)$ is a pure gap at $H\left(P_{1}, \ldots, P_{m}\right)$ whenever $1 \leq \alpha_{i} \leq q-i-s(i=1, \ldots, m)$ (i.e. only have pure gaps "between" $(s r+1,1, \ldots, 1)$ and $(s r+q-1-s, q-2-s, \ldots, q-m-s))$. Let $G=(2 s r+q-1-s) P_{1}+\sum_{i=2}^{m}(q-i-s) P_{i}$ and let $D$ be the sum of the other $q^{3}+1-m$ rational points of $X$. From the work together with $F$. Torres we know that $C_{\Omega}(D, G)$ is an $\left[q^{3}+1-m, k, d\right]$ code with $d \geq \operatorname{deg}(G)-(2 g-2)+m+\sum_{i=1}^{n}\left(p_{i}-n_{i}\right)$

## Total inflection points and pure gaps

Theorem. Let $P_{1}, \ldots, P_{m}$ be total inflection points on $X$. Then $\left(s_{1} r+\alpha_{1}, \ldots, s_{m} r+\alpha_{m}\right)$ is a pure gap of $H\left(P_{1}, \ldots, P_{m}\right)$, whenever $s_{i}, \alpha_{i}$ are integers such that $s_{i} \geq 0,1 \leq \alpha_{i} \leq r-1-i-\sum_{i=1}^{m} s_{i}$, for all $i=1, \ldots, m$, and $\sum_{i=1}^{m} s_{i} \leq r-2-m$.
Application. Take $X \subset \mathbb{P}^{2}(K)$ the Hermitian curve of degree $q+1$ defined over $\mathbb{F}=G F\left(q^{2}\right)$. Let $s$ and $m$ be positive integers such that $s+m \leq q-1$; let $P_{1}, \ldots, P_{m}$ be distinct rational points of $X$.
Take $s_{1}=s, s_{2}=\cdots=s_{m}=0$, from the above theorem we get that $\left(s r+\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m}\right)$ is a pure gap at $H\left(P_{1}, \ldots, P_{m}\right)$ whenever $1 \leq \alpha_{i} \leq q-i-s(i=1, \ldots, m)$ (i.e. only have pure gaps "between" $(s r+1,1, \ldots, 1)$ and $(s r+q-1-s, q-2-s, \ldots, q-m-s))$.
Let $G=(2 s r+q-1-s) P_{1}+\sum_{i=2}^{m}(q-i-s) P_{i}$ and let $D$ be the sum of the other $q^{3}+1-m$ rational points of $X$. From the work together with $F$. Torres we know that $C_{\Omega}(D, G)$ is an $\left[q^{3}+1-m, k, d\right]$ code with $d \geq 2 s(q+1)+m(2 q-2 s-1)-m^{2}-(q+1)(q-2)$.

## Total inflection points and pure gaps

So $d \geq 2 s(q+1)+m(2 q-2 s-1)-m^{2}-(q+1)(q-2)$
and if we take $s \geq(q-1) / 2$ then $\operatorname{deg}(G)>2 g-2$ and $k=g+q^{3}-2 s(q+1)-m(q-s)+m(m-1) / 2$.
We compared $C_{\Omega}(D, G)$ with codes on the Hermitian curve supported on one point and having the same dimension $k$, finding many situations where $C_{\Omega}(D, G)$ has better parameters.

For example, asssume that $q$ is odd and $q \geq 5$, take $m=s=(q-1) / 2$. Then $C_{\Omega}(D, G)$ is an $\left[q^{3}+1-(q-1) / 2, k, d\right]$-code with $k=q^{3}-(5 q+13)(q-1) / 8$ e $d \geq q^{2} / 4+q+3 / 4$.
Taking $F=\left(q^{3}-\left(q^{2} / 8+3 q / 2-5 / 8\right)\right) P$, where $P$ is a rational point of $X$ and $E$ is the sum of the other rational points, we get that $C_{\Omega}(F, E)$ is an $\left[q^{3}, k, d^{\prime}\right]$ code, where $d^{\prime}=q^{2} / 8+3 q / 2-5 / 8$ (from works by Stichtenoth, Yang and Kummar) so that $d-d^{\prime} \geq(q(q-4)+11) / 8$.

## Total inflection points and pure gaps

So $d \geq 2 s(q+1)+m(2 q-2 s-1)-m^{2}-(q+1)(q-2)$
and if we take $s \geq(q-1) / 2$ then $\operatorname{deg}(G)>2 g-2$ and $k=g+q^{3}-2 s(q+1)-m(q-s)+m(m-1) / 2$.

We compared $C_{\Omega}(D, G)$ with codes on the Hermitian curve supported on one point and having the same dimension $k$, finding many situations where $C_{\Omega}(D, G)$ has better parameters.

For example, asssume that $q$ is odd and $q \geq 5$, take $m=s=(q-1) / 2$. Then $C_{\Omega}(D, G)$ is an $\left[q^{3}+1-(q-1) / 2, k, d\right]$-code with $k=q^{3}-(5 q+13)(q-1) / 8$ e $d \geq q^{2} / 4+q+3 / 4$.
Taking $F=\left(q^{3}-\left(q^{2} / 8+3 q / 2-5 / 8\right)\right) P$, where $P$ is a rational point of $X$ and $E$ is the sum of the other rational points, we get that $C_{\Omega}(F, E)$ is an $\left[q^{3}, k, d^{\prime}\right]$ code, where $d^{\prime}=q^{2} / 8+3 q / 2-5 / 8$ (from works by Stichtenoth, Yang and Kummar) so that $d-d^{\prime} \geq(q(q-4)+11) / 8$.

## Total inflection points and pure gaps

So $d \geq 2 s(q+1)+m(2 q-2 s-1)-m^{2}-(q+1)(q-2)$
and if we take $s \geq(q-1) / 2$ then $\operatorname{deg}(G)>2 g-2$ and $k=g+q^{3}-2 s(q+1)-m(q-s)+m(m-1) / 2$.
We compared $C_{\Omega}(D, G)$ with codes on the Hermitian curve supported on one point and having the same dimension $k$, finding many situations where $C_{\Omega}(D, G)$ has better parameters.

For example, asssume that $q$ is odd and $q \geq 5$, take $m=s=(q-1) / 2$. Then $C_{\Omega}(D, G)$ is an $\left[q^{3}+1-(q-1) / 2, k, d\right]$-code with $k=q^{3}-(5 q+13)(q-1) / 8$ e $d \geq q^{2} / 4+q+3 / 4$.
Taking $F=\left(q^{3}-\left(q^{2} / 8+3 q / 2-5 / 8\right)\right) P$, where $P$ is a rational point of $X$ and $E$ is the sum of the other rational points, we get that $C_{\Omega}(F, E)$ is an $\left[q^{3}, k, d^{\prime}\right]$ code, where $d^{\prime}=q^{2} / 8+3 q / 2-5 / 8$ (from works by Stichtenoth, Yang and Kummar) so that $d-d^{\prime} \geq(q(q-4)+11) / 8$.

## Total inflection points and pure gaps

So $d \geq 2 s(q+1)+m(2 q-2 s-1)-m^{2}-(q+1)(q-2)$
and if we take $s \geq(q-1) / 2$ then $\operatorname{deg}(G)>2 g-2$ and $k=g+q^{3}-2 s(q+1)-m(q-s)+m(m-1) / 2$.
We compared $C_{\Omega}(D, G)$ with codes on the Hermitian curve supported on one point and having the same dimension $k$, finding many situations where $C_{\Omega}(D, G)$ has better parameters.


## Total inflection points and pure gaps

So $d \geq 2 s(q+1)+m(2 q-2 s-1)-m^{2}-(q+1)(q-2)$
and if we take $s \geq(q-1) / 2$ then $\operatorname{deg}(G)>2 g-2$ and $k=g+q^{3}-2 s(q+1)-m(q-s)+m(m-1) / 2$.
We compared $C_{\Omega}(D, G)$ with codes on the Hermitian curve supported on one point and having the same dimension $k$, finding many situations where $C_{\Omega}(D, G)$ has better parameters.

For example, asssume that $q$ is odd and $q \geq 5$,


## Total inflection points and pure gaps

So $d \geq 2 s(q+1)+m(2 q-2 s-1)-m^{2}-(q+1)(q-2)$
and if we take $s \geq(q-1) / 2$ then $\operatorname{deg}(G)>2 g-2$ and $k=g+q^{3}-2 s(q+1)-m(q-s)+m(m-1) / 2$.
We compared $C_{\Omega}(D, G)$ with codes on the Hermitian curve supported on one point and having the same dimension $k$, finding many situations where $C_{\Omega}(D, G)$ has better parameters.

For example, asssume that $q$ is odd and $q \geq 5$, take $m=s=(q-1) / 2$.
Then $C_{\Omega}(D, G)$ is an $\left[q^{3}+1-(q-1) / 2, k, d\right]$-code with $k=q^{3}-(5 q+13)(q-1) / 8$ e $d \geq q^{2} / 4+q+3 / 4$.


## Total inflection points and pure gaps

So $d \geq 2 s(q+1)+m(2 q-2 s-1)-m^{2}-(q+1)(q-2)$
and if we take $s \geq(q-1) / 2$ then $\operatorname{deg}(G)>2 g-2$ and $k=g+q^{3}-2 s(q+1)-m(q-s)+m(m-1) / 2$.
We compared $C_{\Omega}(D, G)$ with codes on the Hermitian curve supported on one point and having the same dimension $k$, finding many situations where $C_{\Omega}(D, G)$ has better parameters.

For example, asssume that $q$ is odd and $q \geq 5$, take $m=s=(q-1) / 2$.
Then $C_{\Omega}(D, G)$ is an $\left[q^{3}+1-(q-1) / 2, k, d\right]$-code with $k=q^{3}-(5 q+13)(q-1) / 8$ e $d \geq q^{2} / 4+q+3 / 4$.
Taking $F=\left(q^{3}-\left(q^{2} / 8+3 q / 2-5 / 8\right)\right) P$, where $P$ is a rational point of $X$ and $E$ is the sum of the other rational points,

Stichtenoth, Yang and Kummar) so that $d-d^{\prime} \geq(q(q-4)+11) / 8$.

## Total inflection points and pure gaps

So $d \geq 2 s(q+1)+m(2 q-2 s-1)-m^{2}-(q+1)(q-2)$
and if we take $s \geq(q-1) / 2$ then $\operatorname{deg}(G)>2 g-2$ and $k=g+q^{3}-2 s(q+1)-m(q-s)+m(m-1) / 2$.
We compared $C_{\Omega}(D, G)$ with codes on the Hermitian curve supported on one point and having the same dimension $k$, finding many situations where $C_{\Omega}(D, G)$ has better parameters.

For example, asssume that $q$ is odd and $q \geq 5$, take $m=s=(q-1) / 2$.
Then $C_{\Omega}(D, G)$ is an $\left[q^{3}+1-(q-1) / 2, k, d\right]$-code with $k=q^{3}-(5 q+13)(q-1) / 8$ e $d \geq q^{2} / 4+q+3 / 4$.
Taking $F=\left(q^{3}-\left(q^{2} / 8+3 q / 2-5 / 8\right)\right) P$, where $P$ is a rational point of $X$ and $E$ is the sum of the other rational points, we get that $C_{\Omega}(F, E)$ is an $\left[q^{3}, k, d^{\prime}\right]$ code, where $d^{\prime}=q^{2} / 8+3 q / 2-5 / 8$ (from works by Stichtenoth, Yang and Kummar)

## Total inflection points and pure gaps

So $d \geq 2 s(q+1)+m(2 q-2 s-1)-m^{2}-(q+1)(q-2)$
and if we take $s \geq(q-1) / 2$ then $\operatorname{deg}(G)>2 g-2$ and $k=g+q^{3}-2 s(q+1)-m(q-s)+m(m-1) / 2$.
We compared $C_{\Omega}(D, G)$ with codes on the Hermitian curve supported on one point and having the same dimension $k$, finding many situations where $C_{\Omega}(D, G)$ has better parameters.

For example, asssume that $q$ is odd and $q \geq 5$, take $m=s=(q-1) / 2$.
Then $C_{\Omega}(D, G)$ is an $\left[q^{3}+1-(q-1) / 2, k, d\right]$-code with $k=q^{3}-(5 q+13)(q-1) / 8$ e $d \geq q^{2} / 4+q+3 / 4$.
Taking $F=\left(q^{3}-\left(q^{2} / 8+3 q / 2-5 / 8\right)\right) P$, where $P$ is a rational point of $X$ and $E$ is the sum of the other rational points, we get that $C_{\Omega}(F, E)$ is an $\left[q^{3}, k, d^{\prime}\right]$ code, where $d^{\prime}=q^{2} / 8+3 q / 2-5 / 8$ (from works by Stichtenoth, Yang and Kummar) so that $d-d^{\prime} \geq(q(q-4)+11) / 8$.

## THANK YOU!


[^0]:    9 gaps if $P_{1}$ and $P_{2}$ are not $W$. points of $X$

[^1]:    9 gaps if $P_{1}$ and $P_{2}$ are not $W$. points of $X$

[^2]:    9 gaps if $P_{1}$ and $P_{2}$ are not $W$. points of $X$

