A higher order Multiscale Finite Element Method for num. homog.

Let $\Omega \subset \mathbb{R}^n$ open, bounded, $f \in H^{-1}(\Omega)$.

$$(1) \quad \begin{cases} -\nabla \cdot (a^\varepsilon(x) \nabla u^\varepsilon) = f & \text{in } \Omega \\ \qquad\qquad u^\varepsilon = 0 & \text{on } \partial\Omega \end{cases}$$

$a^\varepsilon(x) = a(\frac{x}{\varepsilon})$, $a \in L^\infty(Y)$ with $Y = (0,1)^n$; a positive, Y-perio

We know: For ε small, u^ε is approximated by

$$u^\varepsilon(x) \approx u^*(x) + \varepsilon \sum_{i=1}^n \chi_i(\tfrac{x}{\varepsilon}) \frac{\partial u^*}{\partial x_i}(x)$$

and $\nabla u^\varepsilon(x) \approx \nabla u^*(x) + \sum_{i=1}^n (\nabla_y \chi_i)(\frac{x}{\varepsilon}) \frac{\partial u^*}{\partial x_i}(x)$

where χ_i is the solution of the cell problem

$$\begin{cases} -\nabla_y \cdot (a(y)(e_i + \nabla_y \chi_i)) = 0 & \text{in } Y \\ \qquad y \to \chi_i(y) & Y\text{-periodic} \end{cases}$$

and u^* solution of the homogenized problem

$$\begin{cases} -\nabla \cdot (A^* \nabla u^*) = f & \text{in } \Omega \\ \qquad u^* = 0 & \text{on } \partial\Omega \end{cases}$$

where $A^* e_i = \int_Y a(y)(e_i + \nabla_y \chi_i)\, dy$

Idea: Approximation of u^ε of the form (1st. order Taylor expansion)

$$u^\varepsilon(x) \approx u^*(x + \varepsilon \chi(\tfrac{x}{\varepsilon})), \quad \chi = (\chi_1, \dots, \chi_n)^T$$

If $\{\phi_i^h\}_i$ classical FE-basis on coarse mesh of mesh size $h > \varepsilon$, introduce oscillating basis (MsFEM) fct. by (composition rule)

$$\phi_i^{\varepsilon,h}(x) = \phi_i^h(x + \varepsilon \chi(\tfrac{x}{\varepsilon}))$$

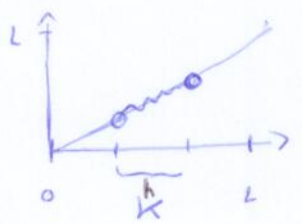
. change of variables: $x \to \tilde{\omega}^\varepsilon(x) := x + \varepsilon \chi(\frac{x}{\varepsilon})$. (standard Galerkin Procedure to variational form. of (1) will

2) Definition of the MsFEM

Let $\mathcal{T}_h$ be a (coarse) conformal triangulation of Ω.
For $i = 1, \dots, n$ define $\hat{\omega}_i^{\varepsilon,h} \in H^1(\Omega)$ locally for each $K \in \mathcal{T}_h$ by

$$(2) \begin{cases} -\nabla \cdot (a^\varepsilon \nabla \hat{\omega}_{i|K}^{\varepsilon,h}) = -\nabla \cdot (A_K^* \nabla x_i) & \text{in } K \\ \hat{\omega}_{i|K}^{\varepsilon,h} = x_i & \text{on } \partial K \end{cases}$$

and $\hat{\omega}^{\varepsilon,h} = (\hat{\omega}_1^{\varepsilon,h}, \dots, \hat{\omega}_n^{\varepsilon,h})^T \in H^1(\Omega, \mathbb{R}^n)$, A_K^* local average of a^ε



(oscillating test-f

Remark: In the following, $\hat{\omega}_i^{\varepsilon,h}$ denotes the exact solution of (2), while $\omega_i^{\varepsilon,h}$ denotes the approximated solution ($\mathbb{P}_{k'}$-Lagrange-FE) on a fine mesh $\mathcal{T}_{h'}^K$ of mesh size h'

$$\boxed{0 < h' < \varepsilon < h < 1}$$

$\mathbb{P}_{k'}$ $\qquad$ $\mathbb{P}_k$

MsFEM - Basis:

Let $V_h \subset H_0^1(\Omega)$ be the space of $\mathbb{P}_k$ - Lagrange FEs on $\mathcal{T}_h$ with basis $\{\phi_l^h\}_{l=1}^{N_h}$. We define the MsFEM basis by

$$\phi_l^{\varepsilon,h}(x) = \phi_l^h \circ \omega^{\varepsilon,h}(x) \qquad l = 1, \dots, N_h$$

Thus, $V_h^\varepsilon := \operatorname{span}\{\phi_l^{\varepsilon,h}\}_{l=1}^{N_h}$ is conforming $\left({}_{"}V_h^\varepsilon = V_h \circ \omega^{\varepsilon,h}{}^{"} \right)$

$$(3) \begin{cases} \text{Find } u_h \circ \omega^{\varepsilon,h} \in V_h^\varepsilon : \\ \quad a^\varepsilon(u_h \circ \omega^{\varepsilon,h}, v_h \circ \omega^{\varepsilon,h}) = \int_\Omega f\, v_h \circ \omega^{\varepsilon,h}\, dx \quad \forall\, v_h \circ \omega^{\varepsilon,h} \in V_h^\varepsilon \end{cases}$$

Note that $V^\varepsilon_h = H_0^1 \circ \hat{\omega}^{\varepsilon,h} \subset H_0^1$ if $\hat{\omega}^{\varepsilon,h} \in W^{1,\infty}(\Omega, \mathbb{R}^M)$ (+Lax-Milgram, unique solution)

Example: $\mathbb{P}_1$ - FEs on $\mathcal{T}_h$ $(k=1)$

p_i basis fct. of V_h

$$\Rightarrow \phi_i^{\varepsilon,h}(x) = p_i \circ \omega^{\varepsilon,h}(x)$$

$$\overset{\text{linear}}{=} p_i(x) + \sum_{j=1}^{n} \left(\omega_j^{\varepsilon,h}(x) - x_j \right) \frac{\partial p_i}{\partial x_j}(x)$$

$$\Rightarrow \nabla \cdot \left(a^\varepsilon \nabla \phi_i^{\varepsilon,h} \right) = \sum_{j=1}^{n} \nabla \cdot \left(a^\varepsilon \nabla \omega_j^{\varepsilon,h} \right) \frac{\partial p_i}{\partial x_j} \text{ in } K \in \mathcal{T}_h$$

choose $A_K^* = \text{const}$ in (2),

then we have for $K \in \mathcal{T}_h$

$$\begin{cases} \nabla \cdot \left(a^\varepsilon \nabla \phi_i^{\varepsilon,h} \right) = 0 & \text{in } K \\ \phi_i^{\varepsilon,h} = p_i & \text{on } \partial K \end{cases}$$

$\leadsto$ coincides with the „standard" MsFEM (Hou, Wu)

Remark: For $k \geq 1$, let n_j, $j = 1, \dots, N_h$ be the nodes associated to the $\mathbb{P}_k$ - FEM on $\mathcal{T}_h$, $\{\phi_i^h\}_{i=1}^{N_h}$ Lagrange - basis of $V_h = \mathbb{P}_k$

It holds

$$\phi_i^h(n_j) = \delta_{ij}$$

and $\quad \phi_i^{\varepsilon,h}(n_j) = \delta_{ij} \quad$ since $\omega^{\varepsilon,h}(x) = x$ on ∂K, $K \in \mathcal{T}_h$

Definition: For $v \in V_h = \mathbb{P}_k$ define the interpolation operators

$$\Pi_h v(x) := \sum_{l=1}^{N_h} v(n_l)\, \phi_l^h(x)$$

$$\Pi_h^\varepsilon v(x) := \sum_{l=1}^{N_h} v(n_l)\, \phi_l^{\varepsilon,h}(x) \qquad \left(= (\Pi_h v) \circ \omega^{\varepsilon,h}(x) \right)$$

~~4~~ 3) Convergence

Lemma 1:

Let $\omega \subset \mathbb{R}^n$ be open, bounded (smooth), $f \in L^2(\omega)$, $g \in H^1(\omega)$.
Let v^ε be the solution of (1) in ω, satisfying $v^\varepsilon = g$ on $\partial\omega$.
Let v^* be the homogenized limit and $\chi_i \in W^{1,\infty}$ the sol. of the cell problem.
Then there exists $C > 0$, indep. of $\varepsilon, \omega, f, g$:

$$\| v^\varepsilon - v^* - \varepsilon \sum_{i=1}^{n} \chi_i(\tfrac{x}{\varepsilon}) \frac{\partial v^*}{\partial x_i}(x) \|_{H^1(\omega)} \leq C \sqrt{\varepsilon} \sqrt{|\partial\omega|} \, \| v^* \|_{W^{2,\infty}(\omega)}$$

Lemma 2:

Let $\tilde{\omega}^\varepsilon(x) = x + \varepsilon \chi(\tfrac{x}{\varepsilon}) \in W^{1,\infty}(\Omega)$, $\chi_i \in W^{1,\infty}(\cancel{Y})$, $u^* \in W^{2,\infty}(\Omega)$
Then there exists $C \neq C(\varepsilon)$:

$$\| u^\varepsilon - u^* \circ \tilde{\omega}^\varepsilon \|_{H_0^1(\Omega)} \leq C\sqrt{\varepsilon}$$

Scetch of the proof:

$$\| \nabla u^\varepsilon - \nabla (u^* \circ \tilde{\omega}^\varepsilon) \|_{L^2(\Omega)^n}$$

$$\leq \| \nabla u^\varepsilon - (\nabla \tilde{\omega}^\varepsilon) \nabla u^* \|_{L^2(\Omega)^n} \qquad \leq C\sqrt{\varepsilon} \text{ (Lemma 1}$$

$$+ \| (\nabla \tilde{\omega}^\varepsilon)(\nabla u^* - (\nabla u^*) \circ \tilde{\omega}^\varepsilon \|_{L^2(\Omega)^n} \qquad \leq C\varepsilon \text{ (Taylor exp.}$$

<u>Theorem:</u> Let u^ε be the exact solution of (1) and

$u_h^\varepsilon = u_h \circ \omega^{\varepsilon,h}$ be the numerical solution of (3).

Assume $u^* \in W^{k+1,\infty}(\Omega)$, $\chi_i \in W^{1,\infty}(\Omega)$, $\hat{\omega}^{\varepsilon,h} \in W^{1,\infty}(\Omega)$

Then there is a constant C indep. of ε and h such that

$$\| u^\varepsilon - u_h^\varepsilon \|_{H_0^1(\Omega)} \leq C \left(h^k + \sqrt{\frac{\varepsilon}{h}} + \left(\frac{h'}{\varepsilon}\right)^{k'} \right)$$

$k=1: \quad (\varepsilon + h + \sqrt{\frac{\varepsilon}{h}} + \dots)$

<u>Proof:</u> From Céa's Lemma, $\exists\, C$ indep. of ε, h s.t.

$$\| u^\varepsilon - u_h^\varepsilon \|_{H_0^1(\Omega)} \leq C \inf_{v_h^\varepsilon \in V_h^\varepsilon} \| u^\varepsilon - v_h^\varepsilon \|_{H_0^1(\Omega)}$$

Let $v_h^\varepsilon = \Pi_h^\varepsilon u^*$, we have

$$\| u^\varepsilon - u_h^\varepsilon \|_{H_0^1(\Omega)} \leq \| u^\varepsilon - \Pi_h^\varepsilon u^* \|_{H_0^1(\Omega)}$$

Define $\tilde{\omega}^\varepsilon(x) := x + \varepsilon \chi(\frac{x}{\varepsilon})$, (rescaled sol. of the cell problem)

We get

$$\| \nabla u^\varepsilon - \nabla(\Pi_h^\varepsilon u^*) \|_{L^2(\Omega)^n}$$

$$\leq C \| \nabla u^\varepsilon - \nabla(u^* \circ \tilde{\omega}^\varepsilon) \|_{L^2(\Omega)^n}$$ (corrector result in periodic. homog.

$$+ \| \nabla(u^* - \Pi_h u^*) \circ \tilde{\omega}^\varepsilon \|_{L^2(\Omega)^n}$$ interpolation result for the course mesh

$$+ \| \nabla(\Pi_h u^* \circ (\tilde{\omega}^\varepsilon - \hat{\omega}^{\varepsilon,h}) \|_{L^2(\Omega)^n}$$ homog. result of th local oscillating test

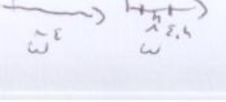
$$+ \| \nabla(\Pi_h u^* \circ (\hat{\omega}^{\varepsilon,h} - \omega^{\varepsilon,h}) \|_{L^2(\Omega)^n}$$ error estimate for th $\mathbb{P}_{k'}$-FEM used to com the local oscillating test

Recapitulate:

$\tilde{\omega}^\varepsilon$,

$\hat{\omega}^{\varepsilon,h}$

$\omega^{\varepsilon,h}$



(i) $\| \nabla u^{\varepsilon} - \nabla u^{*} \circ \tilde{\omega}^{\varepsilon} \|_{L^2(\Omega)^n} \le C\sqrt{\varepsilon}$ by Lemma 2

(ii) $\|(\nabla \tilde{\omega}^{\varepsilon})(\nabla(u^* - \Pi_h u^*) \circ \tilde{\omega}^{\varepsilon}\|_{L^2(\Omega)^n}$

$$\le \|Id + \nabla_y \chi\|_{L^\infty(Y)} \, \|\nabla(u^* - \Pi_h u^*) \circ \tilde{\omega}^{\varepsilon}\|_{L^2(\Omega)^n}$$

We have

$$\nabla(u^* - \Pi_h u^*) \circ \tilde{\omega}^{\varepsilon} \overset{\text{Taylor Exp.}}{=} \nabla(u^* - \Pi_h u^*)(x) + \varepsilon \int_0^1 \nabla\nabla(u^* - \Pi_h u^*)(x + \varepsilon t \chi) \cdot \chi \, dt$$

$\Rightarrow$ (ii) $\le \|Id + \nabla_y \chi\|_{L^\infty(Y)} \left(\|u^* - \Pi_h u^*\|_{H^1(\Omega)} + \varepsilon \|u^* - \Pi_h u^*\|_{W^{2,\infty}(\Omega)} \|\chi\|_{L}\right.$

interp. result for Lagrange Interp. $\le \|Id + \nabla_y \chi\|_{L^\infty(Y)} \, \|u^*\|_{W^{k+1,\infty}(\Omega)} \left(h^k + \varepsilon h^{k-1} \|\chi\|_{L^2(Y)} \right)$

(iii) $\|(\nabla \tilde{\omega}^{\varepsilon} - \nabla \hat{\omega}^{\varepsilon,h})\left(\nabla(\Pi_h u^*) \circ (\tilde{\omega}^{\varepsilon} - \hat{\omega}^{\varepsilon,h})\right)\|_{L^2(\Omega)^n}$

$$\le \|u^*\|_{W^{1,\infty}(\Omega)} \, \|\nabla(\tilde{\omega}^{\varepsilon} - \hat{\omega}^{\varepsilon,h})\|_{L^2(\Omega)^n}$$

It holds

$$\|\nabla(\tilde{\omega}^{\varepsilon} - \hat{\omega}^{\varepsilon,h})\|^2_{L^2(\Omega)^n} = \sum_{K \in \mathcal{T}_h} \|\nabla(\hat{\omega}^{\varepsilon} - \hat{\omega}^{\varepsilon,h})\|^2_{L^2(\Omega)}$$

$$\overset{\text{Lemma 1}}{\le} C \sum_{K \in \mathcal{T}_h} \varepsilon h^{n-1} \overset{\#K = h^{-n}}{=} C \varepsilon \frac{h^{n-1}}{h^n} = C \frac{\varepsilon}{h}$$

(iv) $\|\nabla(\hat{\omega}^{\varepsilon,h} - \omega^{\varepsilon,h})(\nabla \Pi_h u^*) \circ (\hat{\omega}^{\varepsilon,h} - \omega^{\varepsilon,h})\|_{L^2(\Omega)^n}$

$$\le \|u^*\|_{W^{1,\infty}(\Omega)} \, \|\nabla \hat{\omega}^{\varepsilon,h} - \omega^{\varepsilon,h}\|_{L^2(\Omega)^n}$$

We have $\|\nabla(\hat{\omega}^{\varepsilon,h} - \omega^{\varepsilon,h})\|^2_{L^2(\Omega)^n} = \sum_{K \in \mathcal{T}_h} \|\nabla(\hat{\omega}^{\varepsilon,h} - \omega^{\varepsilon,h})\|^2_{L^2(K)^n}$

$$\overset{P_{k'}\text{-interp.}}{\le} C \sum_{K \in \mathcal{T}_h} (h')^{2k'} \underbrace{|\hat{\omega}^{\varepsilon,h}|^2_{H^{k'+1}(K)}}_{\le (C\varepsilon^{-k'}\sqrt{|K|})^2} \qquad \rightsquigarrow \text{(iv)} \le C \varepsilon^{-k'} h^{k'} \frac{h^{\frac{n}{2}}}{h^{\frac{n}{2}}}$$

$\le$

$$\overset{\text{(i)-(iv)}}{\Longrightarrow} \|\nabla u^{\varepsilon} - \nabla(\Pi_h^{\varepsilon} u^*)\| \le C \left(\sqrt{\varepsilon} + (h^k + \varepsilon h^{k-1}) + \sqrt{\frac{\varepsilon}{h}} + \left(\frac{h'}{\varepsilon}\right)^{k'} \right) \square$$