GENERALIZED LINEAR DYNAMIC FACTOR MODELS - A STRUCTURE THEORY

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Modeling of high dimensional time series

Multivariate time series

\[ y_t \in \mathbb{R}^N, t = 1, \ldots, T \]

Sample size \( T \), cross-sectional dimension \( N \)

- "Traditional" approach, e.g. "unstructured" AR modeling:
  "Curse of dimensionality": Dimension of parameter space \( N^2p \) (\( p \): maximal lag). Number of data points \( NT \).

Alternatives:

- Factor models
  - Comovement allows for dimension reduction in the cross-sectional dimension.
Modeling of high dimensional time series

- Time series factor models: Complexity reduction in time and cross-section. Under certain assumptions the dimension of the parameter space is linear in $N$.

Note, there is no symmetry in the time and the cross-sectional dimension: Stationarity in time, “similarity” or “comovement” of time series; permutation invariant.

- Cointegration
- Panel-time series models
- Structural models; e.g. ARX models which are sparse due to “physical“ a priori knowledge.
- “Graphical“ time series models, where the inverse of the spectral density is sparse.
Applications

History:
Psychometrics: Intelligence factors (Burt 1909, Thurstone 1934)

Great range of applications: Signal processing, marketing econometrics, finance econometrics, ...

Recent applications for generalized factor models:
- Forecasting for macrovariables
- Structure and insights for macroeconomics
- Cross-country studies
- Finance
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**GDFM’s - The Model Class**

Generalized linear dynamic factor models (GDFM’s)

- Generalization of:
  - Generalized static linear factor models (Chamberlain, Chamberlain and Rothschild, Econometrica 1983)

- Main features:
  - Dynamics (here in a stationary context)
  - Uncorrelatedness of noise components in cross-section is replaced by weak dependence (risk diversification is possible)
  - Similarity: Information gain by adding additional time series
GDFM’s - The Model Class

Main references for GDFM’s

- Forni, Hallin, Lippi, Reichlin, RES, 2000
- Forni and Lippi, Econometric Theory, 2001
- Forni, Hallin, Lippi, Reichlin, JASA, 2005
- Stock and Watson, JASA, 2002
GDFM’s - The Model Class

\[ y_t^N = \hat{y}_t^N + u_t^N \]

- \( y_t^N \) ... observations
- \( \hat{y}_t^N \) ... latent variables, strongly dependent in the cross-sectional dimension
- \( u_t^N \) ... (wide sense) idiosyncratic noise, weakly dependent

Assumptions:

1. \((\hat{y}_t^N), (u_t^N)\) wide sense stationary with absolutely summable covariances
2. \(\mathbb{E}\hat{y}_t^N u_s^{N'} = 0\)
3. \(\mathbb{E}\hat{y}_t^N = \mathbb{E}u_t^N = 0\)
GDFM’s - The Model Class

Spectral densities:

\[ f_y^N(\lambda) = f_{\hat{y}}^N(\lambda) + f_u^N(\lambda) \]

Asymptotic analysis: \( T \to \infty, N \to \infty \) Sequence of GDFM’s; Nested i.e. elements of \( \hat{y}_t^N \) and \( u_t^N \) do not depend on N
Assumptions

Additional assumptions to separate the latent variables from the noise for \( N \to \infty \):

A1 \( f_{\hat{y}}^N \) is a rational spectral density with constant rank \( q < N \), and of McMillan degree \( 2n < N \); \( q \) and \( n \) do not depend on \( N \)

A2 Weak dependence of \((u_t^N)\): The largest eigenvalue of \( f_u^N \) is uniformly bounded for all frequencies \( \lambda \) and all \( N \)

A3 Strong dependence of \((\hat{y}_t^N)\): The first \( q \) eigenvalues of \( f_{\hat{y}}^N \) diverge to infinity for all frequencies \( \lambda \), as \( N \to \infty \)
Identifiability

GDFM’s are identifiable only for $N \to \infty$:

- The elements of $f_{\hat{y}}^N$ and $\hat{y}_t^N$ are uniquely determined from $y_t^N$ for $N \to \infty$

- Asymptotic equivalence to dynamic PCA
Dynamic PCA (Brillinger 1981)

We want to approximate $f_y(\lambda)$ by a spectral density $\hat{f_y}(\lambda)$ of rank $q$ for all $\lambda$ s.t. $E u_t' u_t$ is minimal

$$f_y(\lambda) = O_1(e^{-i\lambda}) \Omega_1(\lambda) O_1(e^{-i\lambda})^* + O_2(e^{-i\lambda}) \Omega_2(\lambda) O_2(e^{-i\lambda})^*$$

Model:

$$y_t = O_1(z) O_1^*(z) y_t + O_2(z) O_2^*(z) y_t$$

Note

- Here dimension reduction is in cross-section only; even for rational $f_y$, $\hat{f_y}$ may be non-rational
- $O_1(z) O_1^*(z)$ may be non-causal
- Estimation commences from a nonparametric estimator of $f_y$. 
Characterization of GDFM’s

$y_t^N$ follows a generalized dynamic factor model if and only if

- the first $q$ eigenvalues of $f_y^N$ diverge to infinity for all frequencies $\lambda$, as $N \to \infty$

- the $q + 1 - th$ eigenvalue of $f_y^N$ is uniformly bounded for all frequencies $\lambda$ and all $N$
Aims of our Analysis

- Structural insight
- Obtain a state space or an ARMA model and in particular an AR model for \((\hat{y}_t)\) from the second moments of \((y_t)\)
  - estimation of the integer valued parameters such as \(q\) (dimension of dynamic factors), \(r\) (dimension of static factors) and \(n\) (state dimension)
  - estimation of the real valued parameters such as \((F, G, H)\)
- forecasting of \(y_t\) based on forecasts of \(\hat{y}_t\) and eventually of \(u_t\)
The General Framework

Major Steps

- Factorization of $f_{ij}$
- Realization of a “tall“ spectral factor by a state space or an ARMA model
- Emphasise the zeroless case
- Averaging out of (weakly) idiosyncratic noise
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Theorem

(i) Every rational spectral density $f_{\hat{y}}$ of constant rank $q$ can be factorized as in (1) where

$$w(z) = \sum_{j=0}^{\infty} w_j z^j; \quad w_j \in \mathbb{R}^{N \times q}$$

is rational, analytic in $|z| \leq 1$ and has rank $q$ for all $|z| \leq 1$.

(ii) For given $f_{\hat{y}}$, $w$ is unique up to postmultiplication by constant orthogonal matrices.
Wold decomposition

(Smith Mc Millan form)

\[ w = ulv \]

\( u, v \) ... unimodular, polynomial
\( \ell \) ... diagonal, diagonal elements display poles and zeros of \( w \)

\( w \) corresponds to the Wold decomposition:
There exists \((\varepsilon_t), \) white noise, with \( \mathbb{E}\varepsilon_t\varepsilon_t' = 2\pi I, \) s.t.:

\[ \hat{y}_t = w(z)\varepsilon_t = \sum_{j=0}^{\infty} w_j \varepsilon_{t-j} \]

\[ \varepsilon_t = w^-(z)\hat{y}_t \]

causal left inverse \( w^- = v^{-1}(\ell'\ell)^{-1}\ell'u^{-1} \)
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Realization

Realization: Find a system for a transfer function

- State space system - (F, G, H)
- ARMA system
  \[ a(z)\hat{y}_t = b(z)\varepsilon_t \]
  \[ w(z) = a(z)^{-1}b(z) \]
- Right MFD (Lippi)
  \[ w(z) = c(z)d(z)^{-1} \]
ARMA Realizations

\[ a(z) \hat{y}_t = b(z) \varepsilon_t \]

\((a, b)\) left coprime

**Stability:** \(\text{det} \ a(z) \neq 0, |z| \leq 1\)

**Miniphase condition:** \(b(z)\) has full rank, \(|z| \leq 1\)
State space Realizations

\[ x_{t+1} = Fx_t + G\varepsilon_{t+1} \]
\[ \hat{y}_t = Hx_t \]

\( F \in \mathbb{R}^{n \times n}, G \in \mathbb{R}^{n \times q}, H \in \mathbb{R}^{N \times n} \)

\( x_t \) ... state

\( w(z) = H(I - Fz)^{-1}G \)

\( (F, G, H) \) is minimal (i.e. controllable and observable)

Stability: \( |\lambda_{max}(F)| < 1 \)

Miniphase condition:

\[ M(z) = \begin{pmatrix} I - Fz & -G \\ H & 0 \end{pmatrix} \text{ has rank } n + q, |z| \leq 1 \]
(F, G, H) State space systems

The state is unique up to basis changes:

$$\bar{F} = TFT^{-1}, \bar{G} = TG, \bar{H} = HT^{-1}, \det T \neq 0$$

Note that here $x_t$ is a static factor but not necessarily a minimal one. $x_t$ is a minimal static factor if and only if \( rk(H) = n \) holds.
Realization: Kalman-Akaike procedure

\[
\begin{pmatrix}
\hat{y}_t \\
\hat{y}_{t+1|t} \\
\hat{y}_{t+2|t} \\
\vdots \\
\hat{y}_t
\end{pmatrix}
= \begin{pmatrix}
HG & HFG & HF^2G & \ldots \\
HFG & HF^2G & HF^3G & \ldots \\
\vdots & \vdots & \vdots & \ddots \\
E_t & E_{t-1} & E_{t-2} & \ldots
\end{pmatrix}

\mathcal{H} \text{ Hankelmatrix of the transfer function}

\hat{y}_{t+r|t} \ldots \text{ best linear least squares predictor of } \hat{y}_{t+r} \text{ from the infinite past } \hat{y}_t, \hat{y}_{t-1}, \ldots
\[ X_t = \begin{pmatrix} \hat{y}_t \\ \hat{y}_{t+1|t} \\ \hat{y}_{t+2|t} \\ \vdots \end{pmatrix} = S \mathcal{H} E_t^- \]

\[ = S \begin{pmatrix} HFG & HF^2G & \ldots \\ HF^2G & HF^3G & \ldots \\ \vdots & \vdots & \ddots \end{pmatrix} E_{t-1}^- + S \begin{pmatrix} HG \\ HF^2G \\ \vdots \end{pmatrix} \varepsilon_t \]

\[ \text{with} \]

\[ H S \mathcal{H} = \begin{pmatrix} HG & HFG & \ldots \end{pmatrix} \]

S ... Selector matrix
Special choice for S: Select the first basis rows of \( \mathcal{H} \): Echelon form, selection described by Kronecker indices
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Latent Variables and Minimal Static Factors

\[ \hat{y}_t = H T T^{-1} x_t = (H_1, 0) T^{-1} x_t = H_1 z_t \]

\( z_t \) ... \( r \) dimensional minimal static factor, \( rk \ H_1 = r \)

In general \( n \geq r \geq q \);

\( q \) is the dimension of minimal dynamic factors

Note: Minimal static factors are unique up to premultiplication by constant nonsingular matrices
Static factors are obtained from $\Sigma\hat{y} = E\hat{y}_t\hat{y}_t'$:

$$\Sigma\hat{y} = MM', \; M = H_1R, \; M \in \mathbb{R}^{N \times r}, \; \text{rk} \; M = r$$

as:

$$z_t = (M'M)^{-1}M'\hat{y}_t.$$ 

Static factors $z_t$ and latent variables $\hat{y}_t$ are related by a linear static relation and thus have the same dynamics

$$z_t = (M'M)^{-1}M'w(z)\varepsilon_t = k(z)\varepsilon_t$$

$z_t$ has smaller dimension, thus we prefer to model $(z_t)$

Echelon case:

$$S = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}, \; S_1 \in \mathbb{R}^{r \times \infty}, \; S_2 \in \mathbb{R}^{(n-r) \times \infty}$$

$$x_t = S\hat{Y}_t, \; \; z_t = S_1\hat{Y}_t$$
Clearly \((z_t)\) has a rational spectral density.

State space model for \((z_t)\): \((F, G, C)\) where

\[
C = (M' M)^{-1} M' H
\]

Identification of an ARMA model for \((z_t)\), (Zinner, PhD-thesis TU Wien)
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Zeroless Transfer Functions and (Singular) AR Systems

A transfer function \( w(z) \) is called zeroless if all numerator polynomials in the diagonal of the matrix \( \ell \) in the Smith-Mc-Millan form are equal to 1.

Note: \( k(z) \) is zeroless if and only if \( w(z) \) is zeroless.

Theorem (Anderson and Deistler, SICE J. Control 2008)

Consider a rational transfer function \( w(z) \) with minimal state space realization \((F, G, H)\) with state dimension \( n \). If \( N > q \) holds, then the transfer functions are zeroless for generic values of \((F, G, H)\).
Theorem (Anderson and Deistler, CDC, 2008)

The following statements are equivalent:

(i) The stable miniphase spectral factors $k$ of the spectral density $f_z$ of $(z_t)$ are zeroless.

(ii) There exists a polynomial left inverse $k^-$ for $k$.

(iii) $(z_t)$ is a stable AR-process, i.e.

$$z_t = e_1 z_{t-1} + \cdots + e_p z_{t-p} + \nu_t$$

where $\det \left( I - e_1 z - \cdots - e_p z^p \right) \neq 0, \ |z| \leq 1$

and $\text{rk} \Sigma_{\nu} = q, \Sigma_{\nu} = E \nu_t \nu_t'$.
Let

\[
\Gamma_m = \begin{pmatrix}
\gamma_0 & \cdots & \cdots & \gamma_{m-1} \\
\vdots & \gamma_0 & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{m-1} & \cdots & \cdots & \gamma_0
\end{pmatrix}
\]

where \( \gamma_j = E z_{t+j} z'_t \)

If \( \Sigma_\nu \) is nonsingular, then \( \Gamma_m \) is nonsingular for all \( m \).

If \( \Sigma_\nu \) is singular, then \( \Gamma_{p+1} \) will be singular and \( \Gamma_p \) may be singular.

Yule Walker Equations:

\[
(e_1, \ldots, e_p) \Gamma_p = (\gamma_1, \ldots, \gamma_p)
\]

\[
\Sigma_\nu = \gamma_0 - (e_1, \ldots, e_p)(\gamma_1', \ldots, \gamma_p')'
\]
Zeroless Transfer Functions and (Singular) AR Systems

Solution of the Yule Walker equations may not be unique:
Description of the class of all observationally equivalent AR systems for given $p$

Theorem (Anderson and Deistler, CDC, 2008)

(i) Every singular AR system with $\text{rk} \Sigma_{\nu} = q$ can be written as

$$e(z)z_t = f \varepsilon_t, f \in \mathbb{R}^{r \times q}$$

where $(\varepsilon_t)$ is white noise with $E\varepsilon_t\varepsilon'_t = I_q$ and where $e(z)$ and $f$ are relatively left prime.
Theorem (Anderson and Deistler, CDC, 2008)

(ii) Let \((e(z), f)\) be relatively left prime, then the class of all observationally equivalent \((\bar{e}(z), \bar{f})\) satisfying the degree restrictions \(\delta(\bar{e}(z)) \leq p, \delta(\bar{f}) = 0\) is given by

\[ (\bar{e}(z), \bar{f}) = u(z)(e(z), f) \]

where the polynomial matrix \(u(z)\) satisfies

\[ \det u(z) \neq 0, |z| \leq 1 \]
\[ u(0) = I \]
\[ \delta(u(z)e(z)) \leq p \]
\[ \delta(u(z)f) = 0 \]
Theorem (Anderson and Deistler, CDC, 2008)

(iii) $e(z)$ is unique if and only if $\text{rk}(e_p, f) = r$ holds.
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Let $\hat{\gamma}_j^T = 1/T \sum_{t=1}^{T} \hat{y}_{t+j} \hat{y}_t$ and let

$$(\hat{e}_1, \ldots, \hat{e}_p) \hat{\Gamma}_p = (\hat{\gamma}_1, \ldots, \hat{\gamma}_p), \hat{\Gamma}_p = \begin{pmatrix} \hat{\gamma}_0 & \cdots & \cdots & \hat{\gamma}_{p-1} \\ \vdots & \hat{\gamma}_0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\gamma}_{p-1} & \cdots & \cdots & \hat{\gamma}_0 \end{pmatrix}$$

be the corresponding Yule Walker equations.

Typically $\hat{\Gamma}_p$ will be nonsingular, even if $\Gamma_p$ is singular, however, truncation is appropriate in such a case

$$(\hat{e}_1, \ldots \hat{e}_p) = (\hat{\gamma}_1^T, \ldots, \hat{\gamma}_p^T) O_T \Lambda_T^{-1} O_T'$$

where

$$\hat{\Gamma}_p^T = O_T \Lambda_T O_T', \ O_T \in \mathbb{R}^{pr \times s}, \ \Lambda_T \in \mathbb{R}^{s \times s}, \ s = \text{rk} \Gamma_p$$
The Yule Walker Equations

Theorem (Anderson, Deistler, Filler and Zinner, ECC, 2009)

(i) If $\text{rk} \Gamma_p = pr$ holds, then the YW estimators correspond to a stable autoregression

(ii) For $\text{rk} \Gamma_p = s < pr$, the truncation procedure above yields a stable autoregression
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Estimation of Integers

Work in progress:

Estimation of $r, q, p$ and $s$ for the AR case, or $r, n, q$ and the Kronecker indices for the state space case.
Removing the (weakly) idiosyncratic noise from the observations

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Removing the (weakly) idiosyncratic noise from the observations

Note: \( \hat{y}_t \) is not directly observed, how can we get rid of \( u_t \)?

\( N \to \infty \)

There are several approaches.

Here we only describe the (static) PCA based procedure

(i) Commence from

\[
\hat{\Gamma}_y^N = T^{-1} \sum y_t^N y_t^{N'}
\]

Now, under suitable assumptions, for \( T, N \to \infty \) the first \( r \) eigenvalues of \( \hat{\Gamma}_y^N \) tend to infinity, and the other eigenvalues of \( \hat{\Gamma}_y^N \) converge to finite values.
Removing the (weakly) idiosyncratic noise from the observations

This is used to estimate $r$ and $z_t$:

$$\hat{\Gamma}^N_y = O_T \underbrace{\Lambda_T}_{\text{first } r \text{ eigenvalues}} O_T' + \text{remainder}$$

$$\hat{z}_t = O_T' y_t$$

$\hat{r}$ and $\hat{z}_t$ are consistent for $T, N \to \infty$

(ii) Use $\hat{z}_t$ to estimate the AR order $p$ and the autoregressive coefficients
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Aims:

- obtain „structural“ insights
- direct estimation procedure based on estimation of the second moments of the observations (computational simplicity)
- treatment of case more general case (compared to existing literature)
- forecasting

Zeroless transfer functions and spectra make things easier.

Further open questions:

- Properties of the estimators for the autoregression and of the estimators for \((F, G, H)\) beyond consistency
- Properties of the Estimation of the integer-valued parameters such as \(r, q, p, s\)
Thank You