

Local Quadratic Convergence of SQP for Elliptic Optimal Control Problems with Mixed Control-State Constraints

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RICAM-Report 2008-21

LOCAL QUADRATIC CONVERGENCE OF SQP FOR ELLIPTIC OPTIMAL CONTROL PROBLEMS WITH MIXED CONTROL-STATE CONSTRAINTS

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ABSTRACT. Semilinear elliptic optimal control problems with pointwise control and mixed control-state constraints are considered. Necessary and sufficient optimality conditions are given. The equivalence of the SQP method and Newton's method for a generalized equation is discussed. Local quadratic convergence of the SQP method is proved.

1. INTRODUCTION

This paper is concerned with the local convergence analysis of the sequential quadratic programming (SQP) method for the following class of semilinear optimal control problems:

$$\text{Minimize } f(y, u) := \int_{\Omega} \phi(\xi, y(\xi), u(\xi)) d\xi \quad (\mathbf{P})$$

subject to $u \in L^{\infty}(\Omega)$ and the elliptic state equation

$$\begin{aligned} Ay + d(\xi, y) &= u && \text{in } \Omega, \\ y &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (1.1)$$

as well as pointwise constraints

$$\begin{aligned} u &\geq 0 && \text{in } \Omega, \\ \varepsilon u + y &\geq y_c && \text{in } \Omega. \end{aligned} \quad (1.2)$$

Here and throughout, Ω is a bounded domain in \mathbb{R}^N , $N \in \{2, 3\}$, which is convex or has a $C^{1,1}$ boundary $\partial\Omega$. In (1.1), A is an elliptic operator in $H_0^1(\Omega)$ specified below, and ε is a positive number. The bound y_c is a function in $L^{\infty}(\Omega)$.

Problems with mixed control-state constraints are important as Lavrientiev-type regularizations of pointwise state-constrained problems [10–12], but they are also interesting in their own right. Note that in addition to the mixed control-state constraint, a pure control constraint is present on the same domain. Since problem (\mathbf{P}) is nonconvex, different local minima may occur.

SQP methods have proved to be fast solution methods for nonlinear programming problems. A large body of literature exists concerning the analysis of these methods for finite-dimensional problems. For a convergence analysis in a general Banach space setting with equality and inequality constraints, we refer to [2, 3].

The main contribution of this paper is the proof of local quadratic convergence of the SQP method, applied to (\mathbf{P}) . To our knowledge, such convergence results in the context of PDE-constrained optimization are so far only available for purely control-constrained problems [7, 17, 19]. Following [2], we exploit the equivalence between

the SQP and the Lagrange-Newton methods, i.e., Newton's method, applied to a generalized (set-valued) equation representing necessary conditions of optimality. We concentrate on specific issues arising due to the semilinear state equation, e.g., the careful choice of suitable function spaces. An important step is the verification of the so-called strong regularity of the generalized equation, which is made difficult by the simultaneous presence of pure control and mixed control-state constraints (1.2). The key idea was recently developed in [4].

We remark that strong regularity is known to be closely related to second-order sufficient conditions (SSC). For problems with pure control constraints, SSC are well understood and they are close to the necessary ones when so-called strongly active subsets are used, see, e.g., [17, 19, 20]. However, the situation is more difficult for problems with mixed control-state constraints [14, 16] or even pure state constraints. In order to avoid a more technical discussion, we presently employ relatively strong SSC and refer to future work for their refinement. We also refer to an upcoming publication concerning the numerical application of the SQP method to problems of type **(P)**.

The material in this paper is organized as follows. In Section 2, we state our main assumptions and recall some properties about the state equation. Necessary and sufficient optimality conditions for problem **(P)** are stated in Section 3, and their reformulation as a generalized equation is given in Section 4. Section 5 addresses the equivalence of the SQP and Lagrange-Newton methods. Section 6 is devoted to the proof of strong regularity of the generalized equation. Finally, Section 7 completes the convergence analysis of the SQP method. A number of auxiliary results have been collected in the Appendix.

We denote by $L^p(\Omega)$ and $H^m(\Omega)$ the usual Lebesgue and Sobolev spaces [1], and (\cdot, \cdot) is the scalar product in $L^2(\Omega)$ or $[L^2(\Omega)]^N$, respectively. $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$ with zero boundary traces, and $H^{-1}(\Omega)$ is its dual. The continuous embedding of a normed space X into a normed space Y is denoted by $X \hookrightarrow Y$. Throughout, we denote by $B_r^X(x)$ the open ball of radius r around x , in the topology of X . In particular, we write $B_r^\infty(x)$ for the open ball with respect to the $L^\infty(\Omega)$ norm. Throughout, c, c_1 etc. denote generic positive constants whose value may change from instance to instance.

2. ASSUMPTIONS AND PROPERTIES OF THE STATE EQUATION

The following assumptions (A1)–(A4) are taken to hold throughout the paper.

Assumption.

- (A1) Let Ω be a bounded domain in \mathbb{R}^N , $N \in \{2, 3\}$ which is convex or has $C^{1,1}$ boundary $\partial\Omega$. The bound y_c is in $L^\infty(\Omega)$, and $\varepsilon > 0$.
- (A2) The operator $A : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ is defined as $Ay(v) = a[y, v]$, where

$$a[y, v] = ((\nabla v), A_0 \nabla y) + (cy, v).$$

A_0 is an $N \times N$ matrix with Lipschitz continuous entries on $\bar{\Omega}$ such that $\rho^\top A_0(\xi) \rho \geq m_0 |\rho|^2$ holds with some $m_0 > 0$ for all $\rho \in \mathbb{R}^N$ and almost all $\xi \in \bar{\Omega}$. Moreover, $c \in L^\infty(\Omega)$ holds. The bilinear form $a[\cdot, \cdot]$ is not necessarily symmetric but it is assumed to be continuous and coercive, i.e.,

$$\begin{aligned} a[y, v] &\leq \bar{c} \|y\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \\ a[y, y] &\geq \underline{c} \|y\|_{H^1(\Omega)}^2 \end{aligned}$$

for all $y, v \in H_0^1(\Omega)$ with some positive constants \bar{c} and \underline{c} . A simple example is $a[y, v] = (\nabla y, \nabla v)$, corresponding to $A = -\Delta$.

- (A3) $d(\xi, y)$ belongs to the C^2 -class of functions with respect to y for almost all $\xi \in \Omega$. Moreover, d_{yy} is assumed to be a locally bounded and locally Lipschitz-continuous function with respect to y , i.e., the following conditions hold true: there exists $K > 0$ such that

$$|d(\xi, 0)| + |d_y(\xi, 0)| + |d_{yy}(\xi, 0)| \leq K_d,$$

and for any $M > 0$, there exists $L_d(M) > 0$ such that

$$|d_{yy}(\xi, y_1) - d_{yy}(\xi, y_2)| \leq L_d(M) |y_1 - y_2| \quad \text{a.e. in } \Omega$$

for all $y_1, y_2 \in \mathbb{R}$ satisfying $|y_1|, |y_2| \leq M$.

Additionally $d_y(\xi, y) \geq 0$ a.e. in Ω , for all $y \in \mathbb{R}$.

- (A4) The function $\phi = \phi(\xi, y, u)$ is measurable with respect to $\xi \in \Omega$ for each y and u , and of class C^2 with respect to y and u for almost all $\xi \in \Omega$. Moreover, the second derivatives are assumed to be locally bounded and locally Lipschitz-continuous functions, i.e., the following conditions hold: there exist $K_y, K_u, K_{yu} > 0$ such that

$$\begin{aligned} |\phi(\xi, 0, 0)| + |\phi_y(\xi, 0, 0)| + |\phi_{yy}(\xi, 0, 0)| &\leq K_y, & |\phi_{yu}(\xi, 0, 0)| &\leq K_{yu}, \\ |\phi(\xi, 0, 0)| + |\phi_u(\xi, 0, 0)| + |\phi_{uu}(\xi, 0, 0)| &\leq K_u, \end{aligned}$$

Moreover, for any $M > 0$, there exists $L_\phi(M) > 0$ such that

$$\begin{aligned} |\phi_{yy}(\xi, y_1, u_1) - \phi_{yy}(\xi, y_2, u_2)| &\leq L_\phi(M) (|y_1 - y_2| + |u_1 - u_2|), \\ |\phi_{yu}(\xi, y_1, u_1) - \phi_{yu}(\xi, y_2, u_2)| &\leq L_\phi(M) (|y_1 - y_2| + |u_1 - u_2|), \\ |\phi_{uy}(\xi, y_1, u_1) - \phi_{uy}(\xi, y_2, u_2)| &\leq L_\phi(M) (|y_1 - y_2| + |u_1 - u_2|), \\ |\phi_{uu}(\xi, y_1, u_1) - \phi_{uu}(\xi, y_2, u_2)| &\leq L_\phi(M) (|y_1 - y_2| + |u_1 - u_2|) \end{aligned}$$

for all $y_i, u_i \in \mathbb{R}$ satisfying $|y_i|, |u_i| \leq M$, $i = 1, 2$.

In addition, $\phi_{uu}(\xi, y, u) \geq m > 0$ a.e. in Ω , for all $(y, u) \in \mathbb{R}^2$.

In the sequel, we will simply write $d(y)$ instead of $d(\xi, y)$ etc. As a consequence of (A3)–(A4), the Nemyckii operators $d(\cdot)$ and $\phi(\cdot)$ are twice continuously Fréchet differentiable with respect to the $L^\infty(\Omega)$ norms, and their derivatives are locally Lipschitz continuous, see Lemma A.1.

The necessity of using $L^\infty(\Omega)$ norms for general nonlinearities d and ϕ motivates our choice

$$Y := H^2(\Omega) \cap H_0^1(\Omega)$$

as a state space, since $Y \hookrightarrow L^\infty(\Omega)$.

Remark 2.1. In case Ω has only a Lipschitz boundary, our results remain true when Y is replaced by $H_0^1(\Omega) \cap L^\infty(\Omega)$.

Recall that a function $y \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is called a **weak solution** of (1.1) with $u \in L^2(\Omega)$ if $a[y, v] + (d(y), v) = (u, v)$ holds for all $v \in H_0^1(\Omega)$.

Lemma 2.2. Under assumptions (A1)–(A3) and for any given $u \in L^2(\Omega)$, the semilinear equation (1.1) possesses a unique weak solution $y \in Y$. It satisfies the a priori estimate

$$\|y\|_{H^1(\Omega)} + \|y\|_{L^\infty(\Omega)} \leq C_\Omega (\|u\|_{L^2(\Omega)} + 1)$$

with a constant C_Ω independent of u .

Proof. The existence and uniqueness of a weak solution $y \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a standard result [18, Theorem 4.8]. It satisfies

$$\|y\|_{H^1(\Omega)} + \|y\|_{L^\infty(\Omega)} \leq C_\Omega (\|u\|_{L^2(\Omega)} + 1) =: M$$

with some constant C_Ω independent of u . Lemma A.1 implies that $d(y) \in L^\infty(\Omega)$. Using the embedding $L^\infty(\Omega) \hookrightarrow L^2(\Omega)$, we conclude that the difference $u - d(y)$ belongs to $L^2(\Omega)$. Owing to assumption (A1), $y \in H^2(\Omega)$, see for instance [6, Theorem 2.2.2.3]. \square

We will frequently also need the corresponding result for the linearized equation

$$\begin{aligned} Ay + d_y(\bar{y})y &= u & \text{in } \Omega, \\ y &= 0 & \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

Lemma 2.3. *Under assumptions (A1)–(A3) and given $\bar{y} \in L^\infty(\Omega)$, the linearized PDE (2.1) possesses a unique weak solution $y \in Y$ for any given $u \in L^2(\Omega)$. It satisfies the a priori estimate*

$$\|y\|_{H^2(\Omega)} \leq C_\Omega(\bar{y}) \|u\|_{L^2(\Omega)}$$

with a constant $C_\Omega(\bar{y})$ independent of u .

Proof. According to (A3) and Lemma A.1, $d_y(\bar{y})$ is a nonnegative coefficient in $L^\infty(\Omega)$. The claim thus follows again from standard arguments, see, e.g., [6, Theorem 2.2.2.3]. \square

3. NECESSARY AND SUFFICIENT OPTIMALITY CONDITIONS

In this section, we introduce necessary and sufficient optimality conditions for problem **(P)**. For convenience, we define the Lagrange functional

$$\mathcal{L} : Y \times L^\infty(\Omega) \times Y \times L^\infty(\Omega) \times L^\infty(\Omega) \rightarrow \mathbb{R}$$

as

$$\mathcal{L}(y, u, p, \mu_1, \mu_2) = f(y, u) + a[y, p] + (p, d(y) - u) - (\mu_1, u) - (\mu_2, \varepsilon u + y - y_c).$$

Here, μ_i are Lagrange multipliers associated to the inequality constraints, and p is the adjoint state. The existence of regular Lagrange multipliers $\mu_1, \mu_2 \in L^\infty(\Omega)$ was shown in [15, Theorem 7.3], which implies the following lemma:

Lemma 3.1. *Suppose that $(y, u) \in Y \times L^\infty(\Omega)$ is a local optimal solution of **(P)**. Then there exist regular Lagrange multipliers $\mu_1, \mu_2 \in L^\infty(\Omega)$ and an adjoint state $p \in Y$ such that the **first-order necessary optimality conditions***

$$\left. \begin{aligned} \mathcal{L}_y(y, u, p, \mu_1, \mu_2) &= 0, & \mathcal{L}_u(y, u, p, \mu_1, \mu_2) &= 0, & \mathcal{L}_p(y, u, p, \mu_1, \mu_2) &= 0, \\ u &\geq 0, & \mu_1 &\geq 0, & \mu_1 u &= 0, \\ \varepsilon u + y - y_c &\geq 0, & \mu_2 &\geq 0, & \mu_2(\varepsilon u + y - y_c) &= 0 \end{aligned} \right\} \text{ (FON)}$$

hold.

Remark 3.2. *The Lagrange multipliers and adjoint state associated to a local optimal solution of **(P)** need not be unique if the active sets $\{\xi \in \Omega : u = 0\}$ and $\{\xi \in \Omega : \varepsilon u + y - y_c = 0\}$ intersect nontrivially. This situation will be excluded by Assumption (A6) below.*

Conditions **(FON)** are also stated in explicit form in (4.1) below. To guarantee that $x = (y, u)$ with associated multipliers $\lambda = (\mu_1, \mu_2, p)$ is a local solution of **(P)**, we introduce the following **second-order sufficient optimality condition (SSC)**:

There exists a constant $\alpha > 0$ such that

$$\mathcal{L}_{xx}(x, \lambda)(\delta x, \delta x) \geq \alpha \|\delta x\|_{[L^2(\Omega)]^2}^2 \quad (3.1)$$

for all $\delta x = (\delta y, \delta u) \in Y \times L^\infty(\Omega)$ which satisfy the linearized equation

$$\begin{aligned} A\delta y + d_y(y) \cdot \delta y &= \delta u \quad \text{in } \Omega, \\ \delta y &= 0 \quad \text{on } \partial\Omega. \end{aligned} \quad (3.2)$$

In (3.1), the Hessian of the Lagrange functional is given by

$$\mathcal{L}_{xx}(x, \lambda)(\delta x, \delta x) := \int_{\Omega} \begin{pmatrix} \delta y \\ \delta u \end{pmatrix}^\top \begin{pmatrix} \phi_{yy}(y, u) + d_{yy}(y)p & \phi_{yu}(y, u) \\ \phi_{uy}(y, u) & \phi_{uu}(y, u) \end{pmatrix} \begin{pmatrix} \delta y \\ \delta u \end{pmatrix} d\xi.$$

For convenience, we will use the abbreviation

$$X := Y \times L^\infty(\Omega) = H^2(\Omega) \cap H_0^1(\Omega) \times L^\infty(\Omega)$$

in the sequel.

Assumption.

- (A5) We assume that $x^* = (y^*, u^*) \in X$, together with associated Lagrange multipliers $\lambda^* = (p^*, \mu_1^*, \mu_2^*) \in Y \times [L^\infty(\Omega)]^2$, satisfies both **(FON)** and **(SSC)**.

As mentioned in the introduction, we are aware of the fact that there exist weaker sufficient conditions which take into account strongly active sets. However, this further complicates the convergence analysis of SQP and is therefore postponed to later work.

Definition 3.3.

- (a) A pair $x = (y, u) \in X$ is called an **admissible point** if it satisfies (1.1) and (1.2).
 (b) A point $\bar{x} \in X$ is called a **strict local optimal solution in the sense of** $L^\infty(\Omega)$ if there exists $\varepsilon > 0$ such that the inequality $f(\bar{x}) < f(x)$ holds for all admissible $x \in X \setminus \{\bar{x}\}$ with $\|x - \bar{x}\|_{[L^\infty(\Omega)]^2} \leq \varepsilon$.

Theorem 3.4. Under Assumptions (A1)–(A5), there exists $\beta > 0$ and $\varepsilon > 0$ such that

$$f(x) \geq f(x^*) + \beta \|x - x^*\|_{[L^2(\Omega)]^2}^2$$

holds for all admissible $x \in X$ with $\|x - x^*\|_{[L^\infty(\Omega)]^2} \leq \varepsilon$. In particular, x^* is a strict local optimal solution in the sense of $L^\infty(\Omega)$.

Proof. The proof uses the two-norm discrepancy principle, see [8, Theorem 3.5]. Let $x \in X$ be an admissible point, which implies

$$\begin{aligned} a[y, p^*] + (p^*, d(y) - u) &= 0 \\ \text{and } u \geq 0, \quad \varepsilon u + y - y_c &\geq 0 \quad \text{a.e. in } \Omega. \end{aligned}$$

In view of $\mu_1^*, \mu_2^* \geq 0$, we can estimate the cost functional f by the Lagrange functional:

$$f(x) \geq f(x) + a[y, p^*] + (p^*, d(y) - u) - (\mu_1^*, u) - (\mu_2^*, \varepsilon u + y - y_c) = \mathcal{L}(x, \lambda^*). \quad (3.3)$$

The Lagrange functional is twice continuously differentiable with respect to the $L^\infty(\Omega)$ norms, as is easily seen from Lemma A.1. Hence it possesses a Taylor expansion

$$\mathcal{L}(x, \lambda^*) = \mathcal{L}(x^*, \lambda^*) + \mathcal{L}_x(x^*, \lambda^*)(x - x^*) + \mathcal{L}_{xx}(x + \theta(x - x^*), \lambda^*)(x - x^*, x - x^*)$$

for all $x \in X$, where $\theta \in (0, 1)$. Since the pair (x^*, λ^*) satisfies **(FON)**, we have

$$f(x^*) = \mathcal{L}(x^*, \lambda^*) + \mathcal{L}_x(x^*, \lambda^*)(x - x^*),$$

which implies

$$\begin{aligned} \mathcal{L}(x, \lambda^*) &= f(x^*) + \mathcal{L}_{xx}(x^*, \lambda^*)(x - x^*, x - x^*) \\ &\quad + (\mathcal{L}_{xx}(x^* + \theta(x - x^*), \lambda^*) - \mathcal{L}_{xx}(x^*, \lambda^*))(x - x^*, x - x^*). \end{aligned}$$

We cannot use **(SSC)** directly since x satisfies the semilinear equation (1.1) instead of the linearized one (3.2). However, Lemma A.2 implies that there exist $\varepsilon > 0$ and $\alpha' > 0$ such that

$$\begin{aligned} \mathcal{L}(x, \lambda^*) &\geq f(x^*) + \alpha' \|x - x^*\|_{[L^2(\Omega)]^2}^2 \\ &\quad + (\mathcal{L}_{xx}(x^* + \theta(x - x^*), \lambda^*) - \mathcal{L}_{xx}(x^*, \lambda^*))(x - x^*, x - x^*), \end{aligned} \quad (3.4)$$

given that $\|x - x^*\|_{[L^\infty(\Omega)]^2} \leq \varepsilon$. Moreover, the Hessian of the Lagrange functional satisfies the following local Lipschitz condition (see Lemma A.1 and also [18, Lemma 4.24]):

$$\begin{aligned} |(\mathcal{L}_{xx}(x^* + \theta(x - x^*), \lambda^*) - \mathcal{L}_{xx}(x^*, \lambda^*))(x - x^*, x - x^*)| \\ \leq c \|x - x^*\|_{[L^\infty(\Omega)]^2} \|x - x^*\|_{[L^2(\Omega)]^2}^2 \end{aligned} \quad (3.5)$$

for all $\|x - x^*\|_{[L^\infty(\Omega)]^2} \leq \varepsilon$. Summarizing (3.3)–(3.5), we can estimate

$$f(x) \geq f(x^*) + \beta \|x - x^*\|_{[L^2(\Omega)]^2}^2,$$

where

$$\beta := \alpha' - c \|x - x^*\|_{[L^\infty(\Omega)]^2} \geq \alpha' - c\varepsilon > 0$$

when ε is taken sufficiently small. \square

4. GENERALIZED EQUATION

We recall the necessary optimality conditions **(FON)** for problem **(P)**, which read in explicit form

$$\left. \begin{aligned} a[v, p] + (d_y(y), v) + (\phi_y(y, u), v) - (\mu_2, v) &= 0, & v \in H_0^1(\Omega) \\ (\phi_u(y, u), v) - (p, v) - (\mu_1, v) - (\varepsilon\mu_2, v) &= 0, & v \in L^2(\Omega) \\ a[y, v] + (d(y), v) - (u, v) &= 0, & v \in H_0^1(\Omega) \\ \mu_1 \geq 0, & u \geq 0, & \mu_1 u = 0 \\ \mu_2 \geq 0, & \varepsilon u + y - y_c \geq 0, & \mu_2(\varepsilon u + y - y_c) = 0 \end{aligned} \right\} \quad \text{a.e. in } \Omega. \quad (4.1)$$

As was mentioned in the introduction, the local convergence analysis of SQP is based on its interpretation as Newton's method for a **generalized (set-valued) equation**

$$0 \in F(y, u, p, \mu_1, \mu_2) + N(y, u, p, \mu_1, \mu_2) \quad (4.2)$$

equivalent to (4.1). We define

$$K := \{\mu \in L^\infty(\Omega) : \mu \geq 0 \text{ a.e. in } \Omega\},$$

the cone of nonnegative functions in $L^\infty(\Omega)$, and the **dual cone** $N_1 : L^\infty(\Omega) \longrightarrow P(L^\infty(\Omega))$,

$$N_1(\mu) := \begin{cases} \{z \in L^\infty(\Omega) : (z, \mu - \nu) \geq 0 \quad \forall \nu \in K\} & \text{if } \mu \in K, \\ \emptyset & \text{if } \mu \notin K. \end{cases}$$

Here $P(L^\infty)$ denotes the **power set** of $L^\infty(\Omega)$, i.e., the set of all subsets of $L^\infty(\Omega)$. In (4.2), F contains the single-valued part of (4.1), i.e.,

$$F(y, u, p, \mu_1, \mu_2) = \begin{pmatrix} A^*p + d_y(y)p + \phi_y(y, u) - \mu_2 \\ \phi_u(y, u) - p - \mu_1 - \varepsilon\mu_2 \\ Ay + d(y) - u \\ u \\ \varepsilon u + y - y_c \end{pmatrix}.$$

Both A and its formal adjoint A^* are considered here as operators from Y to $L^2(\Omega)$, i.e., $Ay = -\operatorname{div}(A_0 \nabla y) + cy$ and $A^*p = -\operatorname{div}(A_0^\top \nabla p) + cp$ hold. Moreover, N is the set-valued function

$$N(y, u, p, \mu_1, \mu_2) = (\{0\}, \{0\}, \{0\}, N_1(\mu_1), N_1(\mu_2))^\top.$$

Note that the generalized equation (4.2) is nonlinear, since it contains the nonlinear functions d , d_y , ϕ_y and ϕ_u .

Remark 4.1. *Let*

$$\begin{aligned} W &:= Y \times L^\infty(\Omega) \times Y \times L^\infty(\Omega) \times L^\infty(\Omega), \\ Z &:= L^2(\Omega) \times L^\infty(\Omega) \times L^2(\Omega) \times L^\infty(\Omega) \times L^\infty(\Omega). \end{aligned}$$

Then $F : W \longrightarrow Z$ and $N : W \longrightarrow P(Z)$. Owing to Assumptions (A3) and (A4), F is continuously Fréchet differentiable with respect to the $L^\infty(\Omega)$ norms, see Lemma A.1.

Lemma 4.2. *The first-order necessary conditions (4.1) and the generalized equation (4.2) are equivalent.*

Proof. (4.2) \Rightarrow (4.1): This is immediate for the first three components. For the fourth component we have

$$\begin{aligned} & -u \in N_1(\mu_1) \\ \Rightarrow & \mu_1 \in K \quad \text{and} \quad (-u, \mu_1 - \nu) \geq 0 \quad \text{for all } \nu \in K \\ \Rightarrow & \mu_1(\xi) \geq 0 \quad \text{and} \quad -u(\xi)(\mu_1(\xi) - \nu) \geq 0 \quad \text{for all } \nu \geq 0, \quad \text{a.e. in } \Omega. \end{aligned}$$

This implies

$$\begin{aligned} \mu_1(\xi) = 0 & \Rightarrow u(\xi) \geq 0 \\ \mu_1(\xi) > 0 & \Rightarrow u(\xi) = 0, \end{aligned}$$

which shows the first complementarity system in (4.1). The second follows analogously.

(4.1) \Rightarrow (4.2): This is again immediate for the first three components. From the first complementarity system in (4.1) we infer that

$$\begin{aligned} & u(\xi)\nu \geq 0 \quad \text{for all } \nu \geq 0, \quad \text{a.e. in } \Omega \\ \Rightarrow & -u(\xi)(\mu_1(\xi) - \nu) \geq 0 \quad \text{for all } \nu \geq 0, \quad \text{a.e. in } \Omega \\ \Rightarrow & -(u, \mu_1 - \nu) \geq 0 \quad \text{for all } \nu \in K. \end{aligned}$$

In view of $\mu_1 \in K$, this implies $-u \in N_1(\mu_1)$. Again, $-(\varepsilon u + y - y_c) \in N_1(\mu_2)$ follows analogously. \square

5. SQP METHOD

In this section we briefly recall the SQP (sequential quadratic programming) method for the solution of problem (\mathbf{P}) . We also discuss its equivalence with Newton's method, applied to the generalized equation (4.2), which is often called the Lagrange-Newton approach. Throughout the rest of the paper we use the notation

$$w^k := (x^k, \lambda^k) = (y^k, u^k, p^k, \mu_1^k, \mu_2^k) \in W$$

to denote an iterate of either method. SQP methods break down the solution of (\mathbf{P}) into a sequence of quadratic programming problems. At any given iterate w^k , one solves

$$\text{Minimize } f_x(x^k)(x - x^k) + \frac{1}{2} \mathcal{L}_{xx}(x^k, \lambda^k)(x - x^k, x - x^k) \quad (\mathbf{QP}_k)$$

subject to $x = (y, u) \in Y \times L^\infty(\Omega)$, the linear state equation

$$\begin{aligned} Ay + d(y^k) + d_y(y^k)(y - y^k) &= u & \text{in } \Omega, \\ y &= 0 & \text{on } \partial\Omega, \end{aligned} \quad (5.1)$$

and inequality constraints

$$\begin{aligned} u &\geq 0 & \text{in } \Omega, \\ \varepsilon u + y - y_c &\geq 0 & \text{in } \Omega. \end{aligned} \quad (5.2)$$

The solution (which needs to be shown to exist)

$$x = (y, u) \in Y \times L^\infty(\Omega),$$

together with the adjoint state and Lagrange multipliers

$$\lambda = (p, \mu_1, \mu_2) \in Y \times L^\infty(\Omega) \times L^\infty(\Omega),$$

will serve as the next iterate w^{k+1} .

Lemma 5.1. *There exists $R > 0$ such that (\mathbf{QP}_k) has a unique global solution $x = (y, u) \in X$, provided that $(x^k, p^k) \in B_R^\infty(x^*, p^*)$.*

Proof. For every $u \in L^2(\Omega)$, the linearized PDE (5.1) has a unique solution $y \in Y$ by Lemma 2.3. We define the feasible set

$$M^k := \{x = (y, u) \in Y \times L^2(\Omega) \text{ satisfying (5.1) and (5.2)}\}.$$

The set M^k is non-empty, which follows from [4, Lemma 2.3] using $\delta_3 = -d(y^k) + d_y(y^k)y^k$. The proof uses the maximum principle for the differential operator $Ay + d_y(y^k)y$. Clearly, M^k is also closed and convex.

The cost functional of (\mathbf{QP}_k) can be decomposed into quadratic and affine parts in x . Lemma A.3 shows that there exists $R > 0$ and $\alpha'' > 0$ such that

$$\mathcal{L}_{xx}(x^k, \lambda^k)(x, x) \geq \alpha'' \|x\|_{[L^2(\Omega)]^2}^2$$

for all $(y, u) \in X$ satisfying $Ay + d_y(y^k)y = u$ in Ω with homogeneous Dirichlet boundary conditions, provided that $(x^k, p^k) \in B_R^\infty(x^*, p^*)$. This implies that the cost functional is uniformly convex, continuous (i.e., weakly lower semicontinuous) and radially unbounded, which shows the unique solvability of (\mathbf{QP}_k) in $Y \times L^2(\Omega)$. Using the optimality system (5.3) below, we can conclude as in [4, Lemma 2.7] that $u \in L^\infty(\Omega)$. \square

The solution (y, u) of (\mathbf{QP}_k) and its Lagrange multipliers (p, μ_1, μ_2) are characterized by the first order optimality system (compare [4, Lemma 2.5]):

$$\left. \begin{aligned} a[v, p] + (d_y(y^k)p, v) + (\phi_y(y^k, u^k), v) + (\phi_{yu}(y^k, u^k)(u - u^k), v) \\ + ((\phi_{yy}(y^k, u^k) + d_{yy}(y^k)p^k)(y - y^k), v) - (\mu_2, v) = 0, \quad v \in H_0^1(\Omega) \\ (\phi_u(y^k, u^k), v) + (\phi_{uu}(y^k, u^k)(u - u^k), v) \\ + (\phi_{uy}(y^k, u^k)(y - y^k), v) - (p, v) - (\mu_1, v) - (\varepsilon\mu_2, v) = 0, \quad v \in L^2(\Omega) \\ a[y, v] + (d(y^k), v) + (d_y(y^k)(y - y^k), v) - (u, v) = 0, \quad v \in H_0^1(\Omega) \\ \left. \begin{aligned} \mu_1 \geq 0, \quad u \geq 0, \quad \mu_1 u = 0 \\ \mu_2 \geq 0, \quad \varepsilon u + y - y_c \geq 0, \quad \mu_2(\varepsilon u + y - y_c) = 0 \end{aligned} \right\} \text{ a.e. in } \Omega. \end{aligned} \right\} \quad (5.3)$$

Note that due to the convexity of the cost functional, (5.3) is both necessary and sufficient for optimality, provided that $(x^k, p^k) \in B_R^\infty(x^*, p^*)$.

Remark 5.2. *The Lagrange multipliers (μ_1, μ_2) and the adjoint state p in (5.3) need not be unique, compare [4, Remark 2.6]. Non-uniqueness can occur only if μ_1 and μ_2 are simultaneously nonzero on a set of positive measure.*

We recall for convenience the generalized equation (4.2),

$$0 \in F(w) + N(w). \quad (5.4)$$

Given the iterate w^k , Newton's method yields the next iterate w^{k+1} as the solution of the linearized generalized equation

$$0 \in F(w^k) + F'(w^k)(w - w^k) + N(w). \quad (5.5)$$

Analogously to Lemma 4.2, one can show:

Lemma 5.3. *System (5.3) and the linearized generalized equation (5.5) are equivalent.*

6. STRONG REGULARITY

The local convergence analysis of Newton's method (5.5) for the solution of (5.4) is based on a perturbation argument. It will be carried out in Section 7. The main ingredient in the proof is the local Lipschitz stability of solutions $w = w(\eta)$ of

$$0 \in F(\eta) + F'(\eta)(w - \eta) + N(w) \quad (6.1)$$

with respect to the parameter η near w^* . The difficulty arises due to the fact that η enters nonlinearly in (6.1). Therefore, we employ an implicit function theorem due to Dontchev [5] to derive this result. This theorem requires the so-called strong regularity of (5.4), i.e., the Lipschitz stability of solutions $w = w(\delta)$ of

$$\delta \in F(w^*) + F'(w^*)(w - w^*) + N(w) \quad (6.2)$$

with respect to the new perturbation parameter δ , which enters linearly. The parameter δ belongs to the image space of F

$$Z := L^2(\Omega) \times L^\infty(\Omega) \times L^2(\Omega) \times L^\infty(\Omega) \times L^\infty(\Omega),$$

see Remark 4.1. Note that w^* is a solution of both (5.4) and (6.2) for $\delta = 0$.

Definition 6.1 (see [13]). *The generalized equation (5.4) is called **strongly regular** at w^* if there exist radii $r_1 > 0$, $r_2 > 0$ and a positive constant L_δ such that for all perturbations $\delta \in B_{r_1}^Z(0)$, the following hold:*

- (1) the linearized equation (6.2) has a solution $w_\delta = w(\delta) \in B_{r_2}^W(w^*)$
- (2) w_δ is the only solution of (6.2) in $B_{r_2}^W(w^*)$
- (3) w_δ satisfies the Lipschitz condition

$$\|w_\delta - w_{\delta'}\|_W \leq L_\delta \|\delta - \delta'\|_Z \quad \text{for all } \delta, \delta' \in B_{r_1}^Z(0).$$

The verification of strong regularity is based on the interpretation of (6.2) as the optimality system of the following QP problem, which depends on the perturbation δ :

$$\begin{aligned} \text{Minimize } & f_x(x^*)(x - x^*) + \frac{1}{2} \mathcal{L}_{xx}(x^*, \lambda^*)(x - x^*, x - x^*) & (\mathbf{LQP}(\delta)) \\ & - ([\delta_1, \delta_2], x - x^*) \end{aligned}$$

subject to $x = (y, u) \in Y \times L^\infty(\Omega)$, the linear state equation

$$\begin{aligned} Ay + d(y^*) + d_y(y^*)(y - y^*) &= u + \delta_3 \quad \text{in } \Omega, \\ y &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{6.3}$$

and inequality constraints

$$\begin{aligned} u &\geq \delta_4 \quad \text{in } \Omega, \\ \varepsilon u + y - y_c &\geq \delta_5 \quad \text{in } \Omega. \end{aligned} \tag{6.4}$$

As before, it is easy to check that the necessary optimality conditions of $(\mathbf{LQP}(\delta))$ are equivalent to (6.2).

Lemma 6.2. *For any $\delta \in Z$, problem $(\mathbf{LQP}(\delta))$ possesses a unique global solution $x_\delta = (y_\delta, u_\delta) \in X$. If $\lambda_\delta = (p_\delta, \mu_{1,\delta}, \mu_{2,\delta}) \in Y \times L^\infty(\Omega) \times L^\infty(\Omega)$ are associated Lagrange multipliers, then $(x_\delta, \lambda_\delta)$ satisfies (6.2). On other hand, if any $(x_\delta, \lambda_\delta) \in W$ satisfies (6.2), then x_δ is the unique global solution of $(\mathbf{LQP}(\delta))$, and λ_δ are associated adjoint state and Lagrange multipliers.*

Proof. For any given $\delta \in Z$, let us denote by M_δ the set of all $x = (y, u) \in Y \times L^2(\Omega)$ satisfying (6.3) and (6.4). Then M_δ is nonempty (as can be shown along the lines of [4, Lemma 2.3]), convex and closed. Moreover, (A5) implies that the cost functional $f_\delta(x)$ of $(\mathbf{LQP}(\delta))$ satisfies

$$f_\delta(x) \geq \frac{\alpha}{2} \|x\|_{[L^2(\Omega)]^2}^2 + \text{linear terms in } x$$

for all x satisfying (6.3). As in the proof of Lemma 5.1, we conclude that $(\mathbf{LQP}(\delta))$ has a unique solution $x_\delta = (y_\delta, u_\delta) \in X$.

Suppose that $\lambda_\delta = (p_\delta, \mu_{1,\delta}, \mu_{2,\delta}) \in Y \times L^\infty(\Omega) \times L^\infty(\Omega)$ are associated Lagrange multipliers, i.e., the necessary optimality conditions of $(\mathbf{LQP}(\delta))$ are satisfied. As argued above, it is easy to check that then (6.2) holds. On the other hand, suppose that any $(x_\delta, \lambda_\delta) \in W$ satisfies (6.2), i.e., the necessary optimality conditions of $(\mathbf{LQP}(\delta))$. As f_δ is strictly convex, these conditions are likewise sufficient for optimality, and the minimizer x_δ is unique. \square

The proof of Lipschitz stability of solutions for problems of type $(\mathbf{LQP}(\delta))$ has recently been achieved in [4]. The main difficulty consisted in overcoming the non-uniqueness of the associated adjoint state and Lagrange multipliers. We follow the same technique here.

Definition 6.3. *Let $\sigma > 0$ be real number. We define two subsets of Ω ,*

$$\begin{aligned} S_1^\sigma &= \{\xi \in \Omega : 0 \leq u^*(\xi) \leq \sigma\} \\ S_2^\sigma &= \{\xi \in \Omega : 0 \leq \varepsilon u^*(\xi) + y^*(\xi) - y_c(\xi) \leq \sigma\}, \end{aligned}$$

called the **security sets** of level σ for (\mathbf{P}) .

Assumption.

(A6) We require that $S_1^\sigma \cap S_2^\sigma = \emptyset$ for some fixed $\sigma > 0$.

From now on, we suppose (A1)–(A6) to hold. Assumption (A6) implies that the active sets

$$\begin{aligned}\mathcal{A}_1^* &= \{\xi \in \Omega : u^*(\xi) = 0\} \\ \mathcal{A}_2^* &= \{\xi \in \Omega : \varepsilon u^*(\xi) + y^*(\xi) - y_c(\xi) = 0\}\end{aligned}$$

are well separated. This in turn implies the uniqueness of the Lagrange multipliers and adjoint state (p^*, μ_1^*, μ_2^*) . Due to a continuity argument, the same conclusions hold for the solution and Lagrange multipliers of $(\mathbf{LQP}(\delta))$ for sufficiently small δ , as proved in the following theorem.

Theorem 6.4. *There exist $G > 0$ and $L_\delta > 0$ such that $\|\delta\|_Z \leq G\sigma$ implies:*

- (1) *The Lagrange multipliers $\lambda_\delta = (p_\delta, \mu_{1,\delta}, \mu_{2,\delta})$ for $(\mathbf{LQP}(\delta))$ are unique.*
- (2) *For any such δ and δ' , the corresponding solutions and Lagrange multipliers of $(\mathbf{LQP}(\delta))$ satisfy*

$$\|x_{\delta'} - x_\delta\|_{Y \times L^\infty(\Omega)} + \|\lambda_{\delta'} - \lambda_\delta\|_{Y \times L^\infty(\Omega) \times L^\infty(\Omega)} \leq L_\delta \|\delta' - \delta\|_Z. \quad (6.5)$$

Proof. The proof employs the technique introduced in [4], so we will only revisit the main steps here. In contrast to the linear quadratic problem considered in [4], the cost functional and PDE in $(\mathbf{LQP}(\delta))$ are slightly more general. To overcome potential non-uniqueness of Lagrange multipliers, one introduces an auxiliary problem with solutions $(y_\delta^{aux}, u_\delta^{aux})$, in which the inequality constraints (6.4) are considered only on the disjoint sets S_1^σ and S_2^σ , respectively. Then the associated Lagrange multipliers $\mu_{i,\delta}^{aux}$, $i = 1, 2$, and adjoint state p_δ^{aux} are unique, see [4, Lemma 3.1]. For any two perturbations $\delta, \delta' \in Z$ we abbreviate

$$\delta u := u_\delta^{aux} - u_{\delta'}^{aux}$$

and similarly for the remaining quantities. From the optimality conditions of the auxiliary problem one deduces

$$\begin{aligned}\alpha (\|\delta y\|_{L^2(\Omega)}^2 + \|\delta u\|_{L^2(\Omega)}^2) &\stackrel{(A5)}{\leq} \mathcal{L}_{xx}(y^*, u^*)(\delta x, \delta x) \\ &= (\delta'_1 - \delta_1, \delta y) + (\delta'_2 - \delta_2, \delta u) - (\delta'_3 - \delta_3, \delta p) \\ &\quad + (\delta \mu_2, \delta y) + (\delta \mu_1, \delta u) + \varepsilon (\delta \mu_2, \delta u) \\ &\leq (\delta'_1 - \delta_1, \delta y) + (\delta'_2 - \delta_2, \delta u) - (\delta'_3 - \delta_3, \delta p) \\ &\quad + (\delta \mu_1, \delta'_4 - \delta_4) + (\delta \mu_2, \delta'_5 - \delta_5).\end{aligned}$$

The last inequality follows from [4, Lemma 3.3]. Young's inequality yields

$$\begin{aligned}\frac{\alpha}{2} (\|\delta y\|_{L^2(\Omega)}^2 + \|\delta u\|_{L^2(\Omega)}^2) \\ \leq \max \left\{ \frac{2}{\alpha}, \frac{1}{4\kappa} \right\} \|\delta - \delta'\|_{[L^2(\Omega)]^5}^2 + \kappa (\|\delta p\|_{L^2(\Omega)}^2 + \|\delta \mu_1\|_{L^2(\Omega)}^2 + \|\delta \mu_2\|_{L^2(\Omega)}^2),\end{aligned} \quad (6.6)$$

where $\kappa > 0$ is specified below. The difference of the adjoint states satisfies

$$\begin{aligned}a[v, \delta p] + (d_y(y^*) \delta p, v) &= -(\phi_{yy}(y^*, u^*) \delta y, v) - (d_{yy}(y^*) p^* \delta y, v) \\ &\quad - (\phi_{yu}(y^*, u^*) \delta u, v) + (\delta_1 - \delta'_1, v) + (\delta \mu_2, v)\end{aligned} \quad (6.7)$$

for all $v \in H_0^1(\Omega)$. The differences in the Lagrange multipliers are given by

$$\delta\mu_1 = \begin{cases} \phi_{uu}(y^*, u^*) \delta u + \phi_{uy}(y^*, u^*) \delta y - \delta p - (\delta_2 - \delta'_2) & \text{in } S_1^\sigma \\ 0 & \text{in } \Omega \setminus S_1^\sigma \end{cases} \quad (6.8)$$

and

$$\varepsilon\delta\mu_2 = \begin{cases} \phi_{uu}(y^*, u^*) \delta u + \phi_{uy}(y^*, u^*) \delta y - \delta p - (\delta_2 - \delta'_2) & \text{in } S_2^\sigma, \\ 0 & \text{in } \Omega \setminus S_2^\sigma \end{cases} \quad (6.9)$$

The substitution of $\delta\mu_2$ into (6.7) yields

$$\begin{aligned} a[v, \delta p] + (d_y(y^*) \delta p, v) + \frac{1}{\varepsilon}(\delta p, \chi_{S_2^\sigma} \cdot v) \\ = -(\phi_{yy}(y^*, u^*) \delta y, v) - (d_{yy}(y^*) p^*, \delta y) - \phi_{yu}(y^*, u^*) \delta u + (\delta_1 - \delta'_1, v) \\ + \frac{1}{\varepsilon}(\phi_{uu}(y^*, u^*) \delta u, \chi_{S_2^\sigma} \cdot v) + \frac{1}{\varepsilon}(\phi_{uy}(y^*, u^*) \delta y, \chi_{S_2^\sigma} \cdot v) - (\delta_2 - \delta'_2, \chi_{S_2^\sigma} \cdot v). \end{aligned}$$

A standard a priori estimate (compare Lemma 2.3) implies

$$\|\delta p\|_{L^2(\Omega)} \leq \|\delta p\|_Y \leq c (\|\delta y\|_{L^2(\Omega)} + \|\delta u\|_{L^2(\Omega)} + \|\delta_1 - \delta'_1\|_{L^2(\Omega)} + \|\delta_2 - \delta'_2\|_{L^2(\Omega)}).$$

From (6.8) and (6.9), we infer that $\|\delta\mu_1\|_{L^2(\Omega)}$ and $\|\delta\mu_2\|_{L^2(\Omega)}$ can be estimated a similar expression. Plugging these estimates into (6.6), and choosing κ sufficiently small, we get

$$\|\delta y\|_{L^2(\Omega)}^2 + \|\delta u\|_{L^2(\Omega)}^2 \leq c_{aux} \|\delta - \delta'\|_{[L^2(\Omega)]^5}^2.$$

By a priori estimates for the linearized and adjoint PDEs, we immediately obtain Lipschitz stability for δy and thus for δp with respect to the $H^2(\Omega)$ -norm.

The projection formula (compare [4, Lemma 2.7] and also Lemma A.1)

$$\begin{aligned} \mu_{1,\delta}^{aux} + \varepsilon\mu_{2,\delta}^{aux} = \max \left\{ 0, \phi_{uu}(y^*, u^*) \left(\max \left\{ \delta_4, \frac{y_c + \delta_5 - y_\delta^{aux}}{\varepsilon} \right\} - u^* \right) \right. \\ \left. + \phi_{uy}(y^*, u^*) (y_\delta^{aux} - y^*) + \phi_u(y^*, u^*) - p_\delta^{aux} - \delta_2 \right\} \end{aligned}$$

yields the $L^\infty(\Omega)$ -regularity for the Lagrange multipliers $(\mu_{1,\delta}^{aux}, \mu_{2,\delta}^{aux})$ and the control u_δ^{aux} . As in [4, Lemma 3.5], we conclude

$$\|\delta\mu_1 + \varepsilon\delta\mu_2\|_{L^\infty(\Omega)} \leq c \|\delta' - \delta\|_Z.$$

From the optimality system we have

$$\phi_{uu}(y^*, u^*) \delta u = \delta\mu_1 + \varepsilon\delta\mu_2 - \phi_{uy}(y^*, u^*) \delta y + \delta p + (\delta_2 - \delta'_2),$$

which implies by Assumption (A4)

$$m \|\delta u\|_{L^\infty(\Omega)} \leq c (\|\delta\mu_1 + \varepsilon\delta\mu_2\|_{L^\infty(\Omega)} + \|\delta y\|_{L^\infty(\Omega)} + \|\delta p\|_{L^\infty(\Omega)} + \|\delta_2 - \delta'_2\|_{L^\infty(\Omega)})$$

and yields the desired L^∞ -stability for the control of the auxiliary problem.

As in [4, Lemma 4.1], one shows that for $\|\delta\|_Z \leq G\sigma$ (for a certain constant $G > 0$), the solution $(y_\delta^{aux}, u_\delta^{aux})$ of the auxiliary problem coincides with the solution of $(\mathbf{LQP}(\delta))$. Likewise, the Lagrange multipliers and adjoint states of both problems coincide and are Lipschitz stable in $L^\infty(\Omega)$ and Y , respectively (see [4, Lemma 4.4]). \square

Remark 6.5. *Theorem 6.4, together with Lemma 6.2, proves the strong regularity of (5.4) at w^* .*

In order to apply the implicit function theorem, we verify that (6.1) satisfies a Lipschitz condition with respect to η , uniformly in a neighborhood of w^* .

Lemma 6.6. *For any radii $r_3 > 0$, $r_4 > 0$ there exists $L > 0$ such that for any $\eta_1, \eta_2 \in B_{r_3}^W(w^*)$ and for all $w \in B_{r_4}^W(w^*)$ there holds the Lipschitz condition*

$$\|F(\eta_1) + F'(\eta_1)(w - \eta_1) - F(\eta_2) - F'(\eta_2)(w - \eta_2)\|_Z \leq L \|\eta_1 - \eta_2\|_W. \quad (6.10)$$

Proof. Let us denote $\eta_i = (y_i, u_i, p_i, \mu_1^i, \mu_2^i) \in B_{r_3}^W(w^*)$ and $w = (y, u, p, \mu_1, \mu_2) \in B_{r_4}^W(w^*)$, with $r_3, r_4 > 0$ arbitrary. A simple calculation shows

$$\begin{aligned} & F(\eta_1) + F'(\eta_1)(w - \eta_1) - F(\eta_2) - F'(\eta_2)(w - \eta_2) \\ &= (f_1(y_1, u_1, p_1) - f_1(y_2, u_2, p_2), f_2(y_1, u_1) - f_2(y_2, u_2), f_3(y_1) - f_3(y_2), 0, 0)^\top, \end{aligned}$$

where

$$\begin{aligned} f_1(y_i, u_i, p_i) &= d_y(y_i) p + \phi_y(y_i, u_i) + [\phi_{yy}(y_i, u_i) + d_{yy}(y_i) p_i](y - y_i) \\ &\quad + \phi_{yu}(y_i, u_i)(u - u_i) \\ f_2(y_i, u_i) &= \phi_u(y_i, u_i) + \phi_{uy}(y_i, u_i)(y - y_i) + \phi_{uu}(y_i, u_i)(u - u_i) \\ f_3(y_i) &= d(y_i) + d_y(y_i)(y - y_i). \end{aligned}$$

We consider only the Lipschitz condition for f_3 , the rest follows analogously. Using the triangle inequality, we obtain

$$\begin{aligned} \|f_3(y_1) - f_3(y_2)\|_{L^2(\Omega)} &\leq \|d(y_1) - d(y_2)\|_{L^2(\Omega)} + \|d_y(y_1)(y_2 - y_1)\|_{L^2(\Omega)} \\ &\quad + \|(d_y(y_1) - d_y(y_2))(y - y_2)\|_{L^2(\Omega)} \\ &\leq \|d(y_1) - d(y_2)\|_{L^2(\Omega)} + \|d_y(y_1)\|_{L^\infty(\Omega)} \|y_2 - y_1\|_{L^2(\Omega)} \\ &\quad + \|d_y(y_1) - d_y(y_2)\|_{L^\infty(\Omega)} \|y - y_2\|_{L^2(\Omega)}. \end{aligned}$$

The properties of d , see Lemma A.1, imply that $\|d_y(y_1)\|_{L^\infty(\Omega)}$ is uniformly bounded for all $y_1 \in B_{r_3}^\infty(y^*)$. Moreover, $\|y - y_2\|_{L^2(\Omega)} \leq \|y - y^*\|_{L^2(\Omega)} + \|y^* - y_2\|_{L^2(\Omega)} \leq c(r_3 + r_4)$ holds. Together with the Lipschitz properties of d and d_y , see again Lemma A.1, we obtain

$$\|f_3(y_1) - f_3(y_2)\|_{L^2(\Omega)} \leq L \|y_1 - y_2\|_{L^\infty(\Omega)}$$

for some constant $L > 0$. \square

Using Theorem 6.4 and Lemma 6.6, the main result of this section follows directly from Dontchev's implicit function theorem [5, Theorem 2.1]:

Theorem 6.7. *There exist radii $r_5 > 0$, $r_6 > 0$ such that for any parameter $\eta \in B_{r_5}^W(w^*)$, there exists a solution $w(\eta) \in B_{r_6}^W(w^*)$ of (6.1), which is unique in this neighborhood. Moreover, there exists a constant $L_\eta > 0$ such that for each $\eta_1, \eta_2 \in B_{r_5}^W(w^*)$, the Lipschitz estimate*

$$\|w(\eta_1) - w(\eta_2)\|_W \leq L_\eta \|\eta_1 - \eta_2\|_W$$

holds.

7. LOCAL CONVERGENCE ANALYSIS OF SQP

This section is devoted to the local quadratic convergence analysis of the SQP method. As was shown in Section 5, the SQP method is equivalent to Newton's method (5.5), applied to the generalized equation (5.4). It is convenient to carry out the convergence analysis on the level of generalized equations. As mentioned in the previous section, the key property is the local Lipschitz stability of solutions $w(\eta)$ of (6.1) and $w(\delta)$ of (6.2), as proved in Theorems 6.7 and 6.4, respectively. In the proof of our main result, the iterates w^k are considered perturbations of the

solution w^* of (5.4) and play the role of the parameter η . We recall the function spaces

$$\begin{aligned} W &:= Y \times L^\infty(\Omega) \times Y \times L^\infty(\Omega) \times L^\infty(\Omega) \\ Y &:= H^2(\Omega) \cap H_0^1(\Omega) \\ Z &:= L^2(\Omega) \times L^\infty(\Omega) \times L^2(\Omega) \times L^\infty(\Omega) \times L^\infty(\Omega) \end{aligned}$$

Theorem 7.1. *There exists a radius $r > 0$ and a constant $C_{SQP} > 0$ such that for each starting point $w^0 \in B_r^W(w^*)$, the sequence of iterates w^k generated by (5.5) is well-defined in $B_r^W(w^*)$ and satisfy*

$$\|w^{k+1} - w^*\|_W \leq C_{SQP} \|w^k - w^*\|_W^2.$$

Proof. Suppose that the iterate $w^k \in B_r^W(w^*)$ is given. The radius r satisfying $r_5 \geq r > 0$ will be specified below. From Theorem 6.7, we infer the existence of a solution w^{k+1} of (5.5) which is unique in $B_{r_6}^W(w^*)$. That is, we have

$$0 \in F(w^*) + F'(w^*)(w^* - w^*) + N(w^*), \quad (7.1a)$$

$$0 \in F(w^k) + F'(w^k)(w^{k+1} - w^k) + N(w^{k+1}). \quad (7.1b)$$

Adding and subtracting the terms $F'(w^*)(w^{k+1} - w^*)$ and $F(w^*)$ to (7.1b), we obtain

$$\delta^{k+1} \in F(w^*) + F'(w^*)(w^{k+1} - w^*) + N(w^{k+1}) \quad (7.2)$$

where

$$\delta^{k+1} := F(w^*) - F(w^k) + F'(w^*)(w^{k+1} - w^*) - F'(w^k)(w^{k+1} - w^k).$$

From Lemma 6.6 with $\eta_1 := w^*$, $\eta_2 := w^k$, $w := w^{k+1}$, and $r_3 := r_5$, $r_4 := r_6$, we get

$$\|\delta^{k+1}\|_Z \leq L \|w^k - w^*\|_W < Lr, \quad (7.3)$$

where L depends only on the radii. That is, $\|\delta^{k+1}\|_Z \leq G\sigma$ holds whenever

$$r \leq \frac{G\sigma}{L},$$

which we impose on r .

Lemma 6.2 shows that (7.1a) and (7.2) are equivalent to problem $(\mathbf{LQP}(\delta))$ for $\delta = 0$ and $\delta = \delta^{k+1}$, respectively. From Theorem 6.4, we thus obtain

$$\|w^{k+1} - w^*\|_W \leq L_\delta \|\delta^{k+1} - 0\|_Z. \quad (7.4)$$

It remains to verify that $\|\delta^{k+1}\|_Z$ is quadratic in $\|w^k - w^*\|_W$. We estimate

$$\begin{aligned} \|\delta^{k+1}\|_Z &\leq \|F(w^*) - F(w^k) - F'(w^k)(w^* - w^k)\|_Z \\ &\quad + \|(F'(w^*) - F'(w^k))(w^{k+1} - w^*)\|_Z. \end{aligned}$$

As in the proof of Theorem 3.4, the first term is bounded by a constant times $\|w^k - w^*\|_{[L^\infty(\Omega)]^5}^2$. Moreover, the Lipschitz properties of the terms in F' imply that the second term is bounded by a constant times $\|w^k - w^*\|_{[L^\infty(\Omega)]^5} \|w^{k+1} - w^*\|_{[L^2(\Omega)]^5}$. We thus conclude

$$\|\delta^{k+1}\|_Z \leq c_1 \|w^k - w^*\|_W^2 + c_2 \|w^k - w^*\|_W \|w^{k+1} - w^*\|_W, \quad (7.5)$$

where the constants depend only on the radius r_5 . We finally choose r as

$$r = \min \left\{ r_5, \frac{G\sigma}{L}, \frac{1}{L_\delta \max\{2c_2, c_1 + c_2 L_\delta L\}} \right\}.$$

Then (7.3)–(7.5) imply $w^{k+1} \in B_r^W(w^*)$ since

$$\begin{aligned} \|w^{k+1} - w^*\|_W &< L_\delta [c_1 r + c_2 \|w^{k+1} - w^*\|_W] r \\ &\leq L_\delta [c_1 + c_2 L_\delta L] r^2 \leq r. \end{aligned}$$

Moreover, (7.4)–(7.5) yield

$$\|w^{k+1} - w^*\|_W \leq L_\delta c_1 \|w^k - w^*\|_W^2 + c_2 L_\delta r \|w^{k+1} - w^*\|_W$$

and thus

$$\|w^{k+1} - w^*\|_W \leq C_{SQP} \|w^k - w^*\|_W^2$$

holds with $C_{SQP} = \frac{L_\delta c_1}{1 - c_2 L_\delta r}$. \square

Clearly, Theorem 7.1 proves the local quadratic convergence of the SQP method. Recall that the iterates w^k are defined by means of Theorem 6.7, as the *local unique* solutions, Lagrange multipliers and adjoint states of (\mathbf{QP}_k) . Indeed, we can now prove that $w^{k+1} = (x^{k+1}, \lambda^{k+1})$ is *globally unique*, provided that w^k is already sufficiently close to w^* .

Corollary 7.2. *There exists a radius $r' > 0$ such that $w^k \in B_{r'}^W(w^*)$ implies that (\mathbf{QP}_k) has a unique global solution x^{k+1} . The associated Lagrange multipliers and adjoint state $\lambda^{k+1} = (\mu_1^{k+1}, \mu_2^{k+1}, p^{k+1})$ are also unique. The iterate w^{k+1} lies again in $B_{r'}^W(x^*, \lambda^*)$.*

Proof. We first observe that Theorem 7.1 remains valid (with the same constant C_{SQP}) if r is taken to be smaller than chosen in the proof. Here, we set

$$r' = \min \left\{ \sigma, \frac{\sigma}{c_\infty + \varepsilon}, R, r \right\},$$

where R and r are the radii from Lemma 5.1 and Theorem 7.1, respectively, and c_∞ is the embedding constant of $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$.

Suppose that $w^k \in B_{r'}^W(w^*)$ holds. Then Lemma 5.1 implies that (\mathbf{QP}_k) possesses a globally unique solution $x^{k+1} \in Y \times L^\infty(\Omega)$. The corresponding active sets are defined by

$$\begin{aligned} \mathcal{A}_1^{k+1} &:= \{\xi \in \Omega : u^{k+1}(\xi) = 0\} \\ \mathcal{A}_2^{k+1} &:= \{\xi \in \Omega : \varepsilon u^{k+1}(\xi) + y^{k+1}(\xi) - y_c(\xi) = 0\}. \end{aligned}$$

We show that $\mathcal{A}_1^{k+1} \subset S_1^\sigma$ and $\mathcal{A}_2^{k+1} \subset S_2^\sigma$. For almost every $\xi \in \mathcal{A}_1^{k+1}$, we have

$$u^*(\xi) = u^*(\xi) - u^{k+1}(\xi) \leq \|u^* - u^{k+1}\|_{L^\infty(\Omega)} \leq r' \leq \sigma,$$

since Theorem 7.1 implies that $w^{k+1} \in B_{r'}^W(w^*)$ and thus in particular $u^{k+1} \in B_{r'}^\infty(u^*)$. By the same argument, for almost every $\xi \in \mathcal{A}_2^{k+1}$ we obtain

$$\begin{aligned} y^*(\xi) + \varepsilon u^*(\xi) - y_c(\xi) &= y^*(\xi) + \varepsilon u^*(\xi) - y^{k+1}(\xi) - \varepsilon u^{k+1}(\xi) \\ &\leq \|y^* - y^{k+1}\|_{L^\infty(\Omega)} + \varepsilon \|u^* - u^{k+1}\|_{L^\infty(\Omega)} \\ &\leq (c_\infty + \varepsilon) r' \leq \sigma. \end{aligned}$$

Owing to Assumption (A6), the active sets \mathcal{A}_1^{k+1} and \mathcal{A}_2^{k+1} are disjoint, and one can show as in [4, Lemma 3.1] that the Lagrange multipliers μ_1^{k+1}, μ_2^{k+1} and adjoint state p^{k+1} are unique. \square

8. CONCLUSION

We have studied a class of distributed optimal control problems with semilinear elliptic state equation and a mixed control-state constraint as well as a pure control constraint on the domain Ω . We have assumed that (y^*, u^*) is a solution and (p^*, μ_1^*, μ_2^*) are Lagrange multipliers which satisfy second-order sufficient optimality conditions (A5). Moreover, the active sets at the solution were assumed to be well separated (A6). We have shown the local quadratic convergence of the SQP method towards this solution. In particular, we have proved that the quadratic subproblems possess global unique solutions and unique Lagrange multipliers.

APPENDIX A. AUXILIARY RESULTS

In this appendix we collect some auxiliary results. We begin with a standard result for the Nemyckii operators $d(\cdot)$ and $\phi(\cdot)$ whose proof can be found, e.g., in [18, Lemma 4.10, Satz 4.20]. Throughout, we impose Assumptions (A1)–(A5).

Lemma A.1. *The Nemyckii operator $d(\cdot)$ maps $L^\infty(\Omega)$ into $L^\infty(\Omega)$ and it is twice continuously differentiable in these spaces. For arbitrary $M > 0$, the Lipschitz condition*

$$\|d_{yy}(y_1) - d_{yy}(y_2)\|_{L^\infty(\Omega)} \leq L_d(M) \|y_1 - y_2\|_{L^\infty(\Omega)}$$

holds for all $y_i \in L^\infty(\Omega)$ such that $\|y_i\|_{L^\infty(\Omega)} \leq M$, $i = 1, 2$. In particular,

$$\|d_{yy}(y)\|_{L^\infty(\Omega)} \leq K_d + L_d(M) M$$

holds for all $y \in L^\infty(\Omega)$ such that $\|y\|_{L^\infty(\Omega)} \leq M$. The same properties, with different constants, are valid for $d_y(\cdot)$ and $d(\cdot)$. Analogous results hold for ϕ and its derivatives up to second-order, for all $(y, u) \in [L^\infty(\Omega)]^2$ such that $\|y_i\|_{L^\infty(\Omega)} + \|u_i\|_{L^\infty(\Omega)} \leq M$.

The remaining results address the coercivity of the second derivative of the Lagrangian, considered at different linearization points and for perturbed PDEs. Recall that $(x^*, \lambda^*) \in W$ satisfies the second-order sufficient conditions (SSC) with coercivity constant $\alpha > 0$, see (3.1).

Lemma A.2. *There exists $\varepsilon > 0$ and $\alpha' > 0$ such that*

$$\mathcal{L}_{xx}(x^*, \lambda^*)(x - x^*, x - x^*) \geq \alpha' \|x - x^*\|_{[L^2(\Omega)]^2}^2 \quad (\text{A.1})$$

holds for all $x = (y, u) \in Y \times L^\infty(\Omega)$ which satisfy the semilinear PDE (1.1) and $\|x - x^*\|_{[L^\infty(\Omega)]^2} \leq \varepsilon$.

Proof. Let $x = (y, u)$ satisfy (1.1). We define $\delta u = u - u^*$ and $\delta x = (\delta y, \delta u) \in Y \times L^\infty(\Omega)$ by

$$A \delta y + d_y(y^*) \delta y = \delta u \quad \text{on } \Omega$$

with homogeneous Dirichlet boundary conditions. Then the error $e := y^* - y - \delta y$ satisfies the linear PDE

$$A e + d_y(y^*) e = f \quad \text{on } \Omega \quad (\text{A.2})$$

with homogeneous Dirichlet boundary conditions and

$$f := d(y) - d(y^*) - d_y(y^*)(y - y^*).$$

We estimate

$$\begin{aligned} \|f\|_{L^2(\Omega)} &= \left\| \int_0^1 [d_y(y^* + s(y - y^*)) - d_y(y^*)] ds (y - y^*) \right\|_{L^2(\Omega)} \\ &\leq L \int_0^1 s ds \|y - y^*\|_{L^\infty(\Omega)} \|y - y^*\|_{L^2(\Omega)} \\ &\leq \frac{L}{2} \|y - y^*\|_{L^\infty(\Omega)} (\|\delta y\|_{L^2(\Omega)} + \|e\|_{L^2(\Omega)}). \end{aligned}$$

In view of Lemma A.1, $d_y(y^*) \in L^\infty(\Omega)$ holds and it is a standard result that the unique solution e of (A.2) satisfies an a priori estimate

$$\|e\|_{L^\infty(\Omega)} \leq c \|f\|_{L^2(\Omega)}.$$

In view of the embedding $L^\infty(\Omega) \hookrightarrow L^2(\Omega)$ we obtain

$$\|e\|_{L^2(\Omega)} \leq c' \frac{L\varepsilon}{2} (\|\delta y\|_{L^2(\Omega)} + \|e\|_{L^2(\Omega)}).$$

For sufficiently small $\varepsilon > 0$, we can absorb the last term in the left hand side and obtain

$$\|e\|_{L^2(\Omega)} \leq c''(\varepsilon) \|\delta y\|_{L^2(\Omega)}$$

where $c''(\varepsilon) \searrow 0$ as $\varepsilon \searrow 0$. A straightforward application of [9, Lemma 5.5] concludes the proof. \square

Lemma A.3. *There exists $R > 0$ and $\alpha'' > 0$ such that*

$$\mathcal{L}_{xx}(x^k, \lambda^k)(x, x) \geq \alpha'' \|x\|_{[L^2(\Omega)]^2}^2$$

holds for all $(y, u) \in Y \times L^2(\Omega)$:

$$\begin{aligned} Ay + d_y(y^k)y &= u && \text{in } \Omega \\ y &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{A.3}$$

provided that $\|x^k - x^\|_{[L^\infty(\Omega)]^2} + \|p^k - p^*\|_{L^\infty(\Omega)} < R$.*

Proof. Let (y, u) be an arbitrary pair satisfying (A.3) and define $\hat{y} \in Y$ as the unique solution of

$$\begin{aligned} A\hat{y} + d_y(y^*)\hat{y} &= u && \text{in } \Omega \\ \hat{y} &= 0 && \text{on } \partial\Omega, \end{aligned}$$

for the same control u as above. Then $\delta y := y - \hat{y}$ satisfies

$$A\delta y + d_y(y^*)\delta y = (d_y(y^*) - d_y(y^k))y \quad \text{in } \Omega$$

with homogeneous boundary conditions. A standard a priori estimate and the triangle inequality yield

$$\begin{aligned} \|\delta y\|_{L^2(\Omega)} &\leq \|d_y(y^*) - d_y(y^k)\|_{L^\infty(\Omega)} \|y\|_{L^2(\Omega)} \\ &\leq \|d_y(y^*) - d_y(y^k)\|_{L^\infty(\Omega)} (\|\hat{y}\|_{L^2(\Omega)} + \|\delta y\|_{L^2(\Omega)}). \end{aligned}$$

Due to the Lipschitz property of $d_y(\cdot)$ with respect to $L^\infty(\Omega)$, there exists a function $c(R)$ tending to 0 as $R \rightarrow 0$, such that $\|d_y(y^*) - d_y(y^k)\|_{L^\infty(\Omega)} \leq c(R)$, provided that $\|y^k - y^*\|_{L^\infty(\Omega)} < R$. For sufficiently small R , the term $\|\delta y\|_{L^2(\Omega)}$ can be absorbed in the left hand side, and we obtain

$$\|\delta y\|_{L^2(\Omega)} \leq c'(R) \|\hat{y}\|_{L^2(\Omega)},$$

where $c'(R)$ has the same property as $c(R)$. Again, [9, Lemma 5.5] implies that there exists $\alpha_0 > 0$ and $R > 0$ such that

$$\mathcal{L}_{xx}(x^*, \lambda^*)(x, x) \geq \alpha_0 \|x\|_{[L^2(\Omega)]^2}^2,$$

provided that $\|y^k - y^*\|_{L^\infty(\Omega)} < R$.

Note that \mathcal{L}_{xx} depends only on x and the adjoint state p . Owing to its Lipschitz property, we further conclude that

$$\begin{aligned} \mathcal{L}_{xx}(x^k, \lambda^k)(x, x) &= \mathcal{L}_{xx}(x^*, \lambda^*)(x, x) + [\mathcal{L}_{xx}(x^k, \lambda^k) - \mathcal{L}_{xx}(x^*, \lambda^*)](x, x) \\ &\geq \alpha_0 \|x\|_{[L^2(\Omega)]^2}^2 - L \|(x^k, p^k) - (x^*, p^*)\|_{[L^\infty(\Omega)]^3} \|x\|_{[L^2(\Omega)]^2}^2 \\ &\geq (\alpha_0 - LR) \|x\|_{[L^2(\Omega)]^2}^2 =: \alpha'' \|x\|_{[L^2(\Omega)]^2}^2, \end{aligned}$$

given that $(x^k, p^k) \in B_R^\infty(x^*, p^*)$. For sufficiently small R , we obtain $\alpha'' > 0$, which completes the proof. \square

ACKNOWLEDGEMENT

This work was supported by the Austrian Science Fund FWF under project number P18056-N12.

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