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Regularization by fractional filter methods and data smoothing

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Regularization by Fractional Filter Methods and Data Smoothing

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Abstract

This paper is concerned with the regularization of linear ill-posed problems by a combination of data smoothing and fractional filter methods. For the data smoothing, a wavelet shrinkage denoising is applied to the noisy data with known error level δ. For the reconstruction, an approximation to the solution of the operator equation is computed from the data estimate by fractional filter methods. These fractional methods are based on the classical Tikhonov and Landweber method but avoid at least partially the well-known drawback of oversmoothing. Convergence rates as well as numerical examples are presented.

1 Introduction

In this paper, we aim at solving the linear operator equation

$$Kf = g$$

from noisy data g^δ with known error level. We assume that K is a compact operator defined between Hilbert spaces X and Y. Due to compactness, the generalized inverse of K is unbounded and cannot be applied directly to g^δ with $\|g^\delta - g\|_Y \leq \delta$. A well-known approach to deal with such ill-posed problems is given by so-called *regularization methods*, see the textbooks [1, 2].

We start from regularization methods which can be interpreted as filtered versions of the generalized inverse, e.g. the Tikhonov method with L_2-penalty or the iterative Landweber method. In doing so, we have a twofold aim: on the one hand we want to explain and prevent the well-known effect of *oversmoothing* when applying the classical Tikhonov or Landweber method. This is achieved by a modification of the related filter functions. This modification allows to control the amount of damping and leads to *fractional filter methods*. On the other hand we want to combine these fractional filter methods with a data adapted pre-smoothing. The second aim belongs to the theory of two-step methods by which we understand the composition $R \circ S$ of a data smoothing operator S and a reconstruction operator R. The use of data smoothing for the problem of calculating unbounded operators has been examined e.g. in [3]. More recently the problem of denoising (regularization of the unit operator) has

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been studied in [4, 5, 6, 7, 8]. In [3, 4, 5, 6] the smoothing step yields an estimate which is in the range of the operator to be inverted/calculated. In [8] and also in the article at hand the smoothing step yields an estimate in any space in between the space containing the noisy data and the range of the operator to be inverted.

It is well-known that the Tikhonov and Landweber regularization methods oversmooth the solution, i.e., sharp or fine features of the reconstructed function are lost which is extremely troublesome in some medical applications, see [9, 10]. It seems that the effect of oversmoothing occurs whenever the adjoint operator $K^* : Y \to X$ is part of the reconstruction rule. This is the case for the classical Tikhonov and Landweber method, see (1) and (2). As a simple example we consider an operator defined in Sobolev scales. It is a well-known fact that for the Sobolev embedding operator $j : H^s \to L_2$ which smoothes with step size s, the adjoint operator j^* smoothes with step size $2s$, that means $j^* : L_2 \to H^{2s}$, see e.g. [6]. Hence, whenever the adjoint embedding operator is used, there is an additional amount of smoothing. Let us now consider an operator which is continuously invertible between Sobolev scales of stepsize t, i.e., $K : L_2 \leftrightarrow H^t$ but is compact as operator $K : L_2 \to H^r$ with $r < t$. The compact mapping $\tilde{K} : L_2 \to H^r$ can be written as $\tilde{K} = jK$ where only the Sobolev embedding operator $j : H^t \to H^r$ introduces the ill-posedness. But as for $\tilde{K} = jK$ the adjoint of the embedding operator is a part of $\tilde{K}^* = K^* j^*$, this leads to an oversmoothing for K^*. Both Tikhonov and Landweber regularization yield a regularized solution f_{reg} which belongs to $\text{rg}\,(K^*)$ (the range of K^*). The Tikhonov method defines a regularized solution f_α^δ with parameter α by

$$(K^*K + \alpha I) f_\alpha = K^* g^\delta. \tag{1}$$

Equation (1) can be solved for f_α^δ and rewritten as $f_\alpha^\delta = K^*(KK^* + \alpha I)^{-1} g^\delta \in \text{rg}\,(K^*)$. The iterative Landweber method defines a regularized solution f_m^δ by

$$f_m^\delta = \beta \sum_{j=0}^{m-1} (I - \beta K^*K)^j K^* g^\delta. \tag{2}$$

Rewriting equation (2) as $f_m = \sum_{j=0}^{m-1} (I - \beta K^*K)^j K^* g^\delta = \sum_{j=0}^{m-1} K^*(I - \beta KK^*)^j g^\delta$ it follows that also the Landweber method results in a regularized solution in $\text{rg}\,(K^*)$.

In order to introduce *fractional filter methods* we remind the reader of filter-based regularization operators

$$R_\alpha g := \sum_{\sigma_n > 0} F_\alpha(\sigma_n) \sigma_n^{-1} \langle g, v_n \rangle u_n. \tag{3}$$

with a suitable real-valued *filter function* F_α and the singular system $(\sigma_n; u_n, v_n)_n$ of a linear compact operator K. Conditions on the filter function, see (4) and (5), assure that the operator R_α defines an (order-optimal) regularization method.

We modify the filter functions of the Tikhonov and the Landweber method in order to control the amount of damping and to avoid oversmoothing. This is done by applying an exponent $\gamma \in [0, 1]$ to the filter function F_α which results in the *fractional filter function* and the corresponding *fractional filter methods* F_α^γ, see Definition 2.1. We study the fractional version of both the Tikhonov and the Landweber method and show that the filter function F_α^γ assures order optimality of the induced method as long as $\gamma > 1/2$, see Propositions 3.2 and 3.5. In order to use fractional methods with parameter $\gamma \le 1/2$, we use wavelet shrinkage as data-adapted pre-smoothing and prove order optimality of this two-step method.

One popular method of regularizing an inverse problem where the true solution is known to be a smooth function with a few jump discontinuities is to use a total variation (TV) penalty. The total variation regularization was introduced in image processing in [11] and successfully applied to inverse problems where 'blocky' reconstructions are desired [12, 13, 14].
Since the total variation functional is not differentiable, its numerical implementation presents a few challenges, see e.g. [15]. The approach considered here completely stays within the Hilbert space framework and avoids the non-differentiability problems encountered by TV methods.
Another approach to recover solutions which have discontinuities or are spatially inhomogeneous is to use suitable bases for the reconstruction. E.g. wavelet bases provide a good localization in time and space and are thus suitable for the reconstructions of functions with spatially inhomogeneous smoothness properties. Since Besov spaces and wavelet bases are closely related via norm equivalences this can be realized by using a Besov penalty term in Tikhonov regularization. In [16] the regularization of a linear operator with penalty $\|\cdot\|_{B^s_{p,p}}$ was considered. The minimization of the related functional requires the solution of a system of coupled nonlinear equations which is rather hard to tackle. In [16] the original functional was replaced by a sequence of so-called *Surrogate functionals* that are much easier to minimize. The approach considered here only needs the computation of fractional powers of operators (or matrices once the problem is discretized) which can be done and is done by a simple series expansion.
The paper is organized as follows. In Section 2 we give a brief overview on standard theory for filter-based regularization methods and define the fraction methods. In Section 3 we prove that the fractional Tikhonov and the fractional Landweber method with parameter $\gamma > 1/2$ are order optimal. In Section 4 we consider the combination of wavelet shrinkage and fractional methods with $\gamma \leq 1/2$ and prove the order optimality of this method. In Section 5 we present some numerical results for the fractional and the combined methods.

2 Filter methods – standard and fractional

In this section we give a brief overview of standard filter-based regularization methods, see the textbooks [1, 2] for details. We introduce the concept of fractional filter methods and present an example to illustrate the effect of regularization methods based on fractional filter functions.

2.1 Standard regularization results

By a *regularization* or *regularization method* for $K^\dagger$ (the generalized inverse of K) we understand any family of operators

$$\{R_\alpha\}_{\alpha>0}, R_\alpha : Y \to X$$

with the following properties. There exists a mapping $\alpha : \mathbb{R}_+ \times Y \to \mathbb{R}_+$ such that for all $g \in \mathcal{D}(K^\dagger)$ and all $g^\delta \in Y$ with $\|g - g^\delta\| \leq \delta$ it is

$$\lim_{\delta\to 0, g^\delta \to g} R_{\alpha(\delta,g^\delta)} g^\delta = K^\dagger g \ .$$

Filter-based regularization methods are defined by (3). The Tikhonov method can be written in the form (3) with filter function

$$F_\alpha(\sigma) = \frac{\sigma^2}{\sigma^2+\alpha}.$$

The same is true for the Landweber method with filter function

$$F_m(\sigma) = 1-(1-\beta\sigma^2)^m$$

where $0 < \beta < \frac{2}{\|K\|^2}$. For the Landweber method with index $m \in \mathbb{N}$ the regularization parameter is $\alpha \sim 1/m$.
The quality of a regularization method is judged by the error asymptotics of $\|K^\dagger g - R_\alpha g^\delta\|_X$. Convergence rates are achieved under the assumption that the exact solution $f^\dagger$ fulfills a smoothness condition of the form

$$f^\dagger \in \operatorname{rg}\left((K^*K)^{\nu/2}\right) \quad \text{with} \quad \|f^\dagger\|_\nu = \Big(\sum_n \sigma_n^{-2\nu}|\langle f^\dagger, u_n\rangle|^2\Big)^{1/2} \leq \rho\ .$$

A regularization method is called *order optimal* if there is a constant c such that

$$\|f^\dagger - R_\alpha g^\delta\|_X \leq c\delta^{\frac{\nu}{\nu+1}} \cdot \rho^{\frac{1}{\nu+1}}\ .$$

Both the Tikhonov and the Landweber method are of optimal order.
Whether a filter-based method is a regularization method at all can be checked by the following conditions, (4a)-(4c).

$$\sup_n |F_\alpha(\sigma_n)\sigma_n^{-1}| = c(\alpha) < \infty, \tag{4a}$$

$$\lim_{\alpha\to 0} F_\alpha(\sigma_n) = 1 \quad \text{pointwise in } \sigma_n, \tag{4b}$$

$$|F_\alpha(\sigma_n)| \leq c \quad \forall \alpha, \sigma_n. \tag{4c}$$

The order optimality is assured if there are constants $\beta > 0$ and c, c_ν such that

$$\sup_{0<\sigma\leq\sigma_1} |F_\alpha(\sigma)\sigma^{-1}| \leq c\alpha^{-\beta}, \tag{5a}$$

$$\sup_{0<\sigma\leq\sigma_1} |(1-F_\alpha(\sigma))\sigma^{\nu^*}| \leq c_{\nu^*}\alpha^{\beta\nu^*}. \tag{5b}$$

2.2 Fractional regularization methods

In this section we define the *fractional filter function* and the *fractional filter operator*. We further introduce a variant of condition (5a) necessary for dealing with the fractional methods. Then we illustrate the effect of regularization methods based on fractional filter functions by a simple signal processing example.

Definition 2.1. *Let $\gamma \in [0,1]$ and $F_\alpha : \mathbb{R}_+ \to \mathbb{R}$ denote a filter function. Then*

$$F_\alpha^\gamma(x) := (F_\alpha(x))^\gamma \tag{6}$$

is called fractional filter function with parameter γ.
For a given filter function F_α and $\gamma \in [0,1]$ the mapping $R_{\alpha,\gamma} : Y \to X$ with

$$R_{\alpha,\gamma} g = \sum_{\sigma_n>0} F_\alpha^\gamma(\sigma_n)\sigma_n^{-1}\langle g, v_n\rangle u_n \tag{7}$$

is called fractional filter operator with parameter γ.

In the following we assume that F_α is the classical Tikhonov or Landweber filter (with $\alpha \sim 1/m$). Accordingly, for $\gamma = 1$ the operator $R_{\alpha,\gamma}$ is the classical Tikhonov or Landweber regularization operator, whereas for $\gamma = 0$ the operator $R_{\alpha,\gamma}$ is the generalized inverse. In this sense, $R_{\alpha,\gamma}$ interpolates between these operators. The implementation of fractional methods involves the evaluation of terms like $(\alpha I + K^*K)^\gamma$. There are already some methods developed using powers of operators, see [17]. However, these methods start from the standard Tikhonov method and then more smoothing is applied.

In Section 3 we prove that as long as γ varies within $(1/2, 1]$, the fractional Tikhonov as well as the fractional Landweber filter define order optimal regularization methods. The main work has to be done in order to show that condition (5a) is fulfilled where the (fractional) filter function is weighted against σ^{-1}. So the question whether a fractional filter can be order optimal is mainly the question whether F_α^γ is able to control the growths of σ_n^{-1} as $n \to \infty$. To answer this question we introduce a variant of condition (5a): Let $\mu \in [0, 1]$ and $\gamma \in [0, 1]$, we will check which pairs (γ, μ) fulfill

$$\sup_{0<\sigma\leq\sigma_1} |F_\alpha^\gamma(\sigma)\sigma^{-\mu}| \leq c\alpha^{-\beta\mu} \ . \tag{5a'}$$

For $\mu = 1$ this coincides with the standard condition (5a). In Section 3 we will show that (5a) (and thus also (5a') with $\mu = 1$) is fulfilled as long as $\gamma > 1/2$.

For $\mu < 1$ the fractional filter function $F_\alpha^\gamma(\cdot)$ is weighted against $\sigma^{-\mu}$ and we will show in Section 3 that condition (5a') is fulfilled for every pair (γ, μ) with $\gamma > \mu/2$. So for $\mu < 1$ it is no longer necessary that γ is greater than 1/2 and thus more values of γ are admissible. To motivate the modification of the singular values from σ_n^{-1} in condition (5a) to $\sigma_n^{-\mu}$ in condition (5a'), we anticipate the basic idea of Section 4, which is the combination of data smoothing and regularization. First we apply a data smoothing operator $S_\lambda : Y \to \tilde{Y}$ to gain a better estimate $\tilde{g} = S_\lambda g^\delta$ of the exact data from the noisy data g^δ. Second we apply a regularization operator to $\tilde{g}$ to construct an approximation to the solution of $Kf = g$. For the data smoothing or data estimation operator many choices are possible. In Section 4 we will concentrate on wavelet shrinkage. For the moment we think of a *data smoothing operator* as an operator $S : Y \to \tilde{Y}$ which corrects the smoothness properties of the noisy data g^δ, meaning that the image space $\tilde{Y}$ of S should be closer to the range of K than the space Y of the noisy data. A measure of this closeness depends on the problem setting. For an operator K which smoothes with respect to Sobolev spaces, i.e., $K : H^s \to H^{s+t}$, and noisy data $g^\delta \in L_2 = H^0$ the space $\tilde{Y}$ could be any Sobolev space H^τ with $0 < \tau < t$. This kind of pre-smoothing of the data also renders the degree of ill-posedness of the problem, since we face the problem of solving $Kf = g$ from the smoother estimate $\tilde{g}$ instead of g^δ. In the modified condition (5a') this is reflected by the fact that the filter function $F_\alpha^\gamma(\sigma)$ is weighted against $\sigma^{-\mu}$ with $\mu \in [0, 1]$. This condition includes the special cases that no pre-smoothing at all is done ($\mu = 1$) and that the pre-smoothing results in an element of $\mathrm{rg}\,(K)$, ($\mu = 0$). In the last case no regularization is necessary any more and the generalized inverse can be applied directly. For these special cases we also refer to [4] and also to [3] where a smoothing family is used for the stable evaluation of unbounded operators.

The Sobolev embedding operator

We would like to illustrate the effect of regularization methods based on fractional filter functions by a very simple signal processing example. Let us assume that we want to reconstruct a function in H^s from its noisy measurements in L_2. A possible way of doing that is to consider the embedding operator $j_s : H^s \to L_2$. It is well known that the operator is compact in case of a bounded region Ω, and thus the inversion of j_s from noisy data is ill-posed. A way to compute a stable approximation to the solution in H^s is to use Tikhonov regularization, where the approximation of a solution of $j_s x = x$ is computed by solving

$$(j_s^* j_s + \alpha I)x = j_s^* x^\delta \ .$$

In our example we will consider a periodic setting, i.e. $\Omega = [0, 2\pi)$ and

$$x(t) = \sum x_k e^{ikt} \ ,$$

where x_k denote the Fourier coefficients of x. We can form Sobolev spaces H^s by

$$\langle x, y \rangle_s = \sum (1 + k^2)^s x_k y_k \ ,$$

and get $L_2 = H^0$. For the operator j_s we have the following decomposition, e.g. [6]:

Lemma 2.2. *Let $v_k = e^{ik\cdot}$, $u_k = (1 + k^2)^{-s/2} e^{ik\cdot}$, and $\sigma_k = (1 + k^2)^{-s/2}$. Then $\{\sigma_k^2, u_k\}$ forms the eigensystem of $j^* j$ and $\{\sigma_k, u_k, v_k\}$ is the singular system of j_s.*

The effect of the fractional Tikhonov filter is given in the following

Proposition 2.3. *For data $x^\delta \in L_2$, the approximation to the solution of $j_s x = x$ according to the fractional Tikhonov method with parameter γ belongs to $H^{2s\gamma}$.*

Proof. Using the definition of the fractional methods, we have

$$\begin{aligned} x_\alpha^\delta &= \sum_k \left(\frac{\sigma_k^2}{\sigma_k^2 + \alpha} \right)^\gamma \sigma_k^{-1} \langle x^\delta, v_k \rangle u_k \\ &= \sum_k \frac{(1 + k^2)^{-s(\gamma - 1/2)}}{((1 + k^2)^{-s} + \alpha)^\gamma} \langle x^\delta, v_k \rangle (1 + k^2)^{-s/2} e^{ik\cdot} \ . \end{aligned}$$

Thus, the Fourier coefficients of x_α^δ are given by

$$\left(x_\alpha^\delta \right)_k = \frac{(1 + k^2)^{-s\gamma}}{((1 + k^2)^{-s} + \alpha)^\gamma} \langle x^\delta, v_k \rangle \ ,$$

and the H^t -norm is given by

$$\begin{aligned} \|x_\alpha^\delta\|_t^2 &= \sum_k (1 + k^2)^t \frac{(1 + k^2)^{-2s\gamma}}{\left((1 + k^2)^{-s/2} + \alpha\right)^{2\gamma}} |\langle x^\delta, v_k \rangle|^2 \\ &\lesssim \sum_k (1 + k^2)^{t - 2s\gamma} |\langle x^\delta, v_k \rangle|^2 \ . \end{aligned}$$

The series is bounded for any $x^\delta \in L_2$ as long as $t \leq 2s\gamma$ holds. $\square$

In particular, for $\gamma = 1$ (standard Tikhonov) we have $x_\alpha^\delta \in H^{2s}$, i.e., the regularized solution is much smoother than the true solution. This reflects the typical oversmoothing behaviour of the classical Tikhonov method. For $\gamma = 1/2$ both functions have the same smoothness properties, which is desirable in many applications. E.g., assume that the solution is in a Sobolev space H^s that still contains functions with jumps, but functions in H^{2s} are continuous. Then a reconstruction with a fractional method will still allow jumps in the reconstruction, whereas classical methods will always reconstruct continuous approximations. A numerical example for this is given in Section 5.
On the other hand, it can be seen from Proposition 2.3 that the fractional Tikhonov method with $\gamma < 1/2$ cannot be a regularization method at all, as it constructs approximations which has less smoothness than the solution, and thus they cannot converge in the H^s-norm for $\delta \to 0$.
In Section 4 we deal with the parameter range $\gamma < 1/2$. As mentioned above, the fractional Tikhonov method with $\gamma < 1/2$ cannot be a regularization method *on its own*. But together with a pre-smoothing of the data it can work as a proper reconstruction method. With this combination it is possible to do the necessary amount of regularization (but no more than this) and to weigh the influence on the data and the reconstruction part. As we will use noise adapted smoothing methods, also information about the noise structure enters the solution scheme for the ill-posed problem. And consequently, we stay as close to the problem as possible. The numerical performance of this combined method is demonstrated by an example computation in Section 5.
Proposition 2.3 is generalized to

Proposition 2.4. *For data $x^\delta \in L_2$, let x_α^δ be the approximation to the solution of $j_s x = x$ defined by*

$$x_\alpha^\delta = \sum_k F_\alpha(\sigma_k)\sigma_k^{-1}\langle x^\delta, v_k\rangle u_k \ ,$$

where F_α is any filter function defined on the spectrum of the operator j_s. If

$$\sup_{0<\sigma\le\sigma_1} |\sigma^{-t/s} F_\alpha(\sigma)| \le c = c(\alpha, t, s) \tag{8}$$

then x_α^δ belongs to H^t.

Proof. From Lemma 2.2 we know $u_k = \sigma_k e^{ik\cdot}$ for the singular system of j_s. It is

$$x_\alpha^\delta = \sum_k F_\alpha(\sigma_k)\sigma_k^{-1}\langle x^\delta, v_k\rangle u_k = \sum_k F_\alpha(\sigma_k)\langle x^\delta, v_k\rangle e^{ik\cdot} \ .$$

Thus, the Fourier coefficients of x_α^δ are given by $\left(x_\alpha^\delta\right)_k = F_\alpha(\sigma_k)\langle x^\delta, v_k\rangle$. It is

$$\begin{aligned}\|x_\alpha^\delta\|_t^2 &= \sum_k (1+k^2)^t F_\alpha^2(\sigma_k)|\langle x^\delta, v_k\rangle|^2 = \sum_k \sigma_k^{-2t/s} F_\alpha^2(\sigma_k)|\langle x^\delta, v_k\rangle|^2 \\ &\le \left(\sup_{0<\sigma\le\sigma_1} |\sigma^{-t/s}F_\alpha(\sigma)|\right)^2 \sum_k |\langle x^\delta, v_k\rangle|^2 \ .\end{aligned}$$

With (8) it follows

$$\|x_\alpha^\delta\|_t \le c(\alpha, t, s)\|x^\delta\|_Y < \infty \ .$$

□

For the following corollary we anticipate two results of the next section.

Corollary 2.5. *For data $x^\delta \in L_2$, let $x^\delta_{\alpha,\gamma}$ be the approximation to the solution of $j_s x = x$ by either the fractional Tikhonov or the fractional Landweber method (for the iterative Landweber method the regularization parameter is $\alpha \sim 1/m$). Then, $x^\delta_{\alpha,\gamma}$ belongs to H^t as long as $t < 2s\gamma$.*

Proof. With $\mu = t/s$ condition (8) of Proposition 2.4 reads as

$$\sup_{0<\sigma\leq\sigma_1} \sigma^{-\mu} F_\alpha^\gamma(\sigma) \leq c(\alpha,\gamma,\mu) \ .$$

According to Lemma 3.1 and Lemma 3.4 the fractional Tikhonov as well as the fractional Landweber method fulfill this condition as long as $\gamma > \mu/2$. □

We want to remark that this result differs slightly from Proposition 2.3. The approximation is only in H^t for $t < 2s\gamma$ but not for $t = 2s\gamma$. This is due to the fact that the modified filter condition (5a') used in the last proof still allows a rate for the corresponding regularization method whereas the proof of Proposition 2.3 only requires that the supremum exists.

3 Order optimality of fractional filter methods

This section presents two of the main results of the paper. In Propositions 3.2 and 3.5 we prove that the fractional Tikhonov and the fractional Landweber methods are order optimal for all parameters γ in $(1/2, 1]$. The proofs are by straightforward calculation and are presented separately for the fractional Tikhonov and the fractional Landweber method.

3.1 The fractional Tikhonov method

Lemma 3.1. *Let $\gamma \in [0,1]$, $\mu \in [0,1]$ and F_α^γ be the fractional Tikhonov filter. For $\gamma > \mu/2$ it is*

$$\sup_{0<\sigma\leq\sigma_1} |F_\alpha^\gamma(\sigma)\sigma^{-\mu}| \leq c_{\gamma,\mu}\alpha^{-\mu/2}$$

and the fractional Tikhonov filter fulfills (5a') *with $\beta = 1/2$.*

Proof. We define

$$\varphi(\sigma,\gamma,\mu) := \sigma^{-2\mu} F_\alpha^{2\gamma}(\sigma) = \sigma^{-2\mu}\left(\frac{\sigma^2}{\sigma^2+\alpha}\right)^{2\gamma} \ .$$

The statement follows by maximizing φ with respect to σ. It is

$$\varphi'(\sigma) = [(2\gamma-\mu)(\sigma^2+\alpha) - 2\sigma^2\gamma] \cdot h(\sigma)$$

with a function $h \neq 0$. Hence we get as condition for critical points $\sigma^2 = \alpha(2\gamma-\mu)/\mu$. Existence is assured by $2\gamma > \mu$. For $2\gamma > \mu$ the function φ is continuous. Since $\varphi \geq 0$, $\varphi(0) = 0$ and $\lim_{\sigma\to\infty}\varphi(\sigma) = 0$ we get the maximum point $\sigma_*(\gamma,\mu) = \sqrt{\alpha}\sqrt{(2\gamma-\mu)/\mu}$. Inserting σ_* in φ yields the maximum

$$\varphi(\sigma) \leq \varphi(\sigma_*) = \alpha^{-\mu}\tilde{c}_{\gamma,\mu}$$

with the constant

$$\tilde{c}_{\gamma,\mu} = \left(\frac{2\gamma-\mu}{\mu}\right)^{2\gamma-\mu} \cdot \left(\frac{\mu}{2\gamma}\right)^{2\gamma}$$

depending on γ and μ. □

The order optimality of the fractional Tikhonov method with parameter $\gamma \in (1/2, 1]$ is given in the following

Proposition 3.2. *Let $K : X \to Y$ be a compact operator with singular system $(\sigma_n, u_n, v_n)_{n\geq 0}$. Let the exact solution $f^\dagger$ of $Kf = g$ fulfill $\|f^\dagger\|_\nu \leq \rho$. Then for $\gamma \in (1/2, 1]$ the fractional Tikhonov method with the parameter choice*

$$\alpha = \kappa \left(\frac{\delta}{\rho}\right)^{1/2(\nu+1)} , \quad \kappa > 0 \text{ constant}$$

is order optimal for all $0 < \nu < \nu^* = 2$.

Proof. The filter function F_α^γ of the fractional Tikhonov method fulfills conditions (4b) and (4c) with constant $c = 1$. Setting $\mu = 1$ in Lemma 3.1 we get

$$\sup_{0<\sigma\leq\sigma_1} |F_\alpha^\gamma(\sigma)\sigma^{-1}| \leq c\alpha^{-1/2}$$

which is (5a) with $\beta = 1/2$. Hence also (4a) is fulfilled.
It remains to show that the filter F_α^γ fulfills condition (5b). For $\sigma > 0$ we have $0 < F_\alpha(\sigma) = \frac{\sigma^2}{\sigma^2+\alpha} \leq 1$ and thus $F_\alpha^\gamma(\sigma) \geq F_\alpha(\sigma)$ for $\gamma > 0$. Hence

$$(1 - F_\alpha^\gamma(\sigma))\sigma^{\nu^*} \leq (1 - F_\alpha(\sigma))\sigma^{\nu^*}$$

and (5b) follows from the optimality of the classical Tikhonov filter with $\nu^* = 2$. □

Figure 1 (left) shows $\sigma^{-1}F_\alpha^\gamma(\sigma)$ for different values of γ: For $\gamma > 1/2$ a maximum exists for $\sigma \neq 0$, for $\gamma = 1/2$ the term stays bounded and for $\gamma < 1/2$ the influence of σ^{-1} is too strong to be controlled by $F_\alpha^\gamma(\sigma)$.

3.2 The fractional Landweber method

In this section we deal with the fractional Landweber filter,

$$F_m^\gamma(\sigma) = (1 - (1 - \beta_{\text{LW}}\sigma^2)^m)^\gamma ,$$

where the index "LW" is introduced to avoid confusion with the parameter β of the filter condition (5a). We start with an auxiliary result.

Lemma 3.3. *Let $\mu > 0$ and $m \in \mathbb{N}_+$. For $\gamma > \mu/2$ the function*

$$\phi(\tau) = \tau^{-2\mu}[1 - (1 - \tau^2)^m]^{2\gamma}$$

is continuous in $[0, \infty)$. For $\gamma > \mu/2$ and $m \geq 2$ the function ϕ restricted to $[0, \sqrt{2}]$ has a maximum and is bounded by

$$\phi(\tau) \leq m^\mu. \tag{9}$$

For $m = 1$ it is $\phi(\tau) \leq \phi(\sqrt{2}) = 2^{2\gamma-\mu}$.

The proof is straightforward, but long and technical; see the appendix.

Lemma 3.4. *Let $\gamma \in [0,1]$, $\mu \in [0,1]$ and F_m^γ be the fractional Landweber filter. For $\gamma > \mu/2$ it is*

$$\sup_{0<\sigma\leq\sigma_1} |F_m^\gamma(\sigma)\sigma^{-\mu}| \leq \beta_{\mathrm{LW}}^{\mu/2} m^{\mu/2}$$

and the fractional Landweber filter fulfills condition (5a') *with $\beta = 1/2$.*

Proof. We define

$$\varphi(\sigma,\gamma,\mu) := \sigma^{-2\mu} F_m^{2\gamma}(\sigma) = \sigma^{-2\mu}[1-(1-\beta_{\mathrm{LW}}\sigma^2)^m]^{2\gamma}$$

and

$$\tilde{\varphi}(\tau) = \tau^{-2\mu}[1-(1-\tau^2)^m]^{2\gamma}\,.$$

With $\tau^2 := \beta_{\mathrm{LW}}\sigma^2$ it is $\varphi(\sigma) = \beta_{\mathrm{LW}}\tilde{\varphi}(\tau)$. Since $\beta_{\mathrm{LW}} < \frac{2}{\|K\|^2} = \frac{2}{\sigma_1^2}$ we maximize $\tilde{\varphi}$ on the interval $[0,\sqrt{2}]$. For $\gamma > \mu/2$ Lemma 3.3 yields

$$\tilde{\varphi}(\tau) \leq m^\mu.$$

With the factor β_{LW}^μ this results in $\sup_{0<\sigma\leq\sigma_1} \varphi(\sigma,\gamma,\mu) \leq \beta_{\mathrm{LW}}^\mu m^\mu$. □

The order optimality of the fractional Landweber method with parameter $\gamma \in (1/2,1]$ is given in the following

Proposition 3.5. *Let all conditions of Proposition 3.2 be satisfied. Then for $0 < \beta_{\mathrm{LW}} < \frac{2}{\|K\|^2}$ and for $\gamma \in (1/2,1]$ the fractional Landweber method is a regularization method. It is order optimal for all $\nu > 0$ if the iteration is stopped for*

$$m = \left\lfloor \left(\frac{\nu^2}{\beta_{\mathrm{LW}}}\right)^{(\nu+1)} \left(2\frac{\beta_{\mathrm{LW}}}{\nu}e\right)^{-\nu/(\nu+1)} \left(\frac{\rho}{\delta}\right)^{2/(\nu+1)} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to x.

Proof. For $\sigma \in (0,\sigma_1]$ the restriction $0 < \beta_{\mathrm{LW}} < \frac{2}{\|K\|}$ yields $-1 < 1-\beta_{\mathrm{LW}}\sigma^2 < 1$ and conditions (4b) and (4c) follow immediately. Inserting $\mu = 1$ in Lemma 3.4 yields

$$\sup_{0<\sigma\leq\sigma_1} |F_m^\gamma(\sigma)\sigma^{-1}| \leq \beta_{\mathrm{LW}}^{1/2} m^{1/2}\,.$$

Since for the iterative Landweber method the regularization parameter is $\alpha = 1/m$ this is condition (5a) with $\beta = 1/2$. From this also (4a) follows. For condition (5b) we have to estimate

$$\sup_{0<\sigma\leq\sigma_1} |(1-[1-(1-\beta_{\mathrm{LW}}\sigma^2)^m]^\gamma)\sigma^{\nu^*}|.$$

With $x = 1-\beta_{\mathrm{LW}}\sigma^2$ it is $|1-[1-(1-\beta_{\mathrm{LW}}\sigma^2)^m]^\gamma| = |1-(1-x^m)^\gamma|$. Since $0 < \beta_{\mathrm{LW}} < \frac{2}{\sigma_1^2}$ it follows that $|x| \leq 1$. Since for $m > 0$, $\gamma \in [0,1]$ and $-1 \leq x \leq 1$ it is $|1-(1-x^m)^\gamma| \leq |x|^m$, the filter error of the reduced Landweber method is bounded by the filter error of the classical Landweber method,

$$\sup_{0<\sigma\leq\sigma_1} |(1-[1-(1-\beta_{\mathrm{LW}}\sigma^2)^m]^\gamma)\sigma^{\nu^*}| \leq \sup_{0<\sigma\leq\sigma_1} |(1-\beta_{\mathrm{LW}}\sigma^2)^m||\sigma^{\nu^*}|.$$

Hence condition (5b) follows from the optimality of the classical Landweber filter for all $\nu > 0$. □

Figure 1 (right) shows $\sigma^{-1}F_\alpha^\gamma(\sigma)$ for different values of γ: For $\gamma > 1/2$ a maximum exists for $\sigma \neq 0$, for $\gamma = 1/2$ the term stays bounded and for $\gamma < 1/2$ the influence of σ^{-1} is too strong to be controlled by $F_\alpha^\gamma(\sigma)$.

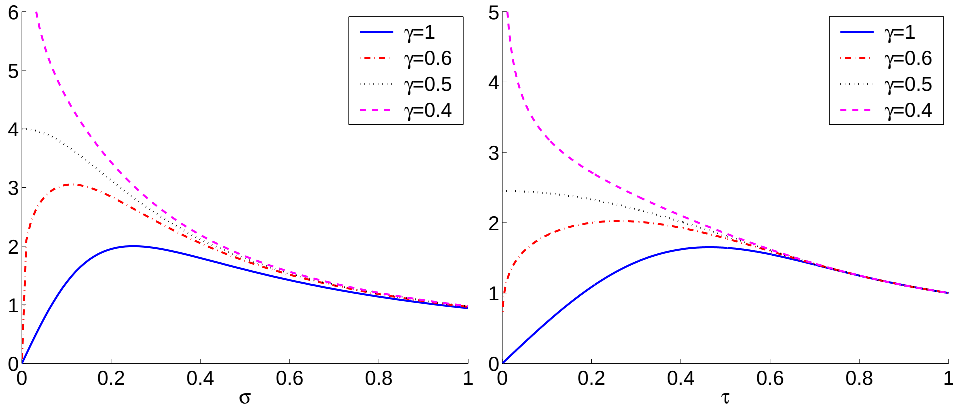


Figure 1: $F_\alpha^\gamma(\sigma)\sigma^{-1}$ for different values of γ; Tikhonov filter (left) and Landweber filter (right).

4 Data smoothing

In this section we consider two-step methods consisting of a data-smoothing step and a reconstruction step, see [5, 8, 7] for some recent work on two-step methods. Given an ill-posed problem $K : X \to Y$ and noisy data $g^\delta \in Y$, we start with a smoothing operator $S_\lambda : Y \to \tilde{Y}$ to get a better estimate $\tilde{g} \in \tilde{Y}$ of the noise-free data. This data estimate is then used as the input for a reconstruction operator $R_\alpha : \tilde{Y} \to X$. The second operator does not need to be a regularization operator by itself, e.g., we will use fractional filter methods with parameter $\gamma \leq 1/2$.
We summarize the concept of two-step methods as follows

$$Y \xrightarrow{\text{smoothing by } S_\lambda} \tilde{Y} \xrightarrow{\text{reconstruction by } R_\alpha} X \ .$$

The space $\tilde{Y}$ can be any space between the data space Y and the range of the operator, $\mathrm{rg}\,(K) \subset \tilde{Y} \subset Y$. This is different from other works, in [3, 4, 5, 6] data smoothing steps have been used which map into the range of the operator.
In order to illustrate the effect of pre-smoothing the data on fractional filter methods we return to the example of the Sobolev embedding operator .

4.1 The Sobolev embedding operator

For the Sobolev embedding operator, $j_s : H^s \to L_2$, we consider a data-smoothing operator S_λ with

$$S_\lambda : L_2 \to H^\eta \quad \text{with} \quad 0 \leq \eta \leq s \ .$$

The operator S_λ is applied to the noisy data $x^\delta \in L_2$ and gives a data estimate $x_\lambda^\delta := S_\lambda(x^\delta)$ in H^η. This data estimate is used as input for a reconstruction method R which we choose to be a fractional filter method $R_{\alpha,\gamma}$. Hence, in the general context of two-step methods we fix the spaces Y and $\tilde{Y}$ to be L_2 and H^η with $0 \le \eta \le s$ and get

$$L_2 \xrightarrow{\text{smoothing by } S_\lambda} H^\eta \xrightarrow{\text{reconstruction by } R_{\alpha,\gamma}} X \ .$$

For the space X we choose the Sobolev space $X = H^t$ with $\eta \le t \le s$. We are now able to extend the result of Proposition 2.4 to two-step methods $S_\lambda R_{\alpha,\gamma}$ with fractional filter reconstruction operator.

Proposition 4.1. *For data $x^\delta \in L_2$ and a data estimate $x^\delta_\lambda \in H^\eta$ with $0 \le \eta \le s$ let $x^\delta_{\alpha,\gamma,\lambda}$ be the minimizer of the Tikhonov or the Landweber method with fractional filter $F^\gamma_\alpha(\cdot)$ and operator j_s. Then $x^\delta_{\alpha,\gamma,\lambda}$ belongs to H^t for every pair (γ, η) with*

$$\eta > t - 2\gamma s \qquad \text{or} \qquad \gamma > (t-\eta)/2s \ . \tag{10}$$

Proof. The Fourier coefficients of

$$x^\delta_{\alpha,\gamma,\lambda} := R_{\alpha,\gamma} S_\lambda(x^\delta) = R_{\alpha,\gamma} x^\delta_\lambda$$

are given by

$$\left(x^\delta_{\alpha,\gamma,\lambda}\right)_k = F^\gamma_\alpha(\sigma_k)\langle x^\delta_\lambda, v_k\rangle \ .$$

Since x^δ_λ is in H^η we know

$$\|x^\delta_\lambda\|^2_\eta = \sum_k (1+k^2)^\eta |\langle x^\delta, v_k\rangle|^2 = \sum_k \sigma^{-2\eta/s} |\langle x^\delta, v_k\rangle|^2 < \infty \ .$$

Hence, for the H^t-norm we get

$$\begin{aligned} \|x^\delta_{\alpha,\gamma,\lambda}\|^2_t &= \sum_k \sigma_k^{-2t/s} \left| \left(x^\delta_{\alpha,\gamma,\lambda}\right)_k \right|^2 \\ &= \sum_k \sigma_k^{-2t/s} F^{2\gamma}_\alpha(\sigma_k) |\langle x^\delta_\lambda, v_k\rangle|^2 \sigma_k^{-2\eta/s} \sigma_k^{2\eta/s} \\ &\le \left(\sup_{0<\sigma\le\sigma_1} \sigma^{-\mu} F^\gamma_\alpha(\sigma)\right)^2 \|x^\delta_\lambda\|^2_\eta \end{aligned} \tag{11}$$

with $\mu = (t-\eta)/s$. From Lemma 3.1 and Lemma 3.4 we know that the supremum exists as long as $\gamma > \mu/2$. □

We want to make a few remarks on the proof. The supremum in (11) is in the form of condition (5a'). The exponent μ depends on η and takes into account a (possible) pre-smoothing of the data: the problem of computing a regularization from the smoothed data estimate is less ill-posed since the reconstruction does not have to deal with σ^{-1} but only with $\sigma^{-\mu}$ with $\mu \in [0, 1]$. For the special case $\eta = 0$ (no pre-smoothing), Proposition 4.1 reduces to Proposition 2.4 and condition (10) becomes $t < 2s\gamma$.

For $\eta \ne 0$, condition (10) allows to weight the two steps of smoothing and reconstruction: let us consider the case that the exact solution x is of smoothness s, hence, $x \in H^s$ and that the smoothing operator maps into the space H^η with $\eta = s/2$. We further assume that the reconstructed solution $x^\delta_{\alpha,\gamma,\lambda}$ should be as smooth as the exact x, i.e., $x^\delta_{\alpha,\gamma,\lambda} \in H^t$ with $t = s$. Then, condition (10) reads as

$$\gamma > \frac{t-\eta}{2s} = (s - s/2)/2s = 1/4 \ .$$

4.2 Convergence rates for two-step methods with fractional filter functions

In this section we present convergence rates for the combination of wavelet shrinkage as data smoothing operation and fractional filter methods as reconstruction operation for general linear operator equations. We aim at a combination of shrinkage and reconstruction that stays as close to the problem as possible. I.e., we neither want to apply too much shrinkage nor oversmooth the solution by using too much of the regularizing filter.

We start with a brief sketch of wavelet shrinkage, for a comprehensive treatment of wavelet analysis we refer the reader to [18, 19, 20, 21], for approximation results and details on wavelet shrinkage we refer to [22, 23, 24, 25, 26]. The basic idea of wavelet shrinkage is as follows: a (noisy) signal g^δ is transformed into a series expansion with respect to an (orthogonal) wavelet basis. Then the wavelet coefficients are filtered and from these filtered coefficients an approximation $\tilde{g}$ to g^δ (and hopefully to the exact g) is obtained by the inverse wavelet transform. Filtering in the wavelet domain can be done in many different ways. A linear filtering is done by cutting off the wavelet expansion which is equivalent to a projection on some wavelet subspace V_j. A nonlinear filtering is done by thresholding the coefficients: The overall assumption is that a signal has some structure (e.g. smooth parts, patterns in varying sizes, etc.) whereas noise, especially white noise, has no structure at all. In computing the wavelet representation of the signal, its structure is recognized and coded as few but very large wavelet coefficients whereas the noise just remains noise and is coded as many but very small coefficients. Thus, by keeping the large coefficients and throwing away the small ones, there is a good chance to keep the signal and to eliminate the noise; this procedure is called *hard shrinkage*. Continuing this idea by subtracting also a small amount of the large coefficients (they also carry noise), directly leads to *soft shrinkage*.

The combination of wavelet shrinkage and fractional filter methods generalizes [8, Theorem 4.5]. The class of admissible problems used in [8] and also in here is characterized as follows. We consider a linear compact operator $K : L_2 \to L_2$ with smoothing property $t > 0$ with respect to Sobolev and Besov spaces, i.e., $K : H^\tau \to H^{\tau+t}$ and $K : B^\tau_{pp} \to B^{\tau+t}_{pp}$ for all $\tau \geq 0$. Given a solution $f \in H^s \cap B^s_{pp}$, an approximation from noisy data $g^\delta = g + \delta \mathrm{d}W$ with $\mathrm{d}W$ a white noise process, is constructed by a two-step method: first, a smoothing operator S_λ given by nonlinear wavelet shrinkage and the linear projection operator P_j on a wavelet subspace with an orthonormal wavelet basis in H^η is applied. Second, a reconstruction operator R_α is applied. Then for properly chosen parameters (threshold λ, projection level j, regularization parameter α) the following quasi-optimal convergence rate is achieved [8, (4.6)]

$$E(\|R_\alpha S_\lambda P_j g^\delta - f\|_{L_2}) = \mathcal{O}((\delta\sqrt{|\log\delta|})^{\frac{2s}{2s+2t+d}}) \ . \tag{12}$$

This result is valid for every order-optimal regularization method, hence also for the fractional Tikhonov as well as the fractional Landweber method with parameter $\gamma > 1/2$. For the smoothness η of the wavelet basis no other condition than $\eta > 0$ is needed. This corresponds to the fact that every order-optimal regularization method can achieve the convergence result on its own without the use of a data pre-smoothing. We know from [8] that the use of wavelet shrinkage results in a smaller regularization parameter α and hence the reconstruction operator can stay closer to the original operator. In this paper we reduce the amount of regularization in the reconstruction part even further to fractional methods $R_{\alpha,\gamma}$ with $\gamma \leq 1/2$. We will show that, keeping the convergence rate (12) fixed, the use of data presmoothing is reflected in the parameter γ: the smoother the wavelet basis is, i.e. the closer η is to t and

the data estimate to the range of the operator K, the smaller is the necessary fraction of the reconstruction filter, i.e. the closer is γ to 0.
The following two lemmata are auxiliary results on the reconstruction operator $R_{\alpha,\gamma}$.

Lemma 4.2. *Let $K : L_2 \to L_2$ be a linear compact operator with smoothing property $t > 0$ and let $R_{\alpha,\gamma}$ be a fractional filter operator with parameter β as in* (5a'). *For $\gamma > (t-\eta)/2t$, we have*

$$\|R_{\alpha,\gamma}\|^2_{H^\eta \to L_2} \lesssim \alpha^{2\beta\frac{\eta-t}{t}} \ .$$

Proof. Using the translation of Sobolev smoothness and source condition as given in [8, Lemma 5.1] yields $\tilde{g} \in H^\eta \Leftrightarrow \tilde{g} \in Y_\nu = \mathrm{rg}\,((K^*K)^{\nu/2})$ with $\nu = \eta/t$. For $R_{\alpha,\gamma} : H^\eta \to L_2$ it is

$$\begin{aligned}
\|R_{\alpha,\gamma}\|^2_{H^\eta \to L_2} &= \sup_{\|h\|_{H^\eta}=1} \sum_{\sigma_n>0} \sigma_n^{-2} F_\alpha^{2\gamma}(\sigma) |\langle h, v_n\rangle_{L_2}|^2 \sigma_n^{2\eta/t}\sigma_n^{-2\eta/t} \\
&\leq \sup_{\|h\|_{H^\eta}=1} \sup_{\sigma>0} \sigma_n^{-2(1-\eta/t)} F_\alpha^{2\gamma}(\sigma) \underbrace{\sum_{\sigma_n>0} \sigma_n^{-2\eta/t} |\langle h, v_n\rangle_{L_2}|^2}_{\simeq \|h\|^2_{H^\eta}=1} \\
&\lesssim \sup_{\sigma>0} \sigma_n^{-2(1-\eta/t)} F_\alpha^{2\gamma}(\sigma) \ .
\end{aligned}$$

With $\mu := 1 - \eta/t$ and $\gamma > \mu/2 = (t-\eta)/2t$ the assertion follows from Lemma 3.1 for the fractional Tikhonov method and from Lemma 3.4 for the fractional Landweber method. □

Lemma 4.3. *Let $K : L_2 \to L_2$ be a linear compact operator with smoothing property $t > 0$ and let $R_{\alpha,\gamma}$ be a fractional filter operator with parameter β as in* (5b). *Let $s \leq t$, $f \in H^s$ and $g = Kf$. The reconstruction error of $R_{\alpha,\gamma}$ is then given by*

$$\|R_{\alpha,\gamma} g - f\|^2_{L_2} \lesssim \alpha^{2\beta s/t} \ .$$

Proof. Using the translation of Sobolev smoothness and source condition as given in [8, Lemma 5.1] yields $f \in H^s \Leftrightarrow f \in \mathrm{rg}\,((K^*K)^{\nu/2})$ with $\nu = s/t$. Hence,

$$\begin{aligned}
\|R_{\alpha,\gamma} g - f\|^2_{L_2} &= \sum_{\sigma_n} (F_\alpha^\gamma(\sigma_n) - 1)^2 |\langle f, v_n\rangle|^2 \sigma_n^{-2s/t}\sigma_n^{2s/t} \\
&\lesssim \sup_{\sigma>0} |F_\alpha^\gamma(\sigma) - 1|^2 \sigma_n^{2s/t} \|f\|^2_{H^s} \\
&\lesssim \alpha^{2\beta s/t} \ .
\end{aligned}$$

The last line follows from Proposition 3.2 and Proposition 3.5 where it is shown that the fractional Tikhonov as well as the fractional Landweber filter fulfill condition (5b). □

The following theorem extends the rate (12) to fractional methods $R_{\alpha,\gamma}$ with $\gamma < 1/2$,

Theorem 4.4. *Let $K : L_2(\Omega) \to L_2(\Omega)$ be a linear compact operator with smoothing property of order $t > 0$ with respect to any Besov space $B^\tau_{p,p}(\mathbb{R}^d)$ and any Sobolev space $H^\tau(\mathbb{R}^d)$. Let $s \geq 0$ with $s + t > 1/2$, $s \leq t$ and let η with $0 \leq \eta \leq t$ be given. We assume that f belongs to $H^s(\mathbb{R}^d) \cap B^s_{p,p}(\mathbb{R}^d)$ with $\frac{1}{p} = \frac{1}{2} \cdot \frac{2t+d}{2\eta+d} + \frac{s}{2\eta+d}$.*

i) Let $\varphi \in H^\eta$ define an orthonormal wavelet basis of $L_2(\mathbb{R}^d)$ of smoothness $\eta \leq t$ such that the degree m of polynomial reproduction in V_j satisfies $m + 1 > t$. Let P_j be the L_2-projection on V_j.

ii) Let S_λ denote wavelet shrinkage with hard thresholding. For a given error level δ the threshold λ is chosen as $\lambda = C\delta\sqrt{|\log\delta|}$.

iii) Let $R_{\alpha,\gamma}$ denote the fractional Tikhonov (or Landweber) operator with parameter $\gamma \in (0,1]$, and with parameter β as in (5a), (5b).

If the projection level j fulfills $2^{-j} \le (\delta\sqrt{|\log\delta|})^{1/(\eta+d/2)}$, the pair of parameters (γ,η) fulfills

$$\gamma > \frac{t-\eta}{2t} \tag{13}$$

and the regularization parameter is chosen according to

$$\begin{aligned} \alpha &\simeq (\delta\sqrt{|\log\delta|})^{\frac{1}{\beta}\frac{2t}{2s+2t+d}} && \text{for } \eta < t \\ \alpha &\simeq (\delta\sqrt{|\log\delta|})^2 && \text{for } \ \eta = t\ , \end{aligned} \tag{14}$$

then the following estimate holds for $f^\delta_{\alpha\gamma\lambda j} := R_{\alpha,\gamma}P_jS_\lambda(g^\delta)$,

$$\mathrm{E}(\|f^\delta_{\alpha\gamma\lambda j} - f\|^2_{L_2}) = \mathcal{O}((\delta\sqrt{|\log\delta|})^{\frac{4s}{2s+2t+d}}).$$

For the proof we will use results given in [8], especially [8, Theorem 4.5].

Proof. In order to estimate the error $E(\|R_{\alpha,\gamma}S_\lambda P_j g^\delta - f\|^2_{L_2})$ we split it into three parts, one each for the shrinkage error, the projection error and the error due to the reconstruction operator,

$$\begin{aligned} &E(\|R_{\alpha,\gamma}S_\lambda P_j g^\delta - f\|^2_{L_2}) \\ &\qquad \lesssim E(\|R_{\alpha,\gamma}(S_\lambda g^\delta - Pjg^\delta)\|^2_{L_2}) + \|R_{\alpha,\gamma}(P_jg-g)\|^2_{L_2} + \|R_{\alpha,\gamma}g - f\|^2_{L_2} \\ &\qquad = \|R_{\alpha,\gamma}\|^2_{H^\eta\to L_2}\left(E(\|S_\lambda g^\delta - Pjg^\delta\|^2_{L_2}) + \|P_jg - g\|^2_{L_2}\right) + \|R_{\alpha,\gamma}g - f\|^2_{L_2}\ . \end{aligned}$$

We want to remark that the proof differs from the one of [8, Theorem 4.5] only in the operator $R_{\alpha,\gamma}$. So from Lemma 4.2 we know

$$\|R_{\alpha,\gamma}\|^2_{H^\eta\to L_2} \lesssim \alpha^{2\beta\frac{\eta-t}{t}}$$

for any pair (γ,η) with condition (13). Exactly as in [8] we get for the first two terms

$$\|R_{\alpha,\gamma}\|^2_{H^\eta\to L_2}\big(E(\|S_\lambda g^\delta - Pjg^\delta\|^2_{L_2}) + \|P_jg - g\|^2_{L_2}\big) \lesssim \alpha^{2\beta\frac{\eta-t}{t}}(\delta\sqrt{|\log\delta|})^{\frac{4(s+t-\eta)}{2s+2t+d}}$$

For the third term we use Lemma 4.3 and get

$$\|R_{\alpha,\gamma}g - f\|^2_{L_2} \lesssim \alpha^{2\beta s/t}\ .$$

For $\eta < t$ we insert the parameter choice rule (14) for α. Assembling all three error bounds yields

$$E(\|R_{\alpha,\gamma}S_\lambda P_j g^\delta - f\|^2_{L_2}) \lesssim (\delta\sqrt{|\log\delta|})^{\frac{4s}{2s+2t+d}}$$

For $\eta = t$ we refer again to the proof of [8, Theorem 4.5] and remark that in this case we have $\|R_{\alpha,\gamma}\|_{H^t\to L_2} \lesssim 1$. □

We want to discuss condition (13) for fixed η and fixed γ respectively.

1. Let η be fix. Then (13) is a condition on γ according to

$$\gamma > \frac{t-\eta}{2t} = \frac{1}{2} - \frac{\eta}{2t} \ .$$

If $\eta = 0$ no shrinkage is done and the regularization operator has to deal with data in L_2. In this case we get $\gamma > 1/2$ which is the well-known condition for order-optimal (fractional) regularization methods. If η increases then γ decreases. The more shrinkage is done the less regularization is necessary. For the case $\eta = t$ we get $\gamma > 0$. In that case the shrinkage estimate is in the range of the operator and the generalized inverse could be applied directly. The condition $\gamma > 0$ (instead of $\gamma = 0$) assures that the convergence rate is achieved.

2. Let γ be fix. Then (13) is a condition on η according to

$$\eta > t(1-2\gamma) \ .$$

Here, the same considerations as in (i) apply. For $\gamma = 0$, i.e. no regularization at all, we get $\eta > t$. I.e. the smoothing operator has to assure $\tilde{g}$ in $\mathrm{rg}\,(K)$ and even a little more ($\eta > t$ not $\eta \geq t$) in order to keep the convergence result valid. For $\gamma > 1/2$ we know that η must be greater than $t(1-2\gamma)$ which is less than 0. Hence for $\gamma > 1/2$, which means order-optimal regularization methods, no shrinkage ($\eta = 0$) has to be applied. For $\gamma \leq 1/2$ the filter function is no longer optimal and the shrinkage estimate has to stand in for the order-optimality.

5 Computation of fractional methods

The use of the fractional methods is of particular interest for problems where classical methods generally oversmooth the solutions. In this case sharp or fine features of the solution are lost which is particularly troublesome in imaging or tomographic applications, where it is of high priority to recover e.g. jumps or discontinuities of the solutions [9, 10]. At the end of this section we show that this effect is reduced by the use of the combined method of wavelet shrinkage and fractional Tikhonov reconstruction.
If the singular value decomposition of an operator is not known explicitly, the numerical implementation of the fractional methods is not as straightforward as for the classical ones. The next proposition presents a formulation of the fractional Tikhonov method using the operator and its adjoint only. For an operator T with eigensystem (λ_n, w_n), a new operator $\psi(T)$ can be defined via a real-valued function ψ on the spectrum $\sigma(T)$ of T, see e.g. [2],

$$\psi(T)x := \sum_n \psi(\lambda_n)\langle x, w_n\rangle w_n \ . \tag{15}$$

In that sense we define fractional powers of K^*K and $K^*K + \alpha I$; the numerical computation of these operator roots is presented in the next section.

Proposition 5.1. *The solution $f_{\alpha,\gamma}$ of the fractional Tikhonov method as given in Definiton 2.1 can be computed according to*

$$(K^*K + \alpha I)^{\gamma}(K^*K)^{1-\gamma} f_{\alpha,\gamma} = K^*g. \tag{16}$$

Proof. The proof is by straightforward calculation. The fractional Tikhonov method with parameter γ defines a regularized solution $f_{\alpha,\gamma}$ as

$$f_{\alpha,\gamma} = \sum_{n>0} \sigma_n^{-1} \left(\frac{\sigma_n^2}{\sigma_n^2 + \alpha} \right)^{\gamma} \langle g, v_n \rangle u_n .$$

With $\sigma(K^*K) = (\sigma_n^2)_n$ and $\sigma(K^*K + \alpha I) = (\sigma_n^2 + \alpha)_n$ the Fourier coefficient $\langle f_{\alpha,\gamma}, u_n \rangle$ of $f_{\alpha,\gamma}$ given by equation (16) is

$$\begin{aligned}
\langle f_{\alpha,\gamma}, u_n \rangle &= \langle (K^*K)^{\gamma-1}(K^*K + \alpha I)^{-\gamma} K^* g, u_n \rangle \\
&= \langle g, K(K^*K + \alpha I)^{-\gamma}(K^*K)^{\gamma-1} u_n \rangle \\
&= \langle g, (\sigma_n^2 + \alpha)^{-\gamma} \sigma_n^{2(\gamma-1)} K u_n \rangle \\
&= \sigma_n^{-1} \left(\frac{\sigma_n^2}{\sigma_n^2 + \alpha} \right)^{\gamma} \langle g, v_n \rangle .
\end{aligned}$$

The assertion follows with $f_{\alpha,\gamma} = \sum_n \langle f_{\alpha,\gamma}, u_n \rangle u_n$. □

For the fractional Landweber method we start from the operator representation of the standard method. Let f_m denote the regularized Landweber solution defined as

$$f_m = \sum_{\sigma_n > 0} F_m(\sigma_n) \sigma_n^{-1} \langle g, v_n \rangle u_n = \sum_{\sigma_n > 0} (1 - (1 - \beta \sigma^2)^m) \sigma_n^{-1} \langle g, v_n \rangle u_n \ .$$

It is well-known, see e.g. [2], that for $m \geq 1$

$$f_m = \sum_{\sigma_n > 0} F_m(\sigma_n) \sigma_n^{-1} \langle g, v_n \rangle u_n = \beta \sum_{j=0}^{m-1} (\mathrm{I} - \beta K^*K)^j K^* g \ . \tag{17}$$

Lemma 5.2. *Let $K : X \to Y$ be a linear compact operator with singular system $(\sigma_n; u_n, v_n)$ and let $0 < \beta < 2/\|K\|^2$. Then the operator $A_m : X \to X$ with*

$$A_m := \beta \sum_{j=0}^{m-1} (\mathrm{I} - \beta K^*K)^j \tag{18}$$

has the eigensystem $(F_m(\sigma_k)\sigma_k^{-2}, u_k)_k$ with $F_m(\sigma) = 1 - (1 - \beta\sigma^2)^m$.

Proof. From (17) we know $A_m K^* g = \sum_{\sigma_n > 0} F_m(\sigma_n) \sigma_n^{-1} \langle g, v_n \rangle u_n$. With $K^* v_k = \sigma_k u_k$ it follows

$$\sigma_k A_m u_k = A_m(K^* v_k) = \sum_{\sigma_n > 0} F_m(\sigma_n) \sigma_n^{-1} \langle v_k, v_n \rangle u_n = F_m(\sigma_k) \sigma_k^{-1} u_k$$

and hence $A_m u_k = F_m(\sigma_k) \sigma_k^{-2} u_k$. □

For the fractional Landweber method we have the following operator representation.

Proposition 5.3. *The solution $f_{m,\gamma}$ of the fractional Landweber method as given in Definiton 2.1 can be computed as the solution of*

$$(K^*K)^{1-\gamma} f = \beta^{\gamma} \Big[\sum_{j=0}^{m-1} (\mathrm{I} - \beta K^*K)^j \Big]^{\gamma} K^* g \ . \tag{19}$$

We remark that for $\gamma = 1$ equation (19) coincides with the classical Landweber method (2) whereas for $\gamma = 0$ equation (19) is identical with the normal equations for $Kf = g$.

Proof. Solving equation (19) for f and using the operator A_m from (18) it is

$$\tilde{f} := (K^*K)^{\gamma-1}\beta^\gamma \Big[\sum_{j=0}^{m-1}(\mathrm{I} - \beta K^*K)^j\Big]^\gamma K^*g = (K^*K)^{\gamma-1}A_m^\gamma K^*g \ .$$

The operator K^*K has the eigensystem $(\sigma_k^2, u_k)_k$. Hence, for $h \in X$ it is with (15)

$$(K^*K)^{\gamma-1}h = \sum_k \sigma_k^{2(\gamma-1)}\langle h, u_k\rangle u_k \ .$$

Computing $h := A_m^\gamma(K^*g)$ yields with the help of Lemma 5.2 and (15)

$$A_m^\gamma(K^*g) = \sum_n (F_m(\sigma_n)\sigma_n^{-2})^\gamma \langle K^*g, u_n\rangle u_n = \sum_n F_m^\gamma(\sigma_n)\sigma_n^{-2\gamma}\sigma_n\langle g, v_n\rangle u_n \ .$$

Hence, it is

$$\begin{aligned}
(K^*K)^{\gamma-1}A_m^\gamma(K^*g) &= \sum_k \sigma_k^{2(\gamma-1)}\langle A_m^\gamma(K^*g), u_k\rangle u_k \\
&= \sum_k \sigma_k^{2(\gamma-1)}\sum_n F_m^\gamma(\sigma_n)\sigma_n^{-2\gamma+1}\langle g, v_n\rangle\langle u_n, u_k\rangle u_k \\
&= \sum_k \sigma_k^{-1}F_m^\gamma(\sigma_k)\langle g, v_k\rangle u_k \ .
\end{aligned}$$

With Definition 2.1 of the fractional filter operator it is

$$\tilde{f} = (K^*K)^{\gamma-1}A_m^\gamma(K^*g) = \sum_k \sigma_k^{-1}F_m^\gamma(\sigma_k)\langle g, v_k\rangle u_k = f_{m,\gamma} \ .$$

□

5.1 Series expansion and numerical realization

We restrict ourselves to the fractional Tikhonov method. To compute an approximation for the fractional powers of the operator in (16) we use the binomial series

$$(1+x)^\gamma = 1 + \gamma x + \frac{\gamma(\gamma-1)}{2!}x^2 + \frac{\gamma(\gamma-1)(\gamma-2)}{3!}x^3 + \ldots, \qquad x \in \mathbb{R},\ |x| < 1. \tag{20}$$

For the numerical realization of the fractional Tikhonov method we have to deal with discretized operators, i.e., matrices. The series expansion is valid for matrices A with $\|A\| < 1$. For the matrix representation of the fractional Tikhonov method, see (16), we have to compute

$$(\alpha I + K^*K)^\gamma \quad \text{and} \quad (K^*K)^{1-\gamma} \ .$$

Since K is linear we can assume $\|K^*K\| < 1$. For the expansion of the first term in (16), $(\alpha I + K^*K)^\gamma$, we consider two cases:

1. For $\alpha \geq 1$ it holds
$$(\alpha I + K^*K)^\gamma = \alpha^\gamma (I + \frac{1}{\alpha} K^*K)^\gamma$$
and the series expansion is used with argument $x = K^*K/\alpha$.

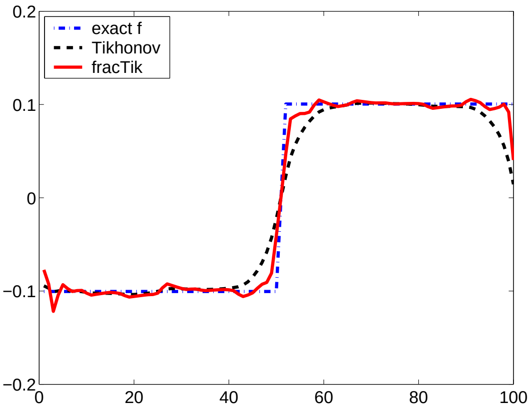
2. For $\alpha < 1$ we use
$$(\alpha I + K^*K)^\gamma = (I + (K^*K - (1-\alpha)I))^\gamma \ .$$
With the help of the singular value decomposition of K and the assumption $\|K^*K\| < 1$ we know that $\|K^*K - (1-\alpha)I\| \leq \max\{1-\alpha, \|K^*K\| - (1-\alpha)\} < 1$. Hence, we use the series expansion (20) with $x = K^*K - (1-\alpha)I$.

The second term in (16) is $(K^*K)^{1-\gamma}$ which is independent of α. It is
$$(K^*K)^{1-\gamma} = (I + (K^*K - I))^{1-\gamma}$$
and we would like to apply the series expansion (20) with $x = K^*K - I$. Since $\|K^*K - I\| \leq \max\{\|K^*K\|, 1\} = 1$ the series expansion might be applied with argument $x = K^*K - I$ but convergence in this case is likely to be slow. The same problem occurs for the first term when $\alpha < 1$ is very small.
For these cases we adapt an idea of [27] where the authors considered fractional powers of matrices. As has been seen, for a matrix A the fractional power A^γ with $\gamma < 1$ can be computed by using $A = I + B$ with a suitable matrix B and application of the series expansion (20). If however, $\|B\| = 1 - \varepsilon \approx 1$, convergence is likely to be slow. To accelerate convergence a weight factor k can be used. We consider $A = k(I + C)$ with $C = C(k) = (1/k)A - I$ and choose $k^* = \operatorname{argmin}_k \|C(k)\|$, see [27]. If we specify the norm to be the Frobenius (or Schur) norm the factor k can be computed explicitly according to
$$k = \sum_i \sum_j a_{ij}^2 / a_{ii} \ .$$
With this renormalization the implementation of the fractional Tikhonov method works fine which is demonstrated by the following test computations.

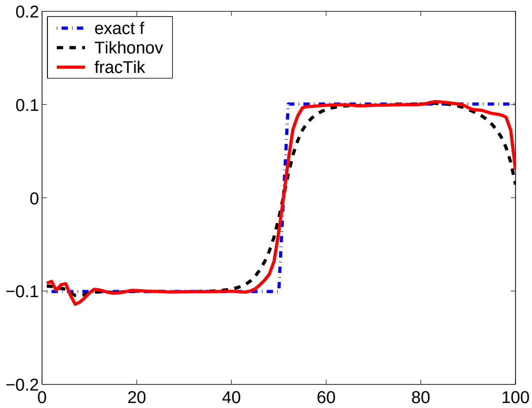
5.2 Test computations

We test the proposed variations of the standard Tikhonov method with two examples. As a first example we consider the integration operator $K : L_2(0,1) \to L_2(0,1)$ with
$$Kf(x) := \int_0^x f(t) \mathrm{d}t \ .$$
The integration operator smoothes one step in the scale of Sobolev spaces, i.e., $K : H^s \to H^{s+t}$ with $t = 1$. Since the classical Tikhonov method recovers smooth functions very well, but fails if the solution to the inverse problem $Kf = g$ has discontinuities, we choose as test function $f^\dagger$ the step function
$$f^\dagger(x) = \begin{cases} -1 & \text{if } x \leq 1/2 \\ 1 & \text{if } x > 1/2 \end{cases} \ .$$
Hence, we have $f^\dagger \in L_2(0,1)$ with the additional smoothness properties
$$f^\dagger \in H^{1/2-\varepsilon} \qquad \text{or} \qquad f^\dagger \in B_{11}^{1-\varepsilon} \ .$$

A similar problem, namely the detection of irregular points by the regularization of numerical differentiation, is studied in [28]. The results of our test computations are documented in the following tables and figures. Computations for the fractional method are always accompanied by computations for the classical method from the same data. Thereby we can check for which fraction of the filter function the reconstruction still achieves the same error bound. So far, no special treatment for the endpoints of the interval has been considered. The regularization parameter α was chosen manually in order to achieve the best possible result.
The combination of wavelet shrinkage and the fractional Tikhonov method yields very good results for low error levels of the data. Figure 2 demonstrates the performance of this combination with parameter $\gamma = 0.1, 0.2$ for 1% relative data noise. One can see that the approximation of the discontinuity is better than with the classical Tikhonov method.



(a) $\gamma = 0.1$, $\|f_{\mathrm{TS}} - f\|_2 = 11.48\%$

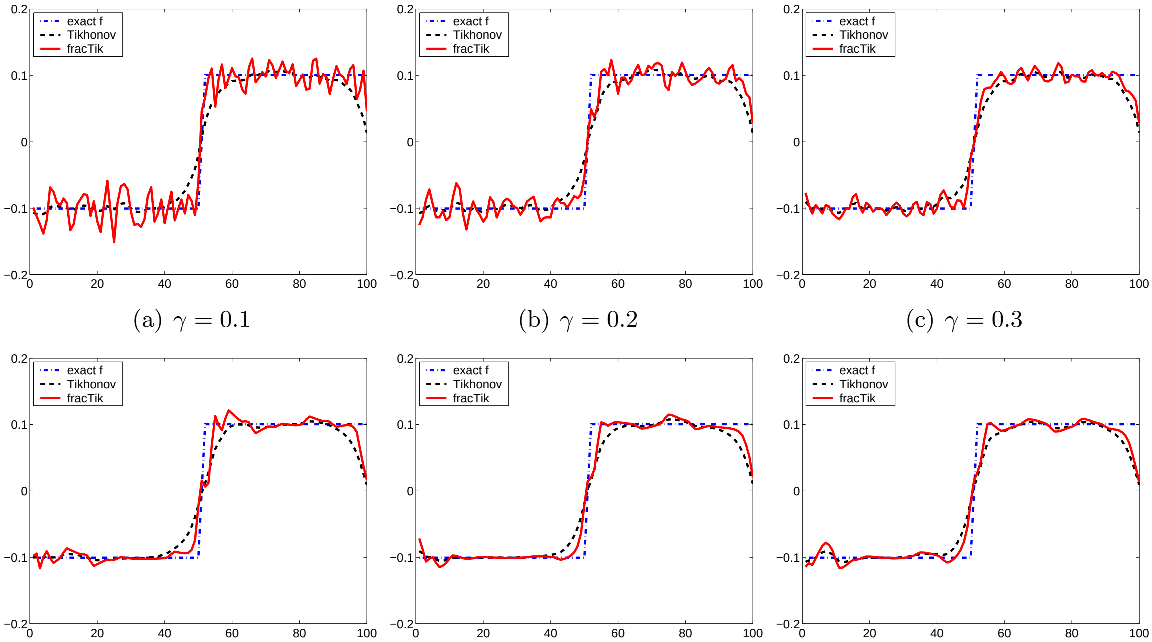


(b) $\gamma = 0.2$, $\|f_{\mathrm{TS}} - f\|_2 = 13.27\%$

Figure 2: Reconstruction of the step function (blue, dash-dotted) from noisy data with 1% relative error level: (black, dashed) classical Tikhonov method and (red, solid) fractional Tikhonov method with wavelet shrinkage.

Figure 3 demonstrates the importance of the smoothing step. In the first row no shrinkage has been applied and the fractional Tikhonov method with non-optimal parameter $\gamma = 0.1, 0.2, 0.3$ has been used. On the one hand, one can still see that the less of the filter is used, i.e. the smaller γ is, the better is the approximation of the discontinuity. On the other hand, the influence of the error in the smooth part of the solution is not controlled satisfactorily and causes heavy oscillations of the solution. These oscillations are either controlled by increasing the parameter γ or by applying a smoothing step first, see also Table 1 and Figure 3.
The following tables demonstrate that the fractional Tikhonov method is order optimal for $\gamma > 1/2$. The reconstruction errors are within the same sizes as for the standard Tikhonov method. For $\gamma < 0.4$ the reconstruction error is growing, the fractional Tikhonov method cannot control the influence of the data error. If shrinkage is applied the combined method reaches the optimal result for all parameter γ and 5% relative data noise. However, for 10% data noise the results for the combined method are no longer satisfactorily. Maybe a finer tuning of the involved parameter constants is necessary.
As a second numerical example, we consider a convolution operator

$$(Kf)(y) = \int_{\mathbb{R}} k(y - x) f(x)\, dx$$



(a) $\gamma = 0.1$ (b) $\gamma = 0.2$ (c) $\gamma = 0.3$

(d) $\gamma = 0.1$ (e) $\gamma = 0.2$ (f) $\gamma = 0.3$

Figure 3: Comparison of reconstructions from noisy data with 2% data noise. Upper row: fractional Tikhonov method with parameter γ and classical Tikhonov method, both without shrinkage. Lower row: fractional Tikhonov method with parameter γ and classical Tikhonov method, both combined with shrinkage.

with known convolution kernel k. If the operator is considered in an L_2- setting, it is easy to see that the adjoint operator K^* is defined via the kernel $\tilde{k}(x) = k(-x)$. Let us first show that convolution operators with suitable kernel k fit into our setting.

Proposition 5.4. *Let $f \in H^s \cap L_1$ and $k \in H^t \cap C^{u+t}$, $0 \le s \le u$, $t > 0$, with derivative $k^{(u+t)} \in L_1$. Then*

$$K : H^s \cap L_1 \to H^{s+t}$$

is a continuous operator.

γ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$\Vert f - f^\delta_{\alpha,\gamma}\Vert$	0.52	0.39	0.26	0.23	0.24	0.21	0.22	0.25	0.27	0.24
$\Vert f - f^\delta_{\alpha,\gamma,\lambda}\Vert$	0.27	0.25	0.26	0.25	0.25	0.26	0.25	0.23	0.26	0.25
$\Vert f - f^\delta_{\alpha}\Vert$										0.25

Table 1: Reconstructions from noisy data with relative error level $\delta = 5\%$. First row: fractional Tikhonov method without shrinkage. Second row: fractional Tikhonov method with shrinkage (db2-wavelet). Last row: standard Tikhonov method.

γ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$\|f - f^\delta_{\alpha,\gamma}\|$	0.97	0.56	0.37	0.31	0.33	0.31	0.30	0.28	0.32	0.26
$\|f - f^\delta_{\alpha,\gamma,\lambda}\|$	0.47	0.46	0.33	0.28	0.28	0.27	0.25	0.28	0.31	0.32
$\|f - f^\delta_{\alpha}\|$										0.29

Table 2: Reconstructions from noisy data with 10% relative noise level. First row: fractional Tikhonov method without shrinkage. Second row: fractional Tikhonov method with shrinkage (db2-wavelet). Last row: standard Tikhonov method.

Proof. By the Fourier convolution theorem it follows

$$\|k * f\|^2_{H^{s+t}} = \int_{\mathbb{R}} (1 + |\omega|^2)^{s+t} |\hat{k}(\omega)|^2 |\hat{f}(\omega)|^2 \, d\omega \ .$$

As $f \in L_1$ we have $|\hat{f}(\omega)| \leq C$, and from $k \in C^{u+t}$ it follows

$$|\omega^{u+t}| \cdot |\hat{k}(\omega)| \leq M \ .$$

Therefore, we get

$$\begin{aligned} \|k * f\|^2_{H^{s+t}} &\leq CM \int_{\mathbb{R}} (1 + |\omega|^2)^{s+(t-u)/2} |\hat{k}(\omega)| |\hat{f}(\omega)| \, d\omega \\ &\leq CM \int_{\mathbb{R}} (1 + |\omega|^2)^{t/2} |\hat{k}(\omega)| \, d\omega \int_{\mathbb{R}} (1 + |\omega|^2)^{2s-u} |\hat{f}(\omega)| \, d\omega \\ &= CM \|k\|^2_{H^t} \|f\|^2_{H^{2s-u}} < \infty \end{aligned}$$

as long as $s \leq u$.

□

Thus, the operator K is smoothing of order t as long as $s \leq u$. By the same techniques, this result can be extended to all s by assuming some more smoothness on f itself:

Proposition 5.5. *Let $f \in H^s \cap C^s$ and $k \in H^t \cap C^t$, $0 < s, t$, with derivatives $f^{(s)}$, $k^{(t)} \in L_1$. Then*

$$K : H^s \cap (C^s, \|\cdot\|_{H^s}) \to H^{s+t}$$

is a continuous operator.

The proof is as above, by using the estimate $|\omega^{u+t}| \cdot |\hat{k}(\omega)| \leq M$ for functions $k \in C^t$.
On the Fourier side, the inverse operator is given by

$$\widehat{K^{-1}g}(\omega) = \frac{\hat{g}(\omega)}{\hat{k}(\omega)} \ ,$$

which can be always carried out if e.g. $\hat{k}(\omega) \neq 0$ for all ω.
Due to the Fourier convolution theorem, the implementation of the Tikhonov fractional filter method is relatively simple. For the standard Tikhonov method, we see immediately that the evaluation of the operator $(\alpha I + K^* K)$ is equivalent on the Fourier side to a multiplication with the function $\hat{k}\hat{\bar{k}} + \alpha$. A similar result holds for the fractional power:

Proposition 5.6. *It is*

$$(\alpha I + K^*K)^\gamma \sim (\hat{k}\hat{\tilde{k}} + \alpha)^\gamma \ .$$

Proof. To prove the assertion, we use the series

$$(1+x)^\gamma = \sum_{n=0}^{\infty} c_n(\gamma) x^n \ ,$$

which converges for $|x| \leq 1$. Using $\alpha I + K^*K = I + (K^*K - (1-\alpha I)$ and

$$\mathcal{F}\{(K^*K - (1-\alpha I)^n f\} = (\hat{k}\hat{\tilde{k}} - (1-\alpha))^n \hat{f}$$

we obtain

$$\begin{aligned}
\mathcal{F}\{(\alpha I + K^*K)^\gamma f\} &= \sum_{n=0}^{\infty} c_n(\gamma) \mathcal{F}\{(K^*K - (1-\alpha I) f\}^n \\
&= \left(\sum_{n=0}^{\infty} c_n(\gamma)(\hat{k}(\omega)\hat{\tilde{k}}(\omega) - (1-\alpha))^n \right) \hat{f}(\omega) \\
&= (1 + (\hat{k}(\omega)\hat{\tilde{k}}(\omega) - (1-\alpha)))^\gamma \hat{f}(\omega) \\
&= (\hat{k}(\omega)\hat{\tilde{k}}(\omega) + \alpha)^\gamma \hat{f}(\omega)
\end{aligned}$$

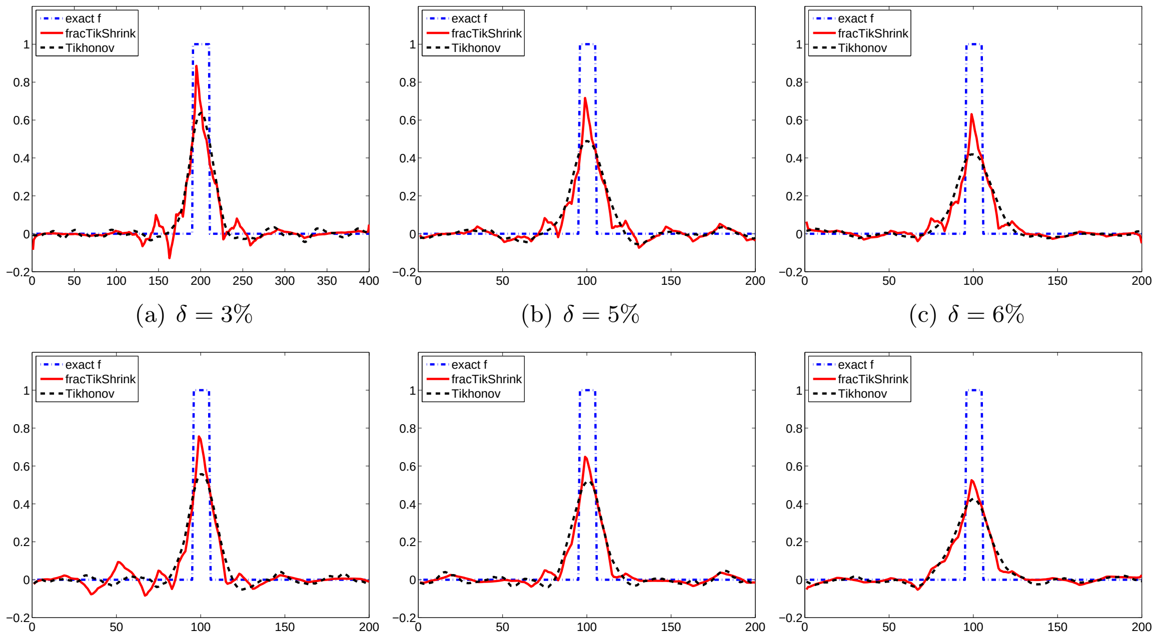
where the series converges. As pointed out above, this can always be achieved by proper normalization. □

With similar arguments we conclude

Proposition 5.7. *It is*

$$\mathcal{F}\{(\alpha I + K^*K)^\gamma (K^*K)^{1-\gamma} f\}(\omega) = \left(\hat{k}(\omega)\hat{\tilde{k}}(\omega) + \alpha\right)^\gamma \left(\hat{k}(\omega)\hat{\tilde{k}}(\omega)\right)^{1-\gamma} \hat{f}(\omega) \ . \qquad (21)$$

We define a specific kernel k by $\hat{k}(\omega) = (1+|\omega|^2)^{-t}$. It is easily seen that $k \in H^\tau$ for all $\tau < 2t$. We choose $t = 1$ and by that have an operator K which smoothes (almost) two steps in the scale of Sobolev spaces. As test function we choose the characteristic function of an interval, $f = \chi_{[a,b]}$ with Sobolev smoothness $1/2 - \varepsilon$. The fractional as well as the standard Tikhonov method are realized in the Fourier domain as given by (21). The restriction to a finite interval when computing the convolution yields an additional truncation error. Reconstructions are done by the combination of fractional Tikhonov with parameter $\gamma = 0.3, 0.4$ and wavelet shrinkage as well as, for comparison, by the standard Tikhonov method. The reconstructions can be seen in Figure 4 whereas Table 3 provides the reconstruction errors. Wavelet shrinkage is done with the sym4-wavelet. The regularization parameter is determined by the Morozov discrepancy principle.
Figure 4 demonstrates that also in this example the standard Tikhonov method results in regularized solutions which are too smooth whereas the fractional Tikhonov method in combination with wavelet shrinkage yields regularized solutions which are closer to the properties of the true solution.



(a) $\delta = 3\%$ (b) $\delta = 5\%$ (c) $\delta = 6\%$

(d) $\delta = 3\%$ (e) $\delta = 5\%$ (f) $\delta = 6\%$

Figure 4: Reconstructions from data with different noise levels. Upper row: fractional Tikhonov method with parameter $\gamma = 0.3$ and wavelet shrinkage compared to standard Tikhonov method. Lower row: fractional Tikhonov method with parameter $\gamma = 0.4$ and wavelet shrinkage compared to standard Tikhonov method.

	$\delta = \|g^\delta - g\|/\|g\|$	0.03	0.04	0.05	0.06
$\gamma = 0.3$	$\|f - f^\delta_{\alpha,\gamma,\lambda}\|/\|f\|$	0.5339	0.5663	0.5877	0.6182
$\gamma = 0.4$	$\|f - f^\delta_{\alpha,\gamma,\lambda}\|/\|f\|$	0.5402	0.5628	0.5838	0.6586
$\gamma = 1$	$\|f - f^\delta_{\alpha}\|/\|f\|$	0.5954	0.6081	0.6626	0.7124

Table 3: Reconstruction errors corresponding to the results given in Figure 4.

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Appendix

Proof of Lemma 3.3. With the expansion $(1-\tau^2)^m = \sum_{k=0}^m (-1)^k \tau^{2k} \binom{m}{k}$ we get

$$\phi(\tau) = \tau^{-2\mu} \left(\sum_{k=1}^{m} (-1)^{k-1} \tau^{2k} \binom{m}{k} \right)^{2\gamma}.$$

Thus for $\gamma > \mu/2$ there is no singularity and the function ϕ is continuous. Hence the function restricted to the interval $[0, \sqrt{2}]$ has a maximum. For $m = 1$ we have $\phi(\tau) = \tau^{2(2\gamma-\mu)}$ and ϕ

is bounded by $\phi(\sqrt{2}) = 2^{2\gamma-\mu}$.
For $m > 1$ the proof of assertion (9) is outlined as follows. We maximize not ϕ itself but ϕ restricted to all its critical points. The description of the critical points is gained by seeking zeros of the derivative of ϕ. Differentiating ϕ with respect to τ yields

$$\phi'(\tau) = \phi(\tau)\underbrace{\left(-2\mu\tau^{-1} + 4\gamma m\tau[1-(1-\tau^2)^m]^{-1}(1-\tau^2)^{m-1}\right)}_{=:h(\tau)} .$$

Hence, seeking zeros of ϕ' amounts to seeking zeros of ϕ and h. Since ϕ is nonnegative and, e.g. $\phi(1) = 1 > 0$, the zeros of ϕ are no candidates for maxima of ϕ. For the zeros of h we assume without loss of generality $\tau \neq 0$ and $1-(1-\tau^2)^m \neq 0$ (all τ with $1-(1-\tau^2)^m = 0$ and $\tau = 0$ are zeros of ϕ). Then $h(\tau) = 0$ is equivalent to

$$(1-\tau^2)^m = \frac{\mu(1-\tau^2)}{\mu(1-\tau^2) + 2\gamma m\tau^2} . \tag{22}$$

With this description of the critical points of ϕ we define $\phi_{\text{critical}} := \phi_{|\text{critical points}}$. We insert (22) in $\phi(\tau) = \tau^{-2\mu}(1-(1-\tau^2)^m)^{2\gamma}$ and get

$$\phi_{\text{critical}}(\tau) = \tau^{-2\mu}\left(\frac{2\gamma m\tau^2}{\mu(1-\tau^2) + 2\gamma m\tau^2}\right)^{2\gamma} .$$

If ϕ_{critical} has a maximum in $[0,\sqrt{2}]$ we have $\phi \leq \max_{[0,\sqrt{2}]}\phi_{\text{critical}}(\tau)$. Differentiating ϕ_{critical} with respect to τ yields

$$\phi'_{\text{critical}}(\tau) = 2\mu\tau^{-1}\phi_{\text{critical}}(\tau)\Big[\underbrace{\frac{2\gamma}{\mu(1-\tau^2) + 2\gamma m\tau^2} - 1}_{=:f(\tau)}\Big] .$$

Because of $\phi_{\text{critical}} > 0$, seeking zeros of ϕ_{critical} results in solving $f(\tau) = 0$ which is equivalent to $2\gamma - \mu = \tau^2(2\gamma m - \mu)$. Thus for $2\gamma > \mu$ critical points exist and are given by

$$\tau_*^2 = \frac{2\gamma - \mu}{2\gamma m - \mu}.$$

For $m > 1$ we have $\tau_*^2 < 1$ and the positive root is in the interval $(0,1) \subset [0,\sqrt{2}]$. In order to check whether $\tau_* = +\sqrt{\frac{2\gamma-\mu}{2\gamma m-\mu}}$ is a maximum of ϕ_{critical} we compute the second derivative. Applying the product rule to $\phi'_{\text{critical}}(\tau) = 2\mu\tau^{-1}\phi_{\text{critical}}(\tau)f(\tau)$ and inserting $\phi'_{\text{critical}}(\tau_*) = 0$ and $f(\tau_*) = 0$ we get

$$\phi''_{\text{critical}}(\tau_*) = 2\mu\tau_*^{-1}\phi_{\text{critical}}(\tau_*)f'(\tau_*) = -8\gamma\mu^2 \cdot \frac{2\gamma m - \mu}{(\mu(1-\tau_*^2) + 2\gamma m\tau_*^2)^2} .$$

With $\mu > 0$ and $\gamma > \mu/2$ it follows $\phi_{\text{critical}}(\tau_*) < 0$. Hence we get the maximum

$$\phi_{\text{critical}}(\tau_*) = \left(\frac{2\gamma - \mu}{2\gamma m - \mu}\right)^{2\gamma-\mu} m^{2\gamma}.$$

It remains to show that

$$\left(\frac{2\gamma - \mu}{2\gamma m - \mu}\right)^{2\gamma-\mu} m^{2\gamma} \leq m^{\mu}.$$

Since $\gamma > \mu/2 > 0$ and $m > 1$ this is equivalent to $(2\gamma-\mu)^{2\gamma-\mu}m^{2\gamma-\mu} \leq (2\gamma m - \mu)^{2\gamma-\mu}$. With the monotony of the power function assertion (9) is proved. □

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