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Multilevel preconditioning of rotated bilinear non-conforming FEM problems

I. Georgiev, J. Kraus, S. Margenov

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Abstract

Preconditioners based on various multilevel extensions of two-level finite element methods (FEM) lead to iterative methods which often have an optimal order computational complexity with respect to the number of degrees of freedom of the system. Such methods were first presented in [6, 7], and are based on (recursive) two-level splittings of the finite element space. The key role in the derivation of optimal convergence rate estimates plays the constant γ in the so-called Cauchy-Bunyakowski-Schwarz (CBS) inequality, associated with the angle between the two subspaces of the splitting. More precisely, the value of the upper bound for $\gamma \in (0, 1)$ is a part of the construction of various multilevel extensions of the related two-level methods.

In this paper algebraic two-level and multilevel preconditioning algorithms for second order elliptic boundary value problems are constructed, where the discretization is done using Rannacher-Turek non-conforming rotated bilinear finite elements on quadrilaterals. An important point to make is that in this case the finite element spaces corresponding to two successive levels of mesh refinement are not nested in general. To handle this, a proper two-level basis is required to enable us to fit the general framework for the construction of two-level preconditioners for conforming finite elements and to generalize the method to the multilevel case.

The proposed variants of hierarchical two-level basis are first introduced in a rather general setting. Then, the involved parameters are studied and optimized. The major contribution of the paper is the derived estimates of the constant γ in the strengthened CBS inequality which is shown to allow the efficient multilevel extension of the related two-level preconditioners. Representative numerical tests well illustrate the optimal complexity of the resulting iterative solver.

KEY WORDS: non-conforming FEM, multilevel preconditioners, hierarchical basis, CBS constant.

1 Introduction

In this paper we consider the elliptic boundary value problem

$$
\begin{aligned}
Lu \equiv -\nabla \cdot (\mathbf{a}(\mathbf{x})\nabla u(\mathbf{x})) &= f(\mathbf{x}) & \text{in} \quad &\Omega, \\
u &= 0 & \text{on} \quad &\Gamma_D, \\
(\mathbf{a}(\mathbf{x})\nabla u(\mathbf{x})) \cdot \mathbf{n} &= 0 & \text{on} \quad &\Gamma_N,
\end{aligned}
\tag{1}
$$

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where Ω is a convex polygonal domain in $\mathbb{R}^2$, $f(\mathbf{x})$ is a given function in $L^2(\Omega)$, the coefficient matrix $\mathbf{a}(\mathbf{x})$ is symmetric positive definite and uniformly bounded in Ω, $\mathbf{n}$ is the outward unit vector normal to the boundary $\Gamma = \partial\Omega$, and $\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_N$. We assume that the elements of the diffusion coefficient matrix $\mathbf{a}(\mathbf{x})$ are a piece-wise smooth functions on $\bar{\Omega}$.

The weak formulation of the above problem reads as follows:
given $f \in L^2(\Omega)$ find $u \in \mathcal{V} \equiv H_D^1(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$, satisfying

$$\mathcal{A}(u, v) = (f, v) \quad \forall v \in H_D^1(\Omega), \quad \text{where} \quad \mathcal{A}(u, v) = \int_\Omega \mathbf{a}(\mathbf{x}) \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) d\mathbf{x}. \qquad (2)$$

We assume that the domain Ω is discretized by the partition $\mathcal{T}_h$ which is obtained by a proper refinement of a given coarser partition $\mathcal{T}_H$. We assume also that $\mathcal{T}_H$ is aligned with the discontinuities of the coefficient $\mathbf{a}(\mathbf{x})$ so that over each element $E \in \mathcal{T}_H$ the coefficients of $\mathbf{a}(\mathbf{x})$ are smooth functions.

The variational problem (2) is then discretized using the finite element method, i.e., the continuous space $\mathcal{V}$ is replaced by a finite dimensional subspace $\mathcal{V}_h$. Then the finite element formulation is: find $u_h \in \mathcal{V}_h$, satisfying

$$\mathcal{A}_h(u_h, v_h) = (f, v_h) \quad \forall v_h \in \mathcal{V}_h, \text{ where } \quad \mathcal{A}_h(u_h, v_h) = \sum_{e \in \mathcal{T}_h} \int_e \mathbf{a}(e) \nabla u_h \cdot \nabla v_h d\mathbf{x}. \qquad (3)$$

Here $\mathbf{a}(e)$ is a piece-wise constant symmetric positive definite matrix, defined by the integral averaged values of $\mathbf{a}(\mathbf{x})$ over each element from the coarser triangulation $\mathcal{T}_H$. We note that in this way strong coefficient jumps across the boundaries between adjacent finite elements from $\mathcal{T}_H$ are allowed.

The resulting discrete problem to be solved is then a linear system of equations

$$A_h \mathbf{u}_h = \mathbf{F}_h, \qquad (4)$$

with A_h and $\mathbf{F}_h$ being the corresponding global stiffness matrix and global right hand side, and h being the discretization (meshsize) parameter for the underlying partition $\mathcal{T}_h$ of Ω.

The aim of this paper is to investigate multilevel preconditioners of optimal complexity for solving the system (4). The general setting and some well-known results for the case of conforming finite elements are summarized in the rest of this section. The next sections are devoted to the study of two-level and multilevel preconditioners for the case of non-conforming Rannacher-Turek finite elements. A unified hierarchical splitting of the FEM spaces is developed, followed by uniform estimates of the related CBS constants. The numerical results that are presented towards the end of the paper are completed by some concluding remarks.

1.1 The two-level setting

We are concerned with the construction of a two-level preconditioner M for A_h, such that the spectral condition number $\varkappa(M^{-1}A_h)$ of the preconditioned matrix $M^{-1}A_h$ is uniformly bounded with respect to the meshsize parameter h, and the possible coefficient jumps.

The classical theory for constructing optimal order two-level preconditioners was first developed in [4, 9], see also [3]. The general framework requires to define two nested finite element spaces $\mathcal{V}_H \subset \mathcal{V}_h$, that correspond to two consecutive (regular) mesh refinements, as illustrated in Figure 1(c) and 1(b). The well studied case of conforming linear finite elements is the starting point in the theory of two-level and multi-level methods.

Let $\mathcal{T}_H$ and $\mathcal{T}_h$ be two successive mesh refinements of the domain Ω, which correspond to $\mathcal{V}_H$ and $\mathcal{V}_h$. Let $\{\phi_H^{(k)}, k = 1, 2, \cdots, N_H\}$ and $\{\phi_h^{(k)}, k = 1, 2, \cdots, N_h\}$ be the standard finite element

(a) Refined mesh   (b) One macro-element   (c) One element
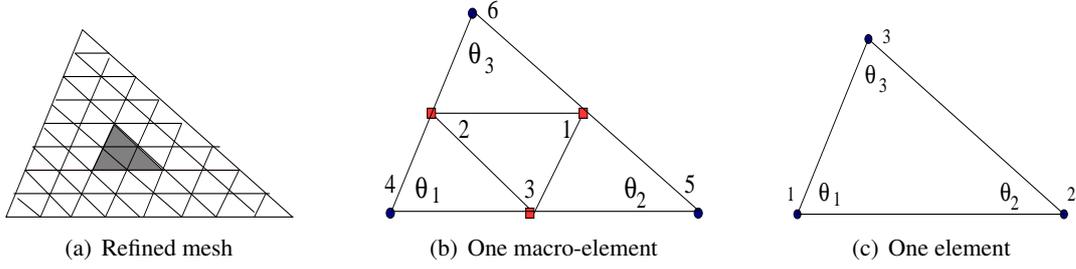
Figure 1: Conforming linear finite elements.

nodal basis functions. We split the meshpoints $\mathbf{n}_h$ from $\mathcal{T}_h$ into two groups: the first group contains the nodes $\mathbf{n}_H$ from $\mathcal{T}_H$ and the second one consists of the rest, where the latter are the newly added node-points $\mathbf{n}_{h\backslash H}$ from $\mathcal{T}_h\backslash\mathcal{T}_H$. Next we define the so-called hierarchical basis functions

$$\{\widetilde{\phi}_h^{(k)}, k = 1, 2, \cdots, N_h\} = \{\phi_H^{(l)} \text{ on } \mathcal{T}_H\} \cup \{\phi_h^{(m)} \text{ on } \mathcal{T}_h\backslash\mathcal{T}_H\}. \tag{5}$$

Let then $\widetilde{A}_h$ be the corresponding hierarchical stiffness matrix. Under the splitting (5) both matrices A_h and $\widetilde{A}_h$ admit in a natural way a two-by-two block structure

$$A_h = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{matrix} \}N_{h\backslash H} \\ \}N_H \end{matrix} \quad , \qquad \widetilde{A}_h = \begin{bmatrix} A_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & A_H \end{bmatrix} \begin{matrix} \}N_{h\backslash H} \\ \}N_H \end{matrix} \quad . \tag{6}$$

As is well-known, there exists a transformation matrix $J = \begin{bmatrix} I_1 & 0 \\ J_{21} & I_2 \end{bmatrix}$, which relates the nodal point vectors for the standard and the hierarchical basis functions as follows,

$$\widetilde{\mathbf{v}} = \begin{bmatrix} \widetilde{\mathbf{v}}_1 \\ \widetilde{\mathbf{v}}_2 \end{bmatrix} = J \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix}, \qquad \begin{matrix} \widetilde{\mathbf{v}}_1 = \mathbf{v}_1 \\ \widetilde{\mathbf{v}}_2 = J_{21}\mathbf{v}_1 + \mathbf{v}_2 \end{matrix} \quad .$$

Remark 1.1 Clearly, the hierarchical stiffness matrix $\widetilde{A}_h$ is more dense than A_h and therefore its action on a vector is computationally more expensive. The transformation matrix J, however, enables us in practical implementations to work with A_h, since $\widetilde{A}_h = JA_hJ^T$.

1.2 Two-level preconditioners and the strengthened Cauchy-Bunyakowski-Schwarz inequality

Consider a general matrix A, which is assumed to be symmetric positive definite and partitioned as in (6). The quality of this partitioning is characterized by the corresponding CBS inequality constant:

$$\gamma = \sup_{\mathbf{v}_1 \in \mathbb{R}^{n_1-n_2}, \, \mathbf{v}_2 \in \mathbb{R}^{n_2}} \frac{\mathbf{v}_1^T A_{12} \mathbf{v}_2}{\left(\mathbf{v}_1^T A_{11} \mathbf{v}_1\right)^{1/2} \left(\mathbf{v}_2^T A_{22} \mathbf{v}_2\right)^{1/2}}, \tag{7}$$

where $n_1 = N_h$ and $n_2 = N_H$.

Consider now two preconditioners to A under the assumptions

$$A_{11} \leq C_{11} \leq (1 + \delta_1)A_{11} \text{ and } A_{22} \leq C_{22} \leq (1 + \delta_2)A_{22}. \tag{8}$$

The inequalities (8) are in a positive semidefinite sense where C_{11} and C_{22} are symmetric and positive definite matrices for some positive constants δ_i, $i = 1, 2$.

3

(a) The symmetric preconditioner of block-diagonal (additive) form is introduced as

$$M_A = \begin{bmatrix} C_{11} & 0 \\ 0 & C_{22} \end{bmatrix}.$$

The following general estimate of the spectral condition number of $M_A^{-1}A$ holds:

$$\varkappa(M_A^{-1}A) \le \frac{1+\gamma}{\Delta_0(1-\gamma)} \tag{9}$$

where γ is the CBS constant in (7) and

$$\Delta_0 = \frac{4(1+\gamma)}{(1+\delta_2)(1+\Delta+\sqrt{(\Delta-1)^2+4\Delta\gamma^2})}, \quad \Delta = \frac{1+\delta_1}{1+\delta_2}.$$

When $C_{11} = A_{11}$ and $C_{22} = A_{22}$, then estimate (9) reduces to

$$\varkappa(M_A^{-1}A) \le \frac{1+\gamma}{1-\gamma}. \tag{10}$$

(b) The full block-matrix factorization preconditioner (multiplicative, or of block Gauss-Seidel form) is based on the exact block-matrix factorization of A,

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & S \end{bmatrix} \begin{bmatrix} I_1 & A_{11}^{-1}A_{12} \\ 0 & I_2 \end{bmatrix}$$

where $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$ is the exact Schur complement of A. The multiplicative preconditioner is then of the form

$$M_F = \begin{bmatrix} C_{11} & 0 \\ A_{21} & C_{22} \end{bmatrix} \begin{bmatrix} I_1 & C_{11}^{-1}A_{12} \\ 0 & I_2 \end{bmatrix}. \tag{11}$$

Then

$$\varkappa(M_F^{-1}A) \le \frac{1}{1-\gamma^2}\left\{1 + \frac{1}{2}\left[\delta_1 + \delta_2 + \sqrt{(\delta_1-\delta_2)^2 + 4\delta_1\delta_2\gamma^2}\right]\right\}. \tag{12}$$

When $C_{22} = A_{22}$, then estimate (12) reduces to

$$\varkappa(M_F^{-1}A) \le \frac{1}{1-\gamma^2}. \tag{13}$$

Detailed proofs of (9), (10) and (12), and an analysis of other versions of constructing M_A and M_F are found, for instance, in [3]. In the hierarchical bases context $\mathcal{V}_1$ and $\mathcal{V}_2$ are subspaces of the finite element space $\mathcal{V}_h$ spanned, respectively, by the basis functions at the new nodes $\{\mathbf{n}_{h\setminus H}\}$ and by the basis functions at the old nodes $\{\mathbf{n}_H\}$. For the strengthened CBS inequality constant, there holds that

$$\gamma = cos(\mathcal{V}_1, \mathcal{V}_2) = \sup_{u \in \mathcal{V}_1, \ v \in \mathcal{V}_2} \frac{\mathcal{A}(u,v)}{\sqrt{\mathcal{A}(u,u)\mathcal{A}(v,v)}} \tag{14}$$

where $\mathcal{A}(\cdot,\cdot)$ is the bilinear form which appears in the variational formulation of the original problem. When $\mathcal{V}_1 \cap \mathcal{V}_2 = \{0\}$, the constant γ is strictly less than one.

As shown in [4], the constant γ can be estimated locally over each finite element (macro-element) $E \in \mathcal{T}_H$, which means that $\gamma = \max_E \gamma_E$, where

$$\gamma_E = \sup_{u \in \mathcal{V}_1(E), \ v \in \mathcal{V}_2(E)} \frac{\mathcal{A}_E(u, v)}{\sqrt{\mathcal{A}_E(u, u)\mathcal{A}_E(v, v)}}, \quad v \neq const.$$

The spaces $\mathcal{V}_k(E)$ above contain the functions from $\mathcal{V}_k$ restricted to E and $\mathcal{A}_E(u, v)$ corresponds to $\mathcal{A}(u, v)$ restricted over the element E of $\mathcal{T}_H$ (see also [14]).

Using the local estimates, it is possible to obtain uniform estimates for γ. In the case of linear conforming finite elements, it is known that γ does not depend on h and on any discontinuities of the coefficients of the bilinear form $\mathcal{A}(\cdot, \cdot)$, as long as they do not occur within any element of the coarse triangulation used. The h-independence means that if we have a hierarchy of refinements of the domain which preserve the properties of the initial triangulation (refinement by congruent triangles, for example), then γ is independent of the level of the refinement as well. For certain implementations, it is shown that γ is independent of anisotropy. Hence, as long as the rate of convergence is bounded by some function of γ, it is independent of various problem and discretization parameters, such as the ones mentioned above.

We stress here, that the above technique is originally developed and straightforwardly applicable for conforming finite elements and nested finite element spaces, i.e., when $\mathcal{V}_H \subset \mathcal{V}_h$.

2 Rannacher-Turek finite elements

Nonconforming finite elements based on *rotated* multilinear shape functions were introduced by Rannacher and Turek [19] as a class of simple elements for the Stokes problem. More generally, the recent activities in the development of efficient solution methods for non-conforming finite element systems are inspired by their attractive properties as a stable discretization tool for illconditioned problems.

The unit square $[-1, 1]^2$ is used as a reference element $\hat{e}$ to define the isoparametric rotated bilinear element $e \in \mathcal{T}_h$. Let $\psi_e : \hat{e} \to e$ be the corresponding bilinear one-to-one transformation, and let the nodal basis functions be determined by the relation
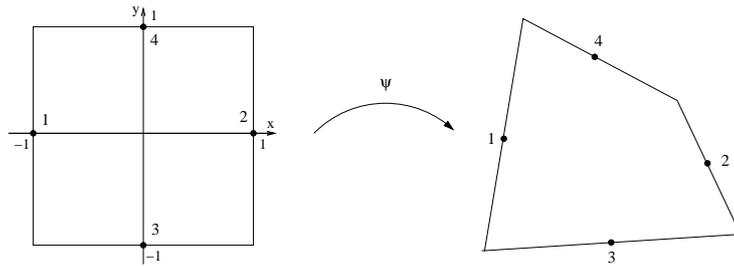

Figure 2: Rotated bilinear finite element.

$$\{\phi_i\}_{i=1}^4 = \{\hat{\phi}_i \circ \psi_e^{-1}\}_{i=1}^4, \qquad \{\hat{\phi}_i\} \in \mathrm{span}\{1, x, y, x^2 - y^2\}.$$

For the variant MP (mid point), $\{\hat{\phi}_i\}_{i=1}^4$ are found by the point-wise interpolation condition

$$\hat{\phi}_i(b_\Gamma^j) = \delta_{ij},$$

5

where $b_\Gamma^j, j = 1, 4$ are the midpoints of the edges of the quadrilateral $\hat{e}$. Then,

$$\hat{\phi}_1(x,y) = \frac{1}{4}(1 - 2x + (x^2 - y^2)), \qquad \hat{\phi}_2(x,y) = \frac{1}{4}(1 + 2x + (x^2 - y^2)),$$

$$\hat{\phi}_3(x,y) = \frac{1}{4}(1 - 2y - (x^2 - y^2)), \qquad \hat{\phi}_4(x,y) = \frac{1}{4}(1 + 2y - (x^2 - y^2)).$$

The variant MV (mid value) corresponds to integral midvalue interpolation conditions. Let $\Gamma_{\hat{e}} = \bigcup_{j=1}^4 \Gamma_{\hat{e}}^j$. Then $\{\hat{\phi}_i\}_{i=1}^4$ are determined by the equality

$$|\Gamma_{\hat{e}}^j|^{-1} \int_{\Gamma_{\hat{e}}^j} \hat{\phi}_i d\Gamma_{\hat{e}}^j = \delta_{ij},$$

which leads to

$$\hat{\phi}_1(x,y) = \frac{1}{8}(2 - 4x + 3(x^2 - y^2)), \qquad \hat{\phi}_2(x,y) = \frac{1}{8}(2 + 4x + 3(x^2 - y^2)),$$

$$\hat{\phi}_3(x,y) = \frac{1}{8}(2 - 4y - 3(x^2 - y^2)), \qquad \hat{\phi}_4(x,y) = \frac{1}{8}(2 + 4y - 3(x^2 - y^2)).$$

Consider now the model anisotropic problem with diagonal coefficient matrix

$$\mathbf{a}(\mathbf{x}) = a(e) \begin{bmatrix} \varepsilon & 0 \\ 0 & 1 \end{bmatrix}. \tag{15}$$

Then, in the case of a square mesh, the element stiffness matrices, corresponding to the variants MP and MV, read as

$$A_{MP}^{(e)} = \frac{a(e)}{3} \begin{bmatrix} 1 + 4\varepsilon & -(2\varepsilon - 1) & -(1 + \varepsilon) & -(1 + \varepsilon) \\ -(2\varepsilon - 1) & 1 + 4\varepsilon & -(1 + \varepsilon) & -(1 + \varepsilon) \\ -(1 + \varepsilon) & -(1 + \varepsilon) & 4 + \varepsilon & -(2 - \varepsilon) \\ -(1 + \varepsilon) & -(1 + \varepsilon) & -(2 - \varepsilon) & 4 + \varepsilon \end{bmatrix}$$

and

$$A_{MV}^{(e)} = \frac{a(e)}{4} \begin{bmatrix} 3 + 7\varepsilon & 3 - \varepsilon & -3(1 + \varepsilon) & -3(1 + \varepsilon) \\ 3 - \varepsilon & 3 + 7\varepsilon & -3(1 + \varepsilon) & -3(1 + \varepsilon) \\ -3(1 + \varepsilon) & -3(1 + \varepsilon) & 7 + 3\varepsilon & -(1 - 3\varepsilon) \\ -3(1 + \varepsilon) & -3(1 + \varepsilon) & -(1 - 3\varepsilon) & 7 + 3\varepsilon \end{bmatrix}.$$

3 Hierarchical two-level splittings

Let us consider two consecutive discretizations $\mathcal{T}_H$ and $\mathcal{T}_h$. Figure 3 illustrates a macro-element obtained after one regular mesh-refinement step. We see that in this case $\mathcal{V}_H$ and $\mathcal{V}_h$ are not nested.

3.1 "First reduce" (FR) two-level splitting

Next, we follow the idea of [12, 17] to define an algebraic two-level preconditioner. Let $\varphi_E = \{\phi_i(x,y)\}_{i=1}^{12}$ be the macro-element vector of the nodal basis functions and A_E be the macro-element stiffness matrix corresponding to $E \in \mathcal{T}_h$. The global stiffness matrix A_h can be written as

$$A_h = \sum_{E \in \mathcal{T}_h} A_E$$

(a) One macro-element   (b) One element

Figure 3: Uniform refinement on a general mesh.

where the summation is understood as the FEM assembly procedure. Let us introduce the following macro-element level transformation matrix

$$
J_E = \frac{1}{2}
\begin{bmatrix}
2 & & & & & & & & & & & \\
& 2 & & & & & & & & & & \\
& & 2 & & & & & & & & & \\
& & & 2 & & & & & & & & \\
& & & & 1 & -1 & & & & & & \\
& & & & & & & & 1 & -1 & & \\
& & & & & & 1 & -1 & & & & \\
& & & & & & & & & & 1 & -1 \\
& & & & & 1 & 1 & & & & & \\
& & & & & & & & 1 & 1 & & \\
& & & & & & 1 & & 1 & & & \\
& & & & & & & & & & 1 & 1 \\
\end{bmatrix}
\tag{16}
$$

which defines locally a two level hierarchical basis $\widetilde{\varphi}_E$, namely, $\widetilde{\varphi}_E = J_E \varphi_E$. The hierarchical two-level macro-element stiffness matrix is then obtained as

$$
\widetilde{A}_E = J_E A_E J_E^T,
$$

and the related global stiffness matrix reads as

$$
\widetilde{A}_h = \sum_{E \in \mathcal{T}_h} \widetilde{A}_E.
$$

We split now the two-level stiffness matrix $\widetilde{A}_h$ into 2×2 block form

$$
\widetilde{A}_h =
\begin{bmatrix}
\widetilde{A}_{11} & \widetilde{A}_{12} \\
\widetilde{A}_{21} & \widetilde{A}_{22}
\end{bmatrix},
\tag{17}
$$

where $\widetilde{A}_{11}$ corresponds to interior nodal unknowns with respect to the macro-elements $E \in \mathcal{T}_h$. The first step of the "First Reduce" (FR) algorithm is to eliminate these unknowns. For this purpose we factor $\widetilde{A}_h$, i.e.,

$$
\widetilde{A}_h =
\begin{bmatrix}
\widetilde{A}_{11} & 0 \\
\widetilde{A}_{21} & B
\end{bmatrix}
\begin{bmatrix}
I_1 & \widetilde{A}_{11}^{-1} \widetilde{A}_{12} \\
0 & I_2
\end{bmatrix},
\tag{18}
$$

7

where $B = \widetilde{A}_{22} - \widetilde{A}_{21}\widetilde{A}_{11}^{-1}\widetilde{A}_{12}$ stands for the Schur complement of this elimination step.

Next we consider a two-level splitting of the matrix B in the block form

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \tag{19}$$

where the first block corresponds to the half-difference basis functions. Now it is easily seen that the matrix B_{22} can be associated with the coarse grid. It is important to note that

$$\ker(B_{E;22}) = \ker(A_e) = \text{span}\{(1,1,1,1)^T\}$$

which allows us to apply a local analysis to estimate the constant γ corresponding the splitting defined by the block partition (19).

For our analysis we proceed as follows:

Step 1: We observe that the first left block of $\widetilde{A}_h$ is a block-diagonal matrix. The diagonal entries of $\widetilde{A}_{11}$ are 4×4 blocks, corresponding to the interior points $\{1, 2, 3, 4\}$, cf. Figure 3, which are not connected to nodes in other macro-elements. Thus, the corresponding unknowns can be eliminated exactly, i.e., to be done locally. Therefore, we first compute the local Schur complements arising from static condensation of the "interior degrees of freedom" and obtain the (8×8) matrix B_E. Next we split B_E as

$$B_E = \begin{bmatrix} B_{E,11} & B_{E,12} \\ B_{E,21} & B_{E,22} \end{bmatrix} \begin{matrix} \}\text{two-level half-difference basis functions} \\ \}\text{two-level half-sum basis functions} \end{matrix}$$

written again in two-by-two block form with blocks of order (4×4).

Step 2: We are now in a position to estimate the CBS constant corresponding to the 2×2 splitting of B. Following the general theory, it suffices to compute the minimal eigenvalue of the generalized eigenproblem

$$S_E \mathbf{v}_E = \lambda_E^{(1)} B_{E,22} \mathbf{v}_E, \quad \mathbf{v}_E \perp (1,1,\ldots,1)^T,$$

where $S_E = B_{E,22} - B_{E,21} B_{E,11}^{-1} B_{E,12}$, and then

$$\gamma^2 \leq \max_{E \in \mathcal{T}_h} \gamma_E^2 = \max_{E \in \mathcal{T}_h} (1 - \lambda_E^{(1)}). \tag{20}$$

3.2 Two-level splitting by differences and aggregates (DA)

Similarly to the FR case, the DA splitting is easily described for one macro-element. If $\phi_1, \ldots, \phi_{12}$ are the standard nodal basis functions for the macro-element, then we define

$$\begin{aligned}
\mathcal{V}(E) &= \text{span}\{\phi_1, \ldots, \phi_{12}\} = \mathcal{V}_1(E) \oplus \mathcal{V}_2(E), \\
\mathcal{V}_1(E) &= \text{span}\{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5 - \phi_6, \phi_9 - \phi_{10}, \phi_7 - \phi_8, \phi_{11} - \phi_{12}\} \\
\mathcal{V}_2(E) &= \text{span}\{\phi_5 + \phi_6 + \sum_{j=1,4} \alpha_{1j}\phi_j, \ \phi_9 + \phi_{10} + \sum_{j=1,4} \alpha_{2j}\phi_j, \\
&\qquad \phi_7 + \phi_8 + \sum_{j=1,4} \alpha_{3j}\phi_j, \ \phi_{11} + \phi_{12} + \sum_{j=1,4} \alpha_{4j}\phi_j\}.
\end{aligned}$$

Using the related transformation matrix J_E,

$$
J_E = \frac{1}{2}
\begin{bmatrix}
2 & & & & & & & & & & & \\
 & 2 & & & & & & & & & & \\
 & & 2 & & & & & & & & & \\
 & & & 2 & & & & & & & & \\
 & & & & 1 & -1 & & & & & & \\
 & & & & & & 1 & -1 & & 1 & -1 & \\
 & & & & & & & & 1 & -1 & & \\
 & & & & & & & & & & 1 & -1 \\
\alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & 1 & 1 & & & & & & \\
\alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} & & & 1 & 1 & & & & \\
\alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} & & & & & 1 & 1 & & \\
\alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} & & & & & & & 1 & 1
\end{bmatrix},
\tag{21}
$$

the vector of the macro-element basis functions $\varphi_E = \{\phi_i\}_{i=1}^{12}$ is transformed to a new hierarchical basis $\tilde{\varphi}_E = \{\tilde{\phi}_i\}_{i=1}^{12} = J_E \varphi_E$. Accordingly, J_E transforms the macro-element stiffness matrix into a hierarchical form

$$
\tilde{A}_E = J_E A_E J_E^T =
\begin{bmatrix}
\tilde{A}_{E,11} & \tilde{A}_{E,12} \\
\tilde{A}_{E,21} & \tilde{A}_{E,22}
\end{bmatrix}
\begin{array}{l}
\tilde{\phi}_i \in \mathcal{V}_1(E) \\
\tilde{\phi}_i \in \mathcal{V}_2(E)
\end{array}.
\tag{22}
$$

Following the local definitions, for the whole finite element space $\mathcal{V}_h$ with the standard nodal finite element basis $\varphi = \{\phi_h^{(i)}\}_{i=1}^{N_h}$ we can similarly construct the new hierarchical basis $\tilde{\varphi} = \{\tilde{\varphi}_h^{(i)}\}_{i=1}^{N_h}$ and the corresponding splitting

$$
\mathcal{V}_h = \mathcal{V}_1 \oplus \mathcal{V}_2.
\tag{23}
$$

The transformation J such that $\tilde{\varphi} = J\varphi$, can be used for transformation of the stiffness matrix A_h to hierarchical form $\tilde{A}_h = J A_h J^T$, which allows preconditioning by the two-level preconditioners based on the splitting (23). Now, we are in a position to analyze the constant

$$
\gamma = \cos(\mathcal{V}_1, \mathcal{V}_2)
$$

for the splitting (23). Again, as in the previous section, we would like to perform this analysis locally, by considering the corresponding problems on macro-elements. For this purpose we need to have satisfied the condition

(i) $\ker(\tilde{A}_{E,22}) = \ker(A_e)$,

which is equivalent to

$$
\sum_{i=1}^{4} \alpha_{ij} = 1, \qquad \forall j \in \{1, 2, 3, 4\}.
\tag{24}
$$

There are obviously various DA splittings satisfying the condition (i). In particular, the variant

$$
\mathcal{A} = [\alpha_{ij}] =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}
$$

could be considered as a direct interpretation of the algorithm from [12, 13], where the case of Crouzeix-Raviart linear non-conforming finite elements is considered. See for some more details about aggregation based preconditioners in the review paper [11].

When the two-level algorithm is recursively generalized to the multilevel case, it is useful if

(ii) $\tilde{A}_{E,22}$ is proportional to A_e.

Such a property holds in a very general setting for the DA splitting of the Crouzeix-Raviart finite element space, see [12]. Unfortunately, it seems to be rather complicated to find a parameter matrix $[\alpha_{ij}]$, which satisfies the condition (ii) in the general case of Rannacher-Turek rotated bilinear finite elements.

4   Uniform estimates of the CBS constant

We study in this section the isotropic model problem where all elements $e \in \mathcal{T}_H$ are squares, and the uniform refinement is as shown in Figure 4. Both splitting algorithms, FR and DA, for both variants MP and MV of rotated bilinear finite elements, are considered.

(a) One macro-element   (b) One element
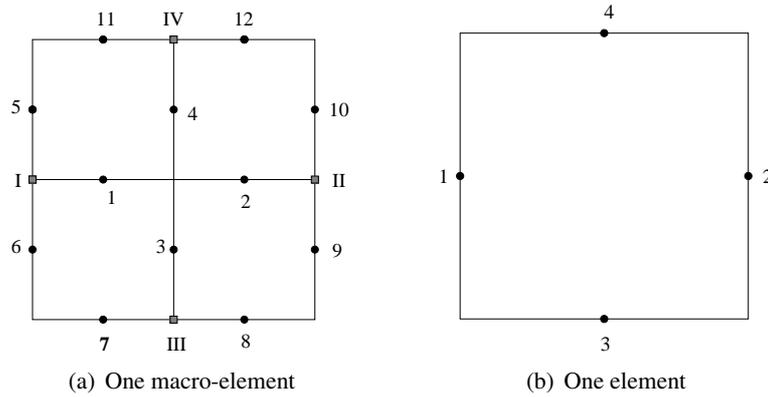
Figure 4: Uniform refinement on a square mesh.

4.1   FR algorithm

Following (20) we compute the local CBS constant and derive the following global estimates which are uniform with respect to the size of the discrete problem and any possible jumps of the coefficients $a(e), e \in \mathcal{T}_H$.

<u>Variant MP</u>:

$$\lambda_E^{(1)} = \frac{5}{7}, \qquad \gamma_E^2 = 1 - \lambda_E^{(1)} = \frac{2}{7},$$

and therefore

$$\gamma_{MP}^2 \leq \frac{2}{7}. \tag{25}$$

<u>Variant MV</u>:

$$\lambda_E^{(1)} = \frac{5}{8}, \qquad \gamma_E^2 = 1 - \lambda_E^{(1)} = \frac{3}{8},$$

and therefore

$$\gamma_{MV}^2 \leq \frac{3}{8}. \tag{26}$$

Let us remind once again, that the obtained estimates hold theoretically for the two-level algorithm only. This is because the matrix B_{22} is only associated with the coarse discretization $e \in \mathcal{T}_H$ and is not proportional to the related element stiffness matrix A_e. As we will see later, the CBS constants have a very stable behaviour in the FR multilevel setting, which has been verified numerically.

4.2   DA algorithm

Due to the symmetry of the model problem, the transformation matrix J_E can be simplified to the form

$$\mathcal{A} = \begin{bmatrix} b & c & a & a \\ c & b & a & a \\ a & a & b & c \\ a & a & c & b \end{bmatrix}. \tag{27}$$

The condition (i) is equivalent to

$$2a + b + c = 1.$$

Let us write the condition (ii) in the form

$$\tilde{A}_{E,22} = pA_e. \tag{28}$$

Then, (ii) is reduced to a system of two nonlinear equations for, say, (b, c), with a parameter p. It appears, that the system for (b, c) has a solution if $p \in [p_0, \infty)$. In such a case, we can optimize the parameter p, so that the related CBS constant is minimal. The obtained results are summarized below. For the related analysis we have used symbolic computations with the computer algebra program MATHEMATICA.

Variant MP:

Lemma 4.1 *There exists a DA two-level splitting satisfying the condition (ii), if and only if,*

$$p \geq \frac{3}{7}.$$

Then, the obtained solutions for (b, c) are invariant with respect to the local CBS constant

$$\gamma_E^2 = 1 - \frac{1}{4p},$$

and for the related optimal splitting

$$\gamma_{MP}^2 \leq \frac{5}{12}. \tag{29}$$

Although the statements of Lemma 4.1. look very simply, the midterm derivations are rather technical, which is just illustrated by the following expressions of one of the similarly looking solutions for (b, c):

$$b = \frac{-1}{70(-729 + 2240p)} \left(24786 - 76160p + 2658\sqrt{\phi(p)} - 7280p\sqrt{\phi(p)} + \sqrt{\phi(p)^3} \right)$$

$$c = \frac{1}{70} \left(6 - \sqrt{\phi(p)} \right)$$

where

$$\phi(p) = -1329 + 3640p - 140\sqrt{63 - 327p + 420p^2}.$$

Variant MV: The same approach is applied to get the estimates below.

Lemma 4.2 *There exists a DA two-level splitting satisfying the condition* (ii)*, if and only if,*

$$p \geq \frac{2}{5}.$$

Then, the obtained solutions for (b, c) *are invariant with respect to the local CBS constant*

$$\gamma_E^2 = 1 - \frac{1}{4p},$$

and for the related optimal splitting

$$\gamma_{MV}^2 \leq \frac{3}{8}. \tag{30}$$

5 Multilevel preconditioning

The multilevel methods have evolved from two-level methods. The straightforward recursive extension leads to the class of hierarchical basis (HB) methods for which the condition number grows in general exponentially with the number of levels ℓ. Therefore, in order to obtain multilevel preconditioners of both additive or multiplicative type, which have optimal convergence rate, i.e.,

$$\varkappa(M^{(\ell)^{-1}}A) = O(1),$$

and optimal computational complexity (linearly proportional to the number of degrees of freedom n_ℓ at the finest discretization level), HB preconditioners are combined with various types of stabilization techniques.

One particular purely algebraic stabilization technique is the so-called Algebraic Multilevel Iteration (AMLI) method, where a specially constructed matrix polynomial P_{β_k} of degree β_k is used on some (all) levels $k = k_0 + 1, \cdots, \ell$. The AMLI methods are originally introduced and studied in a multiplicative form, see [6, 7].

Starting from the coarsest mesh (level 0) with $M_F^{(0)} = A^{(0)}$, the basic idea is to apply the two-level preconditioner (11) recursively at all levels $k = 1, 2, \ldots, \ell$ of mesh refinement, i.e.,

$$M_F^{(k)} = \begin{bmatrix} C_{11}^{(k)} & 0 \\ \widetilde{A}_{21}^{(k)} & C_{22}^{(k)} \end{bmatrix} \begin{bmatrix} I & C_{11}^{(k)^{-1}} \widetilde{A}_{12}^{(k)} \\ 0 & I \end{bmatrix}. \tag{31}$$

Here $C_{11}^{(k)}$ is some preconditioner for the upper left block of the (hierarchical) stiffness matrix

$$\widetilde{A}^{(k)} = \begin{bmatrix} \widetilde{A}_{11}^{(k)} & \widetilde{A}_{12}^{(k)} \\ \widetilde{A}_{21}^{(k)} & \widetilde{A}_{22}^{(k)} \end{bmatrix}$$

at level k and the matrix $C_{22}^{(k)}$ is implicitly defined by the equation

$$C_{22}^{(k)^{-1}} = \left[I - P_\beta \left(M_F^{(k-1)^{-1}} \widetilde{A}^{(k-1)} \right) \right] \widetilde{A}^{(k-1)^{-1}} \tag{32}$$

12

where $M_F^{(k-1)}$ and $\widetilde{A}^{(k-1)}$ denote the multiplicative preconditioner and the stiffness matrix at level $(k-1)$, respectively, and $\widetilde{A}^{(0)} = A^{(0)}$ by definition. Then, as well known from theory [6, 7], a properly shifted Chebyshev polynomial P_β of degree $\beta \in \{2, 3\}$, satisfying the conditions

$$0 \le P_\beta(t) < 1, \quad 0 < t \le 1, \quad P_\beta(0) = 1,$$

can be used in order to stabilize the condition number of the linear AMLI preconditioner.

The main result from this analysis is that the AMLI preconditioner has optimal computational complexity, if $\beta_k = \beta$ and

$$\frac{1}{\sqrt{1-\gamma^2}} < \beta < \rho,$$

where $\rho \approx \dfrac{n_{k+1}}{n_k}$ is the reduction factor of the number of degrees of freedom.

In the case DA, $\gamma_{MP}^2 < 5/12$, $\gamma_{MV}^2 < 3/8$, and the optimality condition is reached for polynomial degrees $\beta \in \{2, 3\}$.

Now, let us turn back to the case FR. The multilevel behaviour of the CBS constant is studied numerically. This means, that at the current coarsening step, the role of the element stiffness matrix is played by the related last obtained block $B_{E,22}$. The obtained results are shown below in both, table and graphic form.

The computed estimates show transparently the stable behaviour of the CBS constant for the FR algorithm. The obtained γ-s are always smaller than the related ones for the DA algorithm. One can also observe a nice one-side convergence to the value of 0.3170 for both, MP and MV, cases, see Figure 5.

The conclusion of the considerations in this section is that the splitting DA provides better opportunities for a systematic theoretical analysis. However, the practical value of the counterpart approach FR seems to be more advantageous.

Table 1: Multilevel behaviour of γ^2 for "First reduce" algorithm

variant	ℓ	$\ell - 1$	$\ell - 2$	$\ell - 3$	$\ell - 4$	$\ell - 5$
MP	0.2857	0.3101	0.3156	0.3167	0.3169	0.3170
MV	0.3750	0.3261	0.3187	0.3173	0.3171	0.3170

Remark 5.1 It is important to note, that the CBS constant is not only used to analyze the related two-level and multilevel preconditioners. It is also involved in the construction of the acceleration matrix polynomial P_β. In other words, the smaller γ means the faster the PCG method convergence in a very general setting.

Remark 5.2 According to our local estimates for γ there is evidence to suggest that the symmetric preconditioner of block-diagonal (additive) form yields an optimal order AMLI method for the DA approach provided third-order stabilization polynomials are employed, i.e., the optimality condition

$$\sqrt{\frac{1+\gamma}{1-\gamma}} < \beta < \rho \tag{33}$$

is met for both – MP and MV – discretization variants if $\beta = 3$.
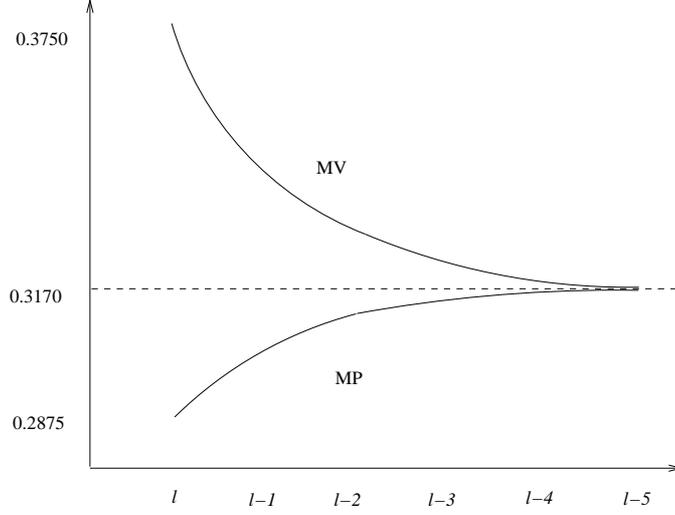
Figure 5: Multilevel behaviour of γ^2 for "First reduce" algorithm

Remark 5.3 The situation for the FR algorithm (according to the multilevel behaviour of γ^2) even indicates the opportunity of stabilizing the condition number of the additive AMLI preconditioner by using second-order matrix polynomials for which (according to (33)) $\gamma^2 < 0.36$ results in a uniformly bounded condition number.

Stabilization techniques for additive multilevel iteration methods and nearly optimal order parameter-free block-diagonal preconditioners of AMLI-type are discussed in References [2, 5, 18].

6 Numerical results

We studied the convergence properties of the preconditioned conjugate gradient (PCG) method using the multiplicative multilevel preconditioner (31) referred to as linear AMLI hereafter. The inverse of the upper left block $C_{11}^{(k)}$ was approximated using an incomplete LU factorization of $\widetilde{A}_{11}^{(k)}$, i.e., $C_{11}^{(k)} = L^{(k)}U^{(k)}$. Assuming that the relation

$$\widetilde{A}_{11}^{(k)} \le C_{11}^{(k)} \le (1+\delta_1)\widetilde{A}_{11}^{(k)} \tag{34}$$

holds (in a positive semidefinite sense) for some $\delta_1 \ge 0$ at all levels $k = 1, 2, \ldots, \ell$ it has been shown in [7] that the coefficients q_0 and q_1 of the optimal stabilization polynomial

$$Q_1(t) = (1 - P_2(t))/t = q_0 + q_1 t,$$

which is evaluated in the linear AMLI W-cycle, are given by

$$q_0 = \frac{2}{-\delta_1 + \sqrt{1 + \delta_1 + \delta_1^2 - \gamma^2}}, \quad q_1 = -\frac{1}{1 - \gamma^2 + \delta_1\left(1 + 2\delta_1 - 2\sqrt{1 + \delta_1 + \delta_1^2 - \gamma^2}\right)}.$$

As we discovered from experiments, even the choice $\delta_1 = 0$, i.e.,

$$q_0 = \frac{2}{\sqrt{1 - \gamma^2}}, \quad q_1 = -\frac{1}{1 - \gamma^2}, \tag{35}$$

14

is applicable for the model problem and was therefore used in our computations. However, this is not based on theory and in general the choice of proper coefficients of the stabilization polynomial will involve bounds of the form (34).

We also did numerical tests with the parameterfree version of variable-step preconditioning [8], referred to herein as nonlinear AMLI. A comparison of both methods can be found in Reference [15]. For the nonlinear variant of AMLI, the W-cycle involves two inner generalized conjugate gradient (GCG) iterations in order to stabilize the procedure at all intermediate levels (except the coarsest one); we used at most 10 vectors in the orthogonalization loop at the fine-grid level (with a restart after every 10 iterations).

In order to confirm the analysis from the previous sections we solved the model problem (15) on the unit square $\Omega = (0,1) \times (0,1)$ for constant coefficient $a(e) = 1$ using Dirichlet boundary conditions and starting with a random initial guess.

The coarsest mesh (at level 0) – involving a direct solve – was chosen to be of size 16×16 elements (resulting in 544 nodes) in all computations. The different refinement levels were taken as it can be seen from Table 2, which contains the inverse of the mesh size and the number of nodes in its second and third row, respectively. The iteration counts as reported from all our experiments correspond to a

Table 2: Refinement levels for model problem on unit square

ℓ	1	2	3	4	5
$1/h$	32	64	128	256	512
N	2112	8320	33024	131584	525312

reduction of the residual norm by a factor 10^6.

The first set of experiments refers to the isotropic model problem.

As we expect, the results for the linear AMLI V-cycle, given in Table 3, show a logarithmic growth, see e.g. [10, 16], of the number of iterations that are required to meet the stopping criterion (when increasing the problem size linearly). The convergence rates for the discretization variants MP and MV are nearly the same, however, the FR splitting yields better results than the DA splitting.

Table 3: Linear AMLI V-cycle: isotropic problem

	$1/h$	32	64	128	256	512
DA/MP		8	10	13	16	19
FR/MP		6	8	9	11	12
DA/MV		8	10	13	16	18
FR/MV		7	9	10	12	14

Using the matrix polynomial (with coefficients (35)) the linear AMLI W-cycle achieves a stabilization of the condition number at a small number of PCG iterations, as can be seen from Table 4.

Comparing these results with those obtained for the nonlinear AMLI W-cycle, which are summarized in Table 5, we find that the two preconditioners take almost the same effect. Since the

Table 4: Linear AMLI W-cycle: isotropic problem

	$1/h$	32	64	128	256	512
DA/MP	$\gamma^2 = 5/12$	8	8	8	8	8
FR/MP	$\gamma^2 = 2/7$	6	6	6	6	6
DA/MV	$\gamma^2 = 3/8$	8	9	9	9	9
FR/MV	$\gamma^2 = 3/8$	7	7	7	7	7

variable-step preconditioner operates in a self-adapting way this indicates that our bounds on the CBS constant are quite sharp.

Table 5: Nonlinear AMLI W-cycle: isotropic problem

$1/h$	32	64	128	256	512
DA/MP	8	8	8	8	8
FR/MP	6	6	6	6	6
DA/MV	8	8	8	8	8
FR/MV	7	7	7	7	7

If we consider the case of moderate anisotropy (up to 1:10) the performance of the linear AMLI preconditioner is still satisfying and the W-cycle can still be run with the same stabilization polynomial as used in the isotropic case. The results for the V-cycle, based on the FR-splitting, are shown in Table 6, those for the W-cycle and the different combinations of the splitting and the discretization variant are summarized in Table 7. Note that the iteration counts reported in Table 7 refer to the fully stabilized multilevel preconditioner, which means that for smaller problem-size one obtains (almost) the same results (like this can be observed for the isotropic problem, cf., Table 4).

Table 6: Linear AMLI V-cycle: orthotropic problem (moderate anisotropy); FR-splitting

	$1/h$	32	64	128	256	512
MP	$\varepsilon = 0.5$	7	9	11	13	15
MP	$\varepsilon = 0.1$	11	14	16	19	22
MV	$\varepsilon = 0.5$	8	10	12	14	16
MV	$\varepsilon = 0.1$	16	19	22	25	29

Finally, we tested the nonlinear AMLI W-cycle for the (uniformly) orthotropic model problem as well as for the case of alternating direction of anisotropy introducing jumps in the coefficients between the four similar subdomains $\Omega_1 = (0, 1/2)^2$, $\Omega_2 = (1/2, 1) \times (0, 1/2)$, $\Omega_3 = (0, 1/2) \times (1/2, 1)$, and

Table 7: Linear AMLI W-cycle: orthotropic problem (moderate anisotropy); $1/h = 512$

	DA/MP	FR/MP	DA/MV	FR/MV
$\varepsilon = 0.5$	10	7	10	8
$\varepsilon = 0.1$	18	12	21	17

$\Omega_4 = (1/2, 1)^2$ of $\Omega = (0, 1)^2$, i.e.,

$$\mathbf{a}(\mathbf{x}) = a(e) \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix} \quad \text{for} \quad \mathbf{x} \in \Omega_1 \cup \Omega_4,$$

$$\mathbf{a}(\mathbf{x}) = a(e) \begin{bmatrix} \varepsilon & 0 \\ 0 & 1 \end{bmatrix} \quad \text{for} \quad \mathbf{x} \in \Omega_2 \cup \Omega_3.$$

The stabilization of the condition number in both cases is achieved with two inner GCG iterations for all of the four considered combinations of the hierarchical splitting and the discretization variant. For that reason Tables 8–9 summarize again only the required number of (outer) iterations for the largest problem, i.e., $1/h = 512$, which appear as a kind of limiting numbers. Moreover, the results show the robustness of the method with respect to jump discontinuities of the entries of the coefficient matrix.

Table 8: Nonlinear AMLI W-cycle: orthotropic problem; $1/h = 512$

	DA/MP	FR/MP	DA/MV	FR/MV
$\varepsilon = 0.5$	9	7	10	8
$\varepsilon = 0.1$	18	11	21	17
$\varepsilon = 0.05$	26	16	34	24
$\varepsilon = 0.01$	78	37	104	59

Table 9: Nonlinear AMLI W-cycle: anisotropic problem with jump discontinuities; $1/h = 512$

	DA/MP	FR/MP	DA/MV	FR/MV
$\varepsilon = 0.5$	9	7	10	8
$\varepsilon = 0.1$	18	12	21	17
$\varepsilon = 0.05$	27	16	34	25
$\varepsilon = 0.01$	79	37	101	59

We observe that (especially) when increasing the ratio of anisotropy the FR splitting yields much better results than the DA approach. In our examples this results in about 20 to 50 percent less iterations to meet the stopping criterion. However, due to the static condensation step in the FR algorithm the computational work per iteration also increases (in our examples about 15 to 20 percent).

This becomes obvious if one compares the number of nonzero entries occurring in the incomplete factorization of the $\widetilde{A}_{11}$-blocks, as shown in Table 10; the FR approach (due to the exact elimination

Table 10: ILU factorization of $\widetilde{A}_{11}$: number of nonzero entries in L (U)

level	9	8	7	6	5
DA	1 438 212	359 688	90 240	22 592	5 664
FR	1 764 868	440 068	109 444	27 072	6 628

of the unknowns associated with the interior nodes of the macro-elements) partially has to embed a complete factorization (of a submatrix) which increases the amount of fill-in terms. Nevertheless, the numerical tests indicate some superiority of the FR approach at least from a practical point of view.

7   Concluding remarks

This study is strongly motivated by the expanding interest in non-conforming finite elements, which are very helpful for solving problems, where the standard conforming elements suffer from so-called locking effects. The success of the non-conforming finite elements can be often explained e.g. by the fact that they produce algebraic systems that are equivalent to the Schur complement system for the Lagrange multipliers arising from the mixed finite element method for Raviart-Thomas elements (see [1]).

In this paper, we presented a general setting of hierarchical splittings of the Rannacher-Turek non-conforming FEM spaces. New estimates for the constant in the strengthened CBS inequality are derived for the considered model problem.

At this stage of the study we underline the following issues.

- The DA algorithm allows for a direct extension of the γ estimate to the multi-level case if the condition (ii) holds. However, the latter is not a trivial problem even for the considered anisotropic model problem. In the general case, we are not able to say anything about the existence of a DA splitting satisfying condition (ii) so far.

- For the FR algorithm, the theoretically derived estimates of the CBS constant γ are not directly applicable to the multilevel case. However, the recursively computed γ-s show a very promising behaviour which could be advantageous from a practical point of view.

- The numerical tests fully confirm the theoretical estimates. Moreover, they indicate some self-stabilization of the FR algorithm and the nonlinear AMLI which is especially important for the case of stronger anisotropy.

8   Acknowledgments

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