The Keller-Segel model for chemotaxis with prevention of overcrowding: Linear vs. nonlinear diffusion

M. Burger, M. Di Francesco, Y. Dolak-Struss

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THE KELLER-SEGEL MODEL FOR CHEMOTAXIS WITH PREVENTION OF OVERCROWDING: LINEAR VS. NONLINEAR DIFFUSION

MARTIN BURGER, MARCO DI FRANCESCO, AND YASMIN DOLAK-STRUSS

Abstract. The aim of this paper is to discuss the effects of linear and nonlinear diffusion in the Keller-Segel model of chemotaxis with volume filling effect. In both cases we first cover the global existence and uniqueness theory of solutions of the Cauchy problem on $\mathbb{R}^d$. Then, we address the large time asymptotic behavior. In the linear diffusion case we provide several sufficient conditions such that the diffusion part dominates and yields decay to zero of solutions. We also provide an explicit decay rate towards self–similarity. Moreover, we prove that no stationary solutions with positive mass exist. In the nonlinear diffusion case we prove that the asymptotic behaviour is fully determined by the diffusivity constant in the model being larger or smaller than the threshold value $\varepsilon = 1$. Below this value we have existence of non-decaying solutions and their convergence (in terms of subsequences) to stationary solutions. For $\varepsilon > 1$ all compactly supported solutions are proven to decay asymptotically to zero, unlike in the classical models with linear diffusion, where the asymptotic behaviour depends on the initial mass.

1. Introduction

This paper focuses on the mathematical analysis of a chemotaxis model in the cases of linear and nonlinear diffusion. The general structure of the model we consider is

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = \nabla \cdot (M(\rho)\nabla \mu(\rho, S)) \\ -\Delta S + S = \rho, \end{cases} \tag{1.1}$$

posed on $\mathbb{R}^d \times \mathbb{R}^+$ subject to the initial condition

$$\rho(x, 0) = \rho_0(x), \quad \rho_0 \in L^1(\mathbb{R}^d), \quad 0 \leq \rho_0(x) \leq 1, \qquad x \in \mathbb{R}^d. \tag{1.2}$$

The mobility term M is given by $M(\rho) = \rho(1-\rho)$. Such a choice takes into account the prevention of overcrowding effect, sometimes also referred to as 'volume–filling' effect (see the motivations and the references below). The potential μ reads

$$\mu(\rho, S) = \frac{\delta E}{\delta \rho}(\rho, S) \tag{1.3}$$

where $\frac{\delta E}{\delta \rho}$ denotes the functional derivative of some energy functional with respect to ρ.

If we model the energy as a combination of a logarithmic entropy and an aggregation part, i.e.,

$$\begin{aligned} E(\rho, S) &= \varepsilon \int_{\mathbb{R}^d} \left(\rho \log \rho + (1-\rho)\log(1-\rho)\right)\, dx \\ &\quad - \int_{\mathbb{R}^d} \rho S \, dx + \frac{1}{2}\int_{\mathbb{R}^d} \left(|\nabla S|^2 + S^2\right)\, dx, \end{aligned} \tag{1.4}$$

with $\varepsilon > 0$, then system (1.1) becomes

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = \nabla \cdot (\varepsilon \nabla \rho - \rho(1-\rho)\nabla S) \\ -\Delta S + S = \rho, \end{cases} \tag{1.5}$$

which is a special case of the Keller-Segel model for chemotaxis, describing the behaviour of a cell population ρ under the influence of the chemical S produced by the cells themselves. Introduced in 1970 [KS70] to describe aggregation of slime mold amoebae, this model has become one of the most widely studied models in mathematical biology. The cell flux on the right hand side of (1.5) comprises two counteracting phenomena: random motion of cells described by Fick's law and cell movement in direction of the gradient of the chemical S. In contrast to the equations presented here, the gradient of S is multiplied by a linear instead of a nonlinear function of ρ in the classical version of the model. An interesting feature of this choice is the fact that solutions can become unbounded in finite time, thus giving rise to concentration phenomena. Whether this blow-up of solutions occurs or not depends typically on the initial data and the space dimension d, and conditions for the blow-up of solutions have been derived by a large number of authors, see for instance [JL92, HV96, HMV97, GZ98, Vel02]. Most studies focus on models where the evolution of the chemical S is governed by a parabolic equation (as in the original Keller-Segel model). We shall often refer to this model as the *fully parabolic* case. Typical alternative formulations for the evolution of S are given either by an elliptic equation in a more general form than that in (1.1) or by the Poisson equation. In the majority of the cases, the model is considered on bounded domains, typically with Neumann boundary conditions. Concerning the case of an unbounded domain, an extensive analysis of the model on $\mathbb{R}^d$, $d \geq 2$, is performed in [CPZ04] (see also the survey [Per04]), where the authors show that the global existence of solutions occurs when the initial value of the $L^{d/2}$ norm is smaller than a certain critical value. In the recent [DP04] it is proved that the critical value 8π of the initial mass produces an optimal threshold between blow up and global existence when $d = 2$. The situation in $d > 2$ is not fully understood, and there is a possible range of choice of initial data where both blow-up and existence can occur. We refer to the introduction of [CPZ04] for a detailed description of the subject. An interesting question, in this framework, is whether is possible to give sense to the evolution of the model after the blow up. This problem is studied in [Vel02, Vel04]. We mention here (and refer the interested reader to) the survey papers [Hor03, Hor04], where more general chemotaxis models are presented, together with an extensive list of results and references.

Although of great mathematical interest, models allowing for the infinite growth of solutions have often been criticized because their biological interpretation is not fully understood. As a consequence of that, several generalizations of the Keller–Segel model, where the formation of singularities is prevented *a priorily*, have been formulated recently. The basic assumption in these models is the existence of a maximal value for the cell density, which is reasonable in certain physical situations (see the introduction of [HP01]) where cells stop aggregating after a certain size of the aggregate has been reached. In [HP01], a chemotaxis model featuring a nonlinear flux term is presented, where the chemotactic response is shut off when a certain cell density has been reached. This model, being of type (1.5), but with a parabolic equation for S, prevents the overcrowding of cells, and the authors prove global existence of solutions under Neumann boundary conditions. In a second paper [PH03], the authors derive this model from a master equation describing a biased random walk on the real line, assuming that the probability of a cell moving to the right or the left depends on the difference in chemical concentration as well as on the cell density. The resulting evolution law for the density is the first equation of (1.1) with mobility $M(\rho) = \rho(1-\rho)$ and logarithmic entropy for ρ in the energy (although there is no clear separation between energy and mobility in [PH03], see also the considerations below on the relation between random mobility and chemotactic sensitivity). Note that the main difference to a standard derivation of the model without prevention of overcrowding is the additional factor $(1-\rho)$ in the mobility and the additional entropy term depending on $(1-\rho)$ in the energy (see also [Wrz04a, Wrz04b], where the same entropy is used to study the asymptotic behaviour in the fully parabolic case in a bounded domain). Intuitively, this change of mobility seems obvious, since the possibility of cells to move freely is limited by the cells around. Therefore, in the case of overcrowding (which happens in our scaling for $\rho \geq 1$), their motion would be stopped. The additional entropy term depending on $(1-\rho)$ is less obvious to interpret, it somehow forces cells to diffuse even in the case of overcrowding. On the other hand if the volume–filling mechanism is thought of as a finite size effect really blocking the cells, then also no diffusion would be allowed and the effect would have consequences only upon mobility but not on the energy.

This argument is one reason why we shall also discuss a different choice in the energy in this paper. More precisely, we shall use a quadratic energy of the form

$$E(\rho,S) \quad = \quad \frac{\varepsilon}{2}\int_{\mathbb{R}^d} \rho^2 \, dx - \int_{\mathbb{R}^d} \rho S \, dx + \frac{1}{2}\int_{\mathbb{R}^d} \left(|\nabla S|^2 + S^2\right) \, dx, \tag{1.6}$$

yielding the degenerate parabolic-elliptic problem

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = \nabla \cdot (\rho(1-\rho)\nabla(\varepsilon\rho - S)) \\ -\Delta S + S = \rho. \end{cases} \tag{1.7}$$

Several reformulations of the Keller–Segel model have been studied recently, with a nonlinear diffusion term in the evolution of the cell density replacing the linear one. For an overview of the biological motivations and the existence theory for such models we refer to [Hor03, Chapter6]. An interesting issue is whether is possible to avoid blow up of solutions by introducing nonlinear diffusion effects without the help of the volume–filling term in the mobility. A study towards this direction has been recently carried out in [Kow05]. In [CC05] a simple threshold condition for the nonlinear diffusion term of the density is established in order to achieve global existence in time (in the special case where the balance law of the cell density is coupled with Poisson equation). We stress that the main feature in our nonlinear model (1.7) is that the random mobility (i. e. the nonlinear diffusion coefficient) is vanishing at the threshold value $\rho = 1$. Such a property is not usually covered in the literature concerning chemotaxis models with nonlinear diffusion.

In addition to the remarks made above concerning the interplay between energy and mobility, another motivation for our choice of the system (1.7) is a simplification of an alternative formulation of the Keller-Segel model recently derived in [BO04]. Due to the physical observation that many chemicals appear both to stimulate directed motion up the chemotactic gradient and to alter the random mobility coefficient, the authors of [BO04] introduced a model based upon a multiphase interpretation of the cell density (in an incompressible material composed of the "phases" cells and water), where a direct relationship between the chemotaxis and random motion coefficients is established. In particular, they both may vanish at the threshold value of the density. A study of the well–posedness theory for models of that type has been started in [LW04], in the fully parabolic case on bounded domains, where the existence of nontrivial stationary solutions has been proven. Using an interpretation as a multiphase system, it is natural to make the connection to Cahn-Hilliard equations (cf. [CH58, CH71]), which have the same structure as the first equation in (1.1), but with the potential μ determined as

$$\mu = -\Delta\rho + W'(\rho)$$

for some double-well potential function W having minimizers at zero and one. The prevention of overcrowding by a mobility $M(\rho) = \rho(1-\rho)$ in the Cahn-Hilliard equation is well motivated from physical arguments and has been studied in detail (cf. [CENC96, CT94, EG96]).

Another argument in favour of (1.7) relies on the asymptotic behaviour, in particular on the existence both of decaying and of non-decaying solutions of (1.7). We recall that the long-time asymptotics of (1.5) in bounded intervals (under Neumann boundary conditions) have been recently studied in [DS05] and, with a parabolic equation for S, in [PH05]. The observed behaviour is a coarsening process reminiscent of phase change models, where plateau-like peaks of the cell density form after a short transition period and then merge exponentially slowly. Numerical studies indicate that in most situations the only stable stationary states are single plateaus located at boundary of the domain. It is therefore not surprising that the behaviour of solutions on the whole space is different. Roughly speaking, the cells are not stopped by any boundaries and therefore the linear model would allow them to spread out rather than aggregate. We shall make this statement more rigorous by several results on the decay of the cell density ρ for large time. For the present moment we emphasize that, to our knowledge, the choice of the system (1.7) is the only known version that achieves stationary profiles on the whole space.

Nonlinear degenerate diffusion models have been used to describe various biological phenomena such as the dispersal of biological populations (cf. e.g. [GM77]) or aggregation of animal population (called "swarming", cf. e.g. [NM83, MCO05, MEK99, TB04]), the latter exhibiting many further analogies to models for chemotaxis. In these applications, as well as in the model considered here, the main advantages of nonlinear diffusions are a finite speed of propagation (which seems more reasonable than infinite speed in particular in biological applications) and, as mentioned above, the existence of nontrivial stationary and non-decaying time-dependent solutions.

In order to make the comparison between the linear and nonlinear diffusion case more concrete, we give an overview of the main results of this paper in the following, which also provides a guideline through the paper for readers rather interested on the results than on the detailed proofs. Concerning existence, we have

- Global existence and uniqueness of weak solutions in the case of linear diffusion (Section 2.1).
- Global existence of weak and entropy solutions, uniqueness of entropy solutions of bounded variations in the case of nonlinear diffusion (Sections 3.1 and 3.2) and finite speed of propagation of the support in 1–d (Section 3.3).

The first step towards the large time behaviour of solutions is an investigation of possible stationary solutions. Here we find a first difference between the two cases, namely

- Non-existence of finite mass stationary solutions different from zero for linear diffusion (Subsection 2.2.1).
- Existence of finite mass stationary solutions different from zero for nonlinear diffusion in the case $\varepsilon < 1$ (Subsection 3.4.2). A partly numerical construction of stationary solutions even indicates their existence for arbitrary mass if $\varepsilon < 1$ (Section 3.5).

The detailed large-time behaviour is described by:

- Decay to zero of solutions in the linear diffusion case in the following two cases:
  - with arbitrary initial mass for $\varepsilon > \frac{1}{4}$ in 1–d (Section 2.2),
  - with small initial mass for arbitrary $\varepsilon > 0$ (Section 2.2).

  In both cases, solutions converge in L^1 towards the self–similar Gaussian solution of the heat equation with variance ε (Section 2.3).
- Existence of non-decaying solutions for $\varepsilon < 1$ in the nonlinear case (Subsection 3.4.1), these solutions converge (along subsequences) to stationary solutions (Subsection 3.4.2). For $\varepsilon > 1$ all compactly supported solutions decay and their support must become unbounded as $t \to \infty$ (Subsection 3.4.3).

The result summarized in the last item above, namely the possibility of achieving two drastically different asymptotic behaviours (i. e. convergence to nontrivial stationary solutions or decay to zero) by simply changing the diffusivity constant ε in (1.7) is the main result of our paper. We stress here that the behaviour of solutions for $\varepsilon > 1$ (i. e. their decay to zero) is independent of any initial parameter such as the total mass or the second moment. This fact constitutes an essential difference between (1.7) and classical models with linear diffusion on the whole $\mathbb{R}^d$. The main technique used throughout the paper is the use of suitable energy functionals or Lyapunov functionals, which are already known to be extremely helpful in order to achieve global existence for general nonlinear Keller–Segel models (see [Hor03]).

2. Linear diffusion

In this section, we first cover the existence and uniqueness theory for weak solutions of model (1.5). We then turn our attention to the asymptotic behaviour for large time. The local existence and uniqueness is achieved by means of a standard fixed point technique. The continuation for any time of the solutions is a due to the existence of the invariant domain $0 \leq \rho \leq 1$, which is consistent with the volume filling assumption discussed in the introduction. Similar results are presented in case of bounded domains in [HP01] (for the fully parabolic model) and in [DS05].

2.1. **Existence theory and preliminaries.** We study the parabolic–elliptic system

$$\begin{cases} \frac{\partial \rho}{\partial t} = \varepsilon \Delta \rho - \nabla \cdot (\rho(1-\rho)\nabla S) \\ -\Delta S + S = \rho, \end{cases} \tag{2.1}$$

where $x \in \mathbb{R}^d$, $d \geq 1$, $t \geq 0$, $\varepsilon > 0$, ρ and S are scalar functions. We recall that the above system can be decoupled in order to get a nonlocal parabolic equation for ρ by means of the convolution representation formula

$$S(x,t) = \int_{\mathbb{R}^d} \mathcal{B}(x-y)\rho(y,t)dy,$$

where $\mathcal{B}$ is the *Bessel potential*

$$\mathcal{B}(x) = \frac{1}{(4\pi)^{d/2}} \int_0^{+\infty} \frac{e^{-t-\frac{|x|^2}{4t}}}{t^{d/2}} dt. \tag{2.2}$$

2.1.1. *Existence of solutions.* Let us consider the Cauchy problem

$$\begin{cases} \frac{\partial \rho}{\partial t} = \varepsilon \Delta \rho - \nabla \cdot (\rho(1-\rho)\nabla(\mathcal{B} * \rho)) \\ \rho(x,0) = \rho_0(x). \end{cases} \tag{2.3}$$

We shall establish a local–in–time existence result for (2.3). First, let us recall some elementary properties of the Bessel potential (2.2) and of the heat kernel

$$\mathcal{G}(x,t) = \frac{1}{(4\pi\varepsilon t)^{d/2}} e^{-\frac{|x|^2}{4\varepsilon t}}, \qquad x \in \mathbb{R}^d, \ t > 0. \tag{2.4}$$

The proof of the following lemma follows by straightforward computations.

Lemma 2.1. *The following estimates are satisfied,*

$$\|\mathcal{B}\|_{L^1(\mathbb{R}^d)} = 1 \tag{2.5}$$

$$\|\nabla \mathcal{B}\|_{L^1(\mathbb{R}^d)} < +\infty \tag{2.6}$$

$$\|\mathcal{G}(\cdot,t)\|_{L^p(\mathbb{R}^d)} \leq C t^{-\frac{d(p-1)}{2p}}, \quad p \geq 1 \tag{2.7}$$

$$\|\nabla \mathcal{G}(\cdot,t)\|_{L^p(\mathbb{R}^d)} \leq C t^{-\frac{d(p-1)}{2p} - \frac{1}{2}}, \quad p \geq 1 \tag{2.8}$$

Theorem 2.2 (Local existence)**.** *Let $\rho_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ (resp. $\rho_0 \in L^\infty(\mathbb{R}^d)$). Then, there exists a unique solution $\rho(x,t)$ to (2.3) such that*

$$\rho \in L^\infty\left([0,T], L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\right)$$

(resp. $\rho \in L^\infty\left([0,T], L^\infty(\mathbb{R}^d)\right)$) for a small enough positive time T.

Proof. Let is first assume $\rho_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Given M, T positive constants , for $\rho \in L^\infty\left([0,T], L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\right)$ we define the norm

$$|||\rho|||_T := \sup_{0 \leq t \leq T} \left[\|\rho(t) - \mathcal{G}(t) * \rho_0\|_{L^1(\mathbb{R}^d)} + \|\rho(t) - \mathcal{G}(t) * \rho_0\|_{L^\infty(\mathbb{R}^d)} \right],$$

the Banach space

$$\mathcal{X}_T^M := \left\{ \rho \in L^\infty\left([0,T], L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\right) \,\Big|\, |||\rho|||_T \le M \right\},$$

and the map $\rho \mapsto \mathcal{T}\rho$

$$(\mathcal{T}\rho)(x,t) = (\mathcal{G} * \rho_0)(x,t) + \int_0^t \int_{\mathbb{R}^d} \nabla \mathcal{G}(x-y, s-t)\left(\rho(1-\rho)\nabla(\mathcal{B} * \rho)\right)(y,s)\,ds\,dy.$$

Thanks to the Duhamel principle for the linear heat equation (and thanks to integration by parts), a fixed point for $\mathcal{T}$ is clearly a weak solution to the Cauchy problem (2.3) with initial datum ρ_0 on the strip $[0,T]\times\mathbb{R}^d$. To establish the existence and uniqueness of the fixed point we first prove that $\mathcal{T}\rho \in \mathcal{X}_T^M$, then we prove that $\mathcal{T} : \mathcal{X}_T^M \to \mathcal{X}_T^M$ is a strict contraction, and we deduce the desired assertion by the Banach fixed point theorem. Let $\rho \in \mathcal{X}_T^M$. Young inequality for convolutions, together with (2.6), (2.7) and (2.8) yield, for any $0 < t < T$,

$$\begin{aligned}
&\|(\mathcal{T}\rho)(t) - \mathcal{G}(t) * \rho_0\|_{L^1(\mathbb{R}^d)} \le \int_0^t \|\nabla\mathcal{G}(t-s)\|_{L^1} \|(\rho(1-\rho)\nabla(\mathcal{B}*\rho))(s)\|_{L^1} ds \\
&\le C \int_0^t (t-s)^{-1/2} \|\rho(1-\rho)(s)\|_{L^\infty} \|\nabla\mathcal{B}\|_{L^1} \|\rho(s)\|_{L^1} ds \\
&\le C \int_0^t (t-s)^{-1/2} (\|\rho(s)\|_{L^\infty} + \|\rho(s)\|_{L^\infty}^2) \|\rho(s)\|_{L^1} ds \\
&\le C(M + \|\rho_0\|_{L^\infty})(M + \|\rho_0\|_{L^\infty} + 1)(M + \|\rho_0\|_{L^1}) T^{1/2}
\end{aligned}$$

Analogously,

$$\begin{aligned}
&\|(\mathcal{T}\rho)(t) - \mathcal{G}(t) * \rho_0\|_{L^\infty(\mathbb{R}^d)} \le \int_0^t \|\nabla\mathcal{G}(t-s)\|_{L^1} \|(\rho(1-\rho)\nabla(\mathcal{B}*\rho))(s)\|_{L^\infty} ds \\
&\le C \int_0^t (t-s)^{-1/2} \|\rho(1-\rho)(s)\|_{L^\infty} \|\nabla\mathcal{B}\|_{L^1} \|\rho(s)\|_{L^\infty} ds \\
&\le C(M + \|\rho_0\|_{L^\infty})(M + \|\rho_0\|_{L^\infty} + 1)(M + \|\rho_0\|_{L^\infty}) T^{1/2}
\end{aligned}$$

The two estimates above ensure $\mathcal{T}\rho \in \mathcal{X}_T^M$ provided T is small enough. We now consider two elements $\rho, \overline{\rho} \in \mathcal{X}_T^M$. For any $0 < t < T$, in a similar fashion as above we have

$$\begin{aligned}
&\|(\mathcal{T}\rho)(t) - (\mathcal{T}\overline{\rho})(t)\|_{L^1(\mathbb{R}^d)} \\
&\le \int_0^t \|\nabla\mathcal{G}(t-s)\|_{L^1} \|(\rho(1-\rho)\nabla(\mathcal{B}*\rho))(s) - (\overline{\rho}(1-\overline{\rho})\nabla(\mathcal{B}*\overline{\rho}))(s)\|_{L^1} ds \\
&\le \int_0^t \|\nabla\mathcal{G}(t-s)\|_{L^1} \|(\rho(1-\rho)\nabla(\mathcal{B}*\rho))(s) - (\overline{\rho}(1-\rho)\nabla(\mathcal{B}*\rho))(s)\|_{L^1} ds \\
&+ \int_0^t \|\nabla\mathcal{G}(t-s)\|_{L^1} \|(\overline{\rho}(1-\rho)\nabla(\mathcal{B}*\rho))(s) - (\overline{\rho}(1-\overline{\rho})\nabla(\mathcal{B}*\rho))(s)\|_{L^1} ds \\
&+ \int_0^t \|\nabla\mathcal{G}(t-s)\|_{L^1} \|(\overline{\rho}(1-\overline{\rho})\nabla(\mathcal{B}*\rho))(s) - (\overline{\rho}(1-\overline{\rho})\nabla(\mathcal{B}*\overline{\rho}))(s)\|_{L^1} ds \\
&\le C(\|\rho_0\|_{L^1}, M) T^{1/2} \sup_{0<t<T} \|\rho(t) - \overline{\rho}(t)\|_{L^1},
\end{aligned} \tag{2.9}$$

and, analogously

$$\|(\mathcal{T}\rho)(t) - (\mathcal{T}\overline{\rho})(t)\|_{L^\infty(\mathbb{R}^d)} \leq C(\|\rho_0\|_{L^\infty}, M)T^{1/2} \sup_{0<t<T} \|\rho(t) - \overline{\rho}(t)\|_{L^\infty}.$$

Hence, by choosing T small, we have

$$|||\mathcal{T}\rho - \mathcal{T}\overline{\rho}|||_T \leq \alpha |||\rho - \overline{\rho}|||_T,$$

for some $0 < \alpha < 1$, which concludes the proof in case $\rho_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. The proof in case $\rho_0 \in L^\infty(\mathbb{R}^d)$ can be obtained by modifying the norm $|||\cdot|||_T$ and the Banach space $\mathcal{X}_T^M$ (by taking into account the L^∞ norm only) and by repeating the same steps as in the previous case. □

Remark 2.3 (Continuation principle)**.** From the implicit representation formula

$$\rho(x,t) = (\mathcal{G} * \rho_0)(x,t) + \int_0^t \int_{\mathbb{R}^d} \nabla\mathcal{G}(x-y,t-s)\,(\rho(1-\rho)\nabla(\mathcal{B}*\rho))\,(y,s)dsdy \tag{2.10}$$

of the local solution ρ, we also deduce that, if T_M is the maximal time of existence of the solution, then there exist $t_j \to T_M^-$ as $j \to \infty$ such that

$$\lim_{j\to\infty} [\|\rho(t_j)\|_{L^1} + \|\rho(t_j)\|_{L^\infty}] = +\infty.$$

Remark 2.4 (Higher regularity)**.** The local–in–time solution $\rho(x,t)$ provided by theorem 2.2 is endowed with (at least) the same regularity with respect to x of the initial datum. To see this, one can modify the space and the norm in the proof of the previous theorem by taking into account some L^p norm of $\nabla\rho$.

A bit of regularity for ρ without the help of any further requirements on the initial datum can be obtained at least in 1–d.

Proposition 2.5 (Regularizing effect)**.** *Let the initial datum $\rho_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then, at any positive time t, the solution $\rho(t)$ of (2.1) is continuous with respect to x.*

Proof. Since $\mathcal{G} * \rho_0$ is a C^∞ function, we only need to prove that $\rho - \mathcal{G} * \rho_0$ is continuous. By formula (2.10), we have for small $h > 0$

$$\begin{aligned}
&(\rho - \mathcal{G}*\rho_0)(x+h,t) - (\rho - \mathcal{G}*\rho_0)(x,t)\\
&= \int_0^t \int_{-\infty}^{+\infty} [\mathcal{G}_x(x+h-y,t-s) - \mathcal{G}_x(x-y,t-s)]\,(\rho(1-\rho)\nabla(\mathcal{B}*\rho))\,(y,s)dsdy.
\end{aligned}$$

We recall that the term $(\rho(1-\rho)\nabla(\mathcal{B}*\rho))$ is (locally) bounded. Moreover, thanks to (2.8) we can find nonnegative functions $H(x,y,t,s)$ and $K \in L^1$ such that

$$|\mathcal{G}_x(x+h-y,t-s) - \mathcal{G}_x(x-y,t-s)| \leq H(x,y,t,s) \leq (t-s)^{-1} K\left(\frac{|x-y|^2}{t-s}\right).$$

Therefore, the limit as $h \to 0$ of the left hand side above is zero in view of Lebesgue's dominated convergence theorem. □

Theorem 2.6 (Continuity with respect to the initial data)**.** *Let ρ and $\overline{\rho}$ two local–in–time solutions to (2.1) with initial data $\rho_0, \overline{\rho}_0 \in L^1 \cap L^\infty$ respectively. Then, for a small T we have*

$$\|\rho(t) - \overline{\rho}(t)\|_{L^1} \leq C(T)\|\rho_0 - \overline{\rho}_0\|_{L^1}, \tag{2.11}$$

for all $t \in [0,T]$.

Proof. The proof can be performed by means of a similar argument as in (2.9). Since, in this case, we have two different initial data, we easily obtain

$$\|\rho(t) - \overline{\rho}(t)\|_{L^1} \leq CT^{1/2} \sup_{0\leq t\leq T} \|\rho(t) - \overline{\rho}(t)\|_{L^1} + \|\rho_0 - \overline{\rho}_0\|_{L^1},$$

where C depends on the two solutions. For small T we have the desired estimate (2.11). □

2.1.2. *Global existence.* In the following, we will be interested in initial data ρ_0 in (2.3) satisfying the assumption

$$0 \leq \rho_0(x) \leq 1, \qquad x \in \mathbb{R}^d. \tag{2.12}$$

Our aim is to prove that condition (2.12) is invariant under the flow induced by the model (2.1). Similar properties are proven in [HP01] for the fully parabolic model and in [DS05] in a bounded domain. Let us start with the following theorem.

Theorem 2.7 (Conservation of the total mass)**.** *Let $\rho_0 \in L^1(\mathbb{R}^d)$ satisfying (2.12). Then,*

$$\int_{\mathbb{R}^d} \rho(x,t)dx = \int_{\mathbb{R}^d} \rho_0(x)dx. \tag{2.13}$$

Proof. Let $\zeta_n = \zeta_n(x)$ be a sequence of cut–off functions such that $\zeta_n(x) = 1$ as $|x| \leq n$, $\zeta_n(x) = 0$ as $|x| \geq n+1$, $0 \leq \zeta_n(x) \leq 1$ as $n \leq |x| \leq n+1$ and $\zeta_n \in C^\infty(\mathbb{R}^d)$. Multiplying the equation in (2.3) by ζ_n, after integration over $\mathbb{R}^d \times [0,T]$ we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} \rho(x,t)\zeta_n dx - \int_{\mathbb{R}^d} \rho_0(x)\zeta_n dx = \int_0^T \int_{\mathbb{R}^d} \rho_t\zeta_n dxdt \\
&= \int_0^T \int_{\mathbb{R}^d} \zeta_n[\varepsilon\Delta\rho - \nabla\cdot(\rho(1-\rho)\nabla S)]dxdt \\
&= \varepsilon \int_0^T \int_{\mathbb{R}^d} \Delta\zeta_n\rho + \int_0^T \int_{\mathbb{R}^d} \rho(1-\rho)\nabla S\cdot\nabla\zeta_n dxdt \ \to 0 \quad \text{as } n \to +\infty
\end{aligned}$$

The last step is justified by dominated convergence theorem. The whole calculations are justified in case of smooth solutions. Hence, we shall assume that the initial datum belongs in some suitable Sobolev space (see remark 2.4). The result for a general initial datum ρ_0 then follows by approximating ρ_0 with a sequence of smooth initial data and by using the continuity result in theorem 2.6 and Fatou's lemma. □

Theorem 2.8 (Global existence)**.** *Assume the initial datum $\rho_0 \in L^1(\mathbb{R}^d)$ satisfies (2.12). Then, there exists a unique global solution to the Cauchy problem (2.3), satisfying*

$$0 \leq \rho(x,t) \leq 1 \quad \textit{for any} \quad (x,t) \in \mathbb{R}^d \times [0,\infty). \tag{2.14}$$

In particular,

$$0 < \rho(x,t) < 1 \quad \textit{if} \quad 0 < \rho_0(x) < 1. \tag{2.15}$$

Proof. Writing (2.3) as $\rho_t + \nabla\rho\cdot\nabla S(1-2\rho) + \rho(1-\rho)(S-\rho) = \varepsilon\Delta\rho$, it can be seen immediately that $\underline{\rho} \equiv 0$ and $\overline{\rho} \equiv 1$ are lower and upper solutions, respectively. By the mean value theorem of multidimensional calculus, the function $\underline{w}(x,t) = \rho - \underline{\rho}$ satisfies the inequality

$$\underline{w}_t + A(x,t)\nabla\underline{w} + B(x,t)\underline{w} - \varepsilon\Delta\underline{w} \geq 0,$$

with bounded coefficients $A(x,t)$ and $B(x,t)$. For $\overline{w}(x,t) = \rho - \overline{\rho}$, the same equations with a reversed inequality sign holds, and the boundedness of ρ follows from the Phragmèn-Lindelöf principle for parabolic equations [PW84]. Together with theorem 2.7 and with remark 2.3, this implies the assertion. □

2.2. **Decay of solutions as $t \to +\infty$.** In this subsection we determine sufficient conditions on the diffusivity constant ε in (2.1) and on the initial datum ρ_0 such that the $L^\infty(\mathbb{R}^d)$–norm of the corresponding solution $\rho(x,t)$ tends to zero as $t \to +\infty$. When such a phenomenon occurs, the diffusion term $\varepsilon\Delta\rho$ in (2.1) becomes dominant. We prove here that this is the case when either the mass is small enough for arbitrary $\varepsilon > 0$ or for *any* mass in case $\varepsilon > 1/4$ in 1–d.

It is already known in case of models without prevention of overcrowding that when the mass of the initial datum is much smaller than some constant depending on ε, then the diffusion term becomes dominant and produces a long time decay of the solution (see [CPZ04, DP04]). For the sake of completeness we shall reproduce the same result for our model in the following proposition, the proof of which being partly the same as in [CPZ04].

Proposition 2.9. *Let $\varepsilon > 0$. Let $\rho_0 \in L^1(\mathbb{R}^d)$ satisfying (2.12). Then, there exists a constant $C(d,\varepsilon)$ depending only on the dimension and on the diffusivity ε, such that, if the total mass satisfies*

$$\int_{\mathbb{R}^d} \rho_0 dx < C(d,\varepsilon), \tag{2.16}$$

then, the solution $\rho(x,t)$ to (2.3) satisfies the decay estimates

$$\|\rho(t)\|_{L^p(\mathbb{R}^d)} \le C(t+1)^{-\frac{d(p-1)}{2p}}, \quad 2 \le p \le \infty. \tag{2.17}$$

Proof. Let us start with the L_p estimates for finite p. By multiplying the equation in (2.3) by $p\rho^{p-1}$ and after integration by parts we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d} \rho^p(x,t)dx = \varepsilon p \int_{\mathbb{R}^d} \rho^{p-1}\Delta\rho dx + p(p-1)\int_{\mathbb{R}^d} (\rho^{p-1} - \rho^p)\nabla S \cdot \nabla\rho dx \\
&\le -\varepsilon\frac{4(p-1)}{p}\int_{\mathbb{R}^d} |\nabla\rho^{p/2}|^2 dx + (p-1)\int_{\mathbb{R}^d} \nabla\rho^p \cdot \nabla S dx \\
&\le -\varepsilon\frac{4(p-1)}{p}\int_{\mathbb{R}^d} |\nabla\rho^{p/2}|^2 dx + (p-1)\int_{\mathbb{R}^d} \rho^{p+1} dx,
\end{aligned}$$

where we have used the a priori estimate $0 \le \rho \le 1$ and $S \ge 0$. By means of the Gagliardo–Nirenberg inequality

$$\int_{\mathbb{R}^d} \rho^{p+1} dx \le C(p,d)\|\rho\|_{L^\alpha(\mathbb{R}^d)} \int_{\mathbb{R}^d} |\nabla\rho^{p/2}|^2 dx,$$

where $\alpha = 1$ for $d = 1,2$, $\alpha = d/2$ for $d > 2$, we easily get

$$\frac{d}{dt}\int_{\mathbb{R}^d} \rho^p(x,t)dx \le -(p-1)\left(\frac{4\varepsilon}{p} - C(p,d)\int_{\mathbb{R}^d} \rho_0 dx\right)\int_{\mathbb{R}^d} |\nabla\rho^{p/2}|^2 dx$$

Hence, for $\int_{\mathbb{R}^d} \rho_0 dx < \frac{4\varepsilon}{pC(p,d)}$, we can write, for some $C > 0$,

$$\frac{d}{dt}\int_{\mathbb{R}^d} \rho^p(x,t)dx + C\int_{\mathbb{R}^d} |\nabla\rho^{p/2}|^2 dx \le 0.$$

Thanks to the following interpolation inequality (see e.g. [EZ91])

$$\|\rho\|_{L^p(\mathbb{R}^d)}^{\frac{(d(p-1)+2)p}{d(p-1)}} \leq \overline{C}(p,d)\|\nabla\rho^{p/2}\|_{L^2(\mathbb{R}^d)}^2\|\rho\|_{L^1(\mathbb{R}^d)}^{\frac{2p}{d(p-1)}}$$

we have

$$\frac{d}{dt}\int_{\mathbb{R}^d}\rho^p(x,t)dx + Cm\left(\int_{\mathbb{R}^d}\rho^p dx\right)^{\frac{d(p-1)+2}{d(p-1)}} \leq 0$$

which implies the desired polynomial time–decay in L^p

$$\|\rho(t)\|_{L^p(\mathbb{R}^d)} \leq C(p,d)(t+1)^{-d(p-1)/2p} \tag{2.18}$$

where the $(t+1)$ instead of t is justified by the global–in–time control of all the L^p norms proven in the previous subsection. In order to obtain the corresponding L^∞ estimate, we employ the implicit representation of the solution ρ provided by Duhamel principle

$$\begin{aligned}\rho(x,2t) &= \mathcal{G}(2t)*\rho(t) + \int_t^{2t}\nabla\mathcal{G}(2t-\sigma)*(\rho(1-\rho)\nabla\mathcal{B}*\rho)(\sigma)d\sigma \\ &= \mathcal{G}(2t)*\rho(t) + \int_0^t\nabla\mathcal{G}(t-s)*(\rho(1-\rho)\nabla\mathcal{B}*\rho)(t+s)ds. \end{aligned}\tag{2.19}$$

By taking the L^∞ norm in (2.19), we obtain

$$\begin{aligned}&\|\rho(2t)\|_{L^\infty(\mathbb{R}^d)} \leq \|\mathcal{G}(t)\|_{L^\infty(\mathbb{R}^d)}\|\rho(t)\|_{L^1(\mathbb{R}^d)} \\ &+\int_0^t\|\nabla\mathcal{G}(t-s)\|_{L^r(\mathbb{R}^d)}\|\rho(1-\rho)\nabla\mathcal{B}*\rho(t+s)\|_{L^{r'}(\mathbb{R}^d)}ds \\ &\leq Ct^{-\frac{d}{2}} + C\int_0^t(t-s)^{-\frac{d(r-1)}{2r}-\frac{1}{2}}\|\rho(t+s)\|_{L^{2r'}(\mathbb{R}^d)}^2ds \\ &\leq Ct^{-\frac{d}{2}} + \int_0^t(t-s)^{-\frac{d(r-1)}{2r}-\frac{1}{2}}(t+s)^{-\frac{d(2r'-1)}{r'}}ds = Ct^{-\frac{d}{2}} + Ct^{\frac{1}{2}-d}\end{aligned}$$

for $r < \frac{d}{d-1}$, $r' = \frac{r}{r-1}$. Of course, since $\|\rho(t)\|_{L^\infty(\mathbb{R}^d)}$ is uniformly bounded and since $d \geq 1$, we have the estimate

$$\|\rho(t)\|_{L^\infty(\mathbb{R}^d)} \leq C(t+1)^{-\frac{d}{2}}.$$

□

In the next proposition we prove that solutions to (2.3) enjoy a time decay rate as in (2.17) no matter how large the mass is, provided $\varepsilon > 1/4$ and $d = 1$. This result constitutes an essential difference of the present model with respect to the classical models where overcrowding phenomena (i. e. blow up) may occur in finite time.

Proposition 2.10. *Let $\varepsilon > 1/4$ and $d = 1$. Let $\rho_0 \in L^1(\mathbb{R})$ satisfying (2.12). Then, the solution $\rho(x,t)$ to (2.3) satisfies the decay estimates*

$$\|\rho(t)\|_{L^p(\mathbb{R})} \leq C(t+1)^{-\frac{(p-1)}{2p}}, \quad 2 \leq p \leq \infty. \tag{2.20}$$

Proof. We start by performing the following L^2 estimate.

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^d}\rho^2(x,t)dx = 2\varepsilon\int_{\mathbb{R}^d}\rho\Delta\rho dx + 2\int_{\mathbb{R}^d}\rho(1-\rho)\nabla S\cdot\nabla\rho dx \\ &\leq -2\varepsilon\int_{\mathbb{R}^d}|\nabla\rho|^2dx + \frac{1}{2}\int_{\mathbb{R}^d}|\nabla\rho\cdot\nabla S|dx\end{aligned}$$

As a consequence of the Young inequality for convolutions we have

$$\int_{\mathbb{R}^d} |\nabla\rho \cdot \nabla S| dx \leq \int_{\mathbb{R}^d} |\nabla\rho|^2 dx$$

and therefore

$$\frac{d}{dt}\int_{\mathbb{R}^d} \rho^2(x,t)dx \leq -2\left(\varepsilon - \frac{1}{4}\right)\int_{\mathbb{R}^d} |\nabla\rho|^2 dx.$$

Then, by means of the Gagliardo–Nirenberg inequality, as in the previous proposition, we get the decay in L^2

$$\|\rho(t)\|_{L^2(\mathbb{R})} \leq C(t+1)^{-1/4}.$$

By taking the L^4 norm in the representation (2.19), in a similar fashion as in the previous proposition, we obtain

$$\begin{aligned}
\|\rho(2t)\|_{L^4(\mathbb{R})} &\leq C(t+1)^{-3/8} + \int_0^t \|\nabla\mathcal{G}(t-s)\|_{L^4}\|\rho\nabla\mathcal{B} * \rho(t+s)\|_{L^1} ds \\
&\leq C(t+1)^{-3/8} + \int_0^t (t-s)^{-7/8}(t+s)^{-1/2} ds \leq C(t+1)^{-3/8} + Ct^{-3/8}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\|\rho(2t)\|_{L^\infty(\mathbb{R})} &\leq C(t+1)^{-1/2} + \int_0^t \|\nabla\mathcal{G}(t-s)\|_{L^2}\|\rho\nabla\mathcal{B} * \rho(t+s)\|_{L^2} ds \\
&\leq C(t+1)^{-1/2} + \int_0^t (t-s)^{-3/4}(t+s)^{-3/4} ds \leq C(t+1)^{-1/2}.
\end{aligned}$$

The remaining L^p estimates are easily obtained by interpolation. □

2.2.1. *Some remarks and the nonexistence of stationary solutions.* Clearly, an open question is whether solutions to (2.3) decay for any ε and for arbitrarily large masses. In bounded domains, for $d = 1$ and Neumann boundary conditions, system (2.1) has been shown to decay to the constant solution if $\varepsilon > \frac{1}{4}$, but if ε is small enough, stationary, periodic solutions in $L^1((0, L))$ exist (see [DS05] and [PH05]). However, one can easily prove that there exist no nonzero stationary solutions to (2.1) in $L^1(\mathbb{R}^d)$ in the case of unbounded domains: We define the energy functional $E(\rho, S)$ of system (2.1) by

$$E = \frac{1}{2}\int (|\nabla S|^2 + S^2)dx - \int \rho S dx + \varepsilon\int [\rho\log\rho + (1-\rho)\log(1-\rho)]dx, \quad (2.21)$$

with

$$\frac{\partial E}{\partial\rho} = -S + \varepsilon\log\frac{\rho}{(1-\rho)} \quad \text{and} \quad \frac{\partial E}{\partial S} = 0. \quad (2.22)$$

Rewriting the first equation of (2.1) as

$$\frac{\partial\rho}{\partial t} = \nabla\cdot\left(\rho(1-\rho)\nabla\frac{\partial E}{\partial\rho}\right), \quad (2.23)$$

and differentiating the energy with respect to time, we obtain

$$\begin{aligned}
\frac{dE}{dt} =& \frac{\partial E}{\partial S}\frac{\partial S}{\partial t} + \frac{\partial E}{\partial\rho}\frac{\partial\rho}{\partial t} = \int \nabla\cdot\left(\rho(1-\rho)\nabla\frac{\partial E}{\partial\rho}\right)\frac{\partial E}{\partial\rho}dx \\
=& -\int \rho(1-\rho)\left|\nabla\frac{\partial E}{\partial\rho}\right|^2 dx \leq 0.
\end{aligned}$$

Hence, the energy is decreasing in time, and the stationary state $\frac{dE}{dt} = 0$ is only reached if $\rho = 0$, $\rho = 1$ or $\frac{\partial E}{\partial \rho} = const.$, the latter implying that the stationary solution (ρ, S) should satisfy, for some positive constant C,

$$\frac{\rho}{1-\rho} = e^{\frac{S+C}{\varepsilon}},$$

in some open set $\Omega \subset \mathbb{R}^d$, with $\rho = 0$ in some point of $\partial\Omega$. However, this is incompatible with S being bounded, because of the continuity of ρ stated in proposition 2.5.

2.3. **Self–similar long time behaviour.** The aim of this subsection is to prove that, under suitable assumptions on the initial datum ρ_0, the solution to (2.3) behaves asymptotically like the fundamental solution of the heat equation. A similar result concerning self–similar asymptotic behaviour for the Keller–Segel model without the volume filling effect, where the evolution of S is described by Poisson's equation has been proven by J. Dolbeault and A. Blanchel (private communication).

To perform this task we employ the entropy dissipation method (see [AMTU01, CT00]). The long time decay properties of the solution $\rho(x,t)$ are a crucial ingredient in the arguments below. We shall prove our result by assuming *a priori* that the solution $\rho(x,t)$ satisfies the decay estimate

$$\|\rho(t)\|_{L^\infty(\mathbb{R}^d)} \leq C(t+1)^{-\frac{d}{2}} \tag{2.24}$$

which we proved to be fulfilled under the assumptions in propositions 2.9 and 2.10.

2.3.1. *Preliminaries.* Our first step is the following standard (mass–preserving) time dependent rescaling

$$\begin{cases} \rho(x,t) = R(t)^{-\frac{d}{2}} v(y,s) \\ y = R(t)^{-\frac{1}{2}} x \\ s = \frac{1}{2}\log R(t) \\ R(t) = 2t+1. \end{cases} \tag{2.25}$$

Then, it is easily seen that $v(y,s)$ satisfies the Cauchy problem

$$\begin{cases} \frac{\partial v}{\partial s} = \nabla\cdot(\varepsilon\nabla v + yv) - e^{-ds}\nabla\cdot(v(1-e^{-s}v)\mathcal{B}_s * \nabla v) \\ v(y,0) = \rho_0(y) \end{cases} \tag{2.26}$$

where

$$\mathcal{B}_s(y) = e^s\mathcal{B}(e^s y).$$

For future reference we write equation (2.26) as follows

$$\frac{\partial v}{\partial s} = \varepsilon\nabla\cdot\left(v\nabla\left(\log v + \frac{|y|^2}{2\varepsilon}\right)\right) - e^{-ds}\nabla\cdot(v(1-e^{-s}v)\mathcal{B}_s * \nabla v). \tag{2.27}$$

We remark that the fundamental solution $\mathcal{G}(x,t)$ of the heat equation in rescaled variables depends only on y; more precisely, it is given by

$$\mathcal{U}_m(y) = Ce^{-\frac{|y|^2}{2\varepsilon}},$$

where C depends on the mass m of ρ_0. Moreover, we recall that $\mathcal{U}_m$ satisfies the elliptic equation $\nabla\cdot(\varepsilon\nabla\mathcal{U}_m + y\mathcal{U}_m) = 0$. We shall make use of the classical entropy functional

$$\mathrm{E}(v) = \int_{\mathbb{R}^d} v(y)\log v(y)dy + \frac{1}{2\varepsilon}\int_{\mathbb{R}^d}|y|^2 v(y)dy, \tag{2.28}$$

and of the *relative entropy*

$$\mathrm{RE}(v|\mathcal{U}_m) = \mathrm{E}(v) - \mathrm{E}(\mathcal{U}_m).$$

We observe that, thanks to the alternative form (2.27), equation (2.26) can be written as

$$\frac{\partial v}{\partial s} = \varepsilon \nabla \cdot \left(v \nabla \frac{\delta \mathrm{E}(v)}{\delta v} \right) - e^{-ds} \nabla \cdot (v(1 - e^{-s} v) \mathcal{B}_s * \nabla v), \tag{2.29}$$

where $\frac{\delta \mathrm{E}}{\delta v}$ is the first variation of the functional E. Once the mass m is fixed, the relative entropy functional attains zero as minimum value at the ground state $\mathcal{U}_m$ (see [CT00, MV00]). The following inequality (see [AMTU00] for the proof) establishes a connection between the convergence in relative entropy and the convergence in L^1.

Theorem 2.11 (Csiszár–Kullback inequality)**.** *Let $v \in L^1(\mathbb{R}^d)$ having mass m and such that $\mathrm{E}(v) < +\infty$. Then, there exists a fixed constant C (depending on m) such that*

$$\|v - \mathcal{U}_m\|^2 \leq C \mathrm{RE}(v|\mathcal{U}_m). \tag{2.30}$$

For future reference, we recall the following logarithmic Sobolev inequality due to Gross [Gro75] and subsequently generalized in [AMTU01].

Theorem 2.12 (Logarithmic Sobolev inequality)**.** *Let $v \in L^1(\mathbb{R}^d)$ having mass m and such that $\mathrm{E}(v) < +\infty$. Then, the following inequality is satisfied*

$$\mathrm{RE}(v|\mathcal{U}_m) \leq \frac{\varepsilon}{2} \mathrm{I}(v|\mathcal{U}_m), \tag{2.31}$$

where

$$\mathrm{I}(v|\mathcal{U}_m) = \int_{\mathbb{R}^d} v \left| \nabla \left(\log \frac{v}{\mathcal{U}_m} \right) \right|^2 dy = \int_{\mathbb{R}^d} v \left| \nabla \left(\log v + \frac{|y|^2}{2\varepsilon} \right) \right|^2 dy$$

is called (relative) Fisher information*.*

Finally, we remark that assumption (2.24) for ρ in the new variables $v(y,s)$ reads

$$\|v(s)\|_{L^\infty(\mathbb{R}^d)} \leq C \tag{2.32}$$

for some fixed $C > 0$ depending only on the initial datum.

2.3.2. *Trend to self–similarity.* In this subsection we employ the tools introduced above to prove the asymptotic self–similar behaviour of solutions satisfying (2.24).

Theorem 2.13. *Let $\rho \in L^1(\mathbb{R}^d)$ satisfying (2.12) and such that $\mathrm{E}(\rho_0) < \infty$, where $\mathrm{E}(\rho_0)$ is defined in (2.28). Suppose that the corresponding solution $\rho(x,t)$ to (2.3) satisfies the time decay condition (2.24). Then,*

$$\|\rho(t) - \mathcal{G}(t)\|_{L^1(\mathbb{R}^d)} = \begin{cases} o(t^{-\frac{1}{2}+\delta}) & \text{for arbitrary } 0 < \delta \ll 1 \quad \text{if } d = 1 \\ O(t^{-\frac{1}{2}}) & \text{if } d > 1, \end{cases} \tag{2.33}$$

where $\mathcal{G}$ is the fundamental solution of the heat equation (2.4) with same mass as ρ_0.

Proof. In what follows, we shall denote a generic positive constant independent on s by C. Let us multiply equation (2.26) by $\log v(y) + \frac{|y|^2}{2\varepsilon}$ and integrate over $\mathbb{R}^d$. Then, integration by parts and conservation of the total mass yield

$$
\begin{aligned}
\frac{d}{ds}\mathrm{E}(v(s)|\mathcal{U}_m) &= -\varepsilon\int_{\mathbb{R}^d} v\left|\nabla\left(\log\frac{v}{\mathcal{U}_m}\right)\right|^2 dy + Ce^{-ds}\int_{\mathbb{R}^d} v\nabla\mathcal{B}_s * v\cdot\nabla\left(\log v + \frac{|y|^2}{2\varepsilon}\right)dy \\
&\le -\varepsilon\mathrm{I}(v|\mathcal{U}_m) + Ce^{-ds}\left(\int_{\mathbb{R}^d} v\,|\nabla\mathcal{B}_s * v|^2\,dy\right)^{1/2}\left(\int_{\mathbb{R}^d} v\left|\nabla\left(\log v + \frac{|y|^2}{2\varepsilon}\right)\right|^2 dy\right)^{1/2} \\
&=: -\varepsilon\mathrm{I}(v|\mathcal{U}_m) + J.
\end{aligned}
\tag{2.34}
$$

We now estimate the term J by using (2.32) and Young's inequality.

$$
\begin{aligned}
J &\le Ce^{-ds}\left(\int_{\mathbb{R}^d}\frac{1}{v}\,|\nabla v|^2\,dy\right)^{1/2} I(v|\mathcal{U}_m)^{1/2} \\
&= Ce^{-ds}\left(I(v|\mathcal{U}_m) + 2d\int_{\mathbb{R}^d} v - \frac{1}{\varepsilon}\int_{\mathbb{R}^d} v|y|^2 dy\right)^{1/2} I(v|\mathcal{U}_m)^{1/2} \\
&\le Ce^{-ds}\left(I(v|\mathcal{U}_m) + I(v|\mathcal{U}_m)^{1/2}\right).
\end{aligned}
$$

Therefore, for fixed $C_1, C_2 > 0$, inequality (2.34) implies,

$$
\frac{d}{ds}\mathrm{E}(v(s)|\mathcal{U}_m) \le -(\varepsilon - C_1 e^{-ds})I(v|\mathcal{U}_m) + C_2 e^{-ds} I(v|\mathcal{U}_m)^{1/2}
$$

and, for an arbitrarily small $\delta > 0$,

$$
\frac{d}{ds}\mathrm{E}(v(s)|\mathcal{U}_m) \le -(\varepsilon - Ce^{-ds} - Ce^{-2\delta s})I(v|\mathcal{U}_m) + Ce^{-2ds+2\delta s}.
$$

For $s \ge s_0(\varepsilon)$ we have $\varepsilon - Ce^{-ds} - Ce^{-2\delta s} > 0$ and we can use inequality (2.31) to obtain

$$
\frac{d}{ds}\mathrm{E}(v(s)|\mathcal{U}_m) \le -\frac{2(\varepsilon - Ce^{-ds} - Ce^{-2\delta s})}{\varepsilon}\mathrm{E}(v(s)|\mathcal{U}_m) + Ce^{-2ds+2\delta s}.
$$

Hence, thanks to the variation of constants formula we obtain the following decay rates as $s \to +\infty$ (here $0 < \delta \ll 1$)

$$
\mathrm{E}(v(s)|\mathcal{U}_m) = \begin{cases} O(e^{-2(1-\delta)s}) & \text{if } d = 1 \\ O(e^{-2s}) & \text{if } d > 1. \end{cases}
\tag{2.35}
$$

Finally, by means of the Csiszár–Kullback inequality (2.30) and by using the original variables $\rho(x,t)$ according to (2.25), we obtain the desired estimate (2.33). □

3. Nonlinear diffusion

In this section we focus our attention on the nonlinear model

$$
\begin{cases} \dfrac{\partial\rho}{\partial t} = \nabla\cdot(\rho(1-\rho)\nabla(\varepsilon\rho - S)) \\ -\Delta S + S = \rho, \end{cases}
\tag{3.1}
$$

subject to the initial condition

$$
\rho(x,0) = \rho_0(x), \quad \rho_0 \in L^1(\mathbb{R}^d), \quad 0 \le \rho_0(x) \le 1, \qquad x \in \mathbb{R}^d.
\tag{3.2}
$$

3.1. **Existence of Weak Solutions.** We start by providing a suitable definition of weak solutions. In order to simplify the notation, we fix $\varepsilon = 1$ throughout this section, the value of ε not being relevant in the existence theory. We denote

$$A(\rho) = \int_0^\rho \xi(1-\xi)d\xi = \frac{\rho^2}{2} - \frac{\rho^3}{3}, \quad \mathcal{A}(\rho) = \int_0^\rho A(\xi)d\xi = \frac{\rho^3}{6} - \frac{\rho^4}{12}.$$

Definition 3.1. *A function $\rho \in L^2([0,+\infty) \times \mathbb{R}^d)$ is called a* weak solution *of the Cauchy problem (3.1)–(3.2) on $\mathbb{R}^d \times [0,T]$ (T eventually $+\infty$) if the following conditions are satisfied,*

(i) $\mathcal{A}(\rho) \in L^\infty([0,T]; L^1(\mathbb{R}^d))$
(ii) $\nabla A(\rho) \in L^2([0,T] \times \mathbb{R}^d)$
(iii) $0 \le \rho(x,t) \le 1$ *almost everywhere in* $[0,T] \times \mathbb{R}^d$.
(iv) *For all* $\varphi \in C_c^\infty([0,T) \times \mathbb{R}^d)$.*, the following relation holds*

$$\int_0^T \int_{\mathbb{R}^d} \rho\varphi_t dxdt - \int_0^T \int_{\mathbb{R}^d} \nabla A(\rho) \cdot \nabla\varphi dxdt + \int_0^T \int_{\mathbb{R}^d} \rho(1-\rho)\nabla S \cdot \nabla\varphi dxdt + \int_{\mathbb{R}^d} \rho(x,0)\varphi(x,0)dx = 0,$$

where $S = S[\rho]$ is the unique H^1 solution to $-\Delta S + S = \rho$.

In order to prove local–in–time existence of weak solutions to the Cauchy problem (3.1)–(3.2), we use the following strategy. We first regularize the parabolic equation in (3.1) by adding a small linear diffusion term and we solve the Cauchy–Dirichlet problem on a bounded domain via a priori estimates and Schauder's fixed point theorem. At this level, we prove that the condition $0 \le \rho \le 1$ is preserved. We then extend the result to the Cauchy problem (3.1)–(3.2) by a compactness argument. Such a procedure is inspired by a similar argument in [BCM03]. The main technical problem in our case relies on the degeneracy of the nonlinear diffusion coefficient at the threshold value $\rho = 1$, which renders existence a non trivial issue. We overcome this difficulty by including the invariant domain property $0 \le \rho \le 1$ in our definition of solutions. This property prevents the overrunning of the mobility coefficient $\rho(1-\rho)$ into the backward diffusion range.

Remark 3.2. Unlike the notion of weak solution given in definition 3.1 for problem (3.1)–(3.2), we shall usually refer to an L^2 distributional solution of any of the various approximating problems treated below as a *weak solution.*

3.1.1. *Non degenerate approximation in a bounded domain.* Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary. For fixed $T > 0$ we denote as usual $\Omega_T = \Omega \times [0,T]$. For small $\mu, \nu > 0$, we study the approximating IBV problem

$$\begin{cases} \dfrac{\partial\rho}{\partial t} = \nabla \cdot (a_{\mu,\nu}(\rho)\nabla\rho - b_\nu(\rho)\nabla S) & \text{as } (x,t) \in \Omega_T \\ -\Delta S + S = \rho, & \text{as } (x,t) \in \Omega_T, \\ \rho(x,t) = S(x,t) = 0 & \text{as } x \in \partial\Omega, \\ \rho(x,0) = \rho_0(x), \end{cases} \tag{3.3}$$

where $a_{\mu,\nu}(\rho) = \mu + b_\nu(\rho)$ and $b_\nu(\rho)$ is a uniformly bounded (with respect to ν), smooth approximation of the function

$$\rho \in \mathbb{R} \mapsto [\rho(1-\rho)]^+$$

as $\nu \to 0^+$, such that $b_\nu(\rho) \geq 0$ for all ρ and $b_\nu(\rho) > 0$ if and only if $\rho \in (0,1)$. In order to prove existence of solutions for small T, we first define the operator

$$L^2([0,T];H^1(\Omega)) \ni S \mapsto \mathcal{U}(S)$$

as follows

$$\mathcal{U}(S) = \rho \quad \text{where } \rho \text{ solves} \quad \begin{cases} \dfrac{\partial \rho}{\partial t} = \nabla \cdot (a_{\mu,\nu}(\rho)\nabla\rho - b_\nu(\rho)\nabla S) & \text{as } \ (x,t) \in \Omega_T \\ \rho(x,t) = S(x,t) = 0 & \text{as } x \in \partial\Omega, \\ \rho(x,0) = \rho_0(x). \end{cases} \tag{3.4}$$

For a given $S \in L^2([0,T];H^1(\Omega))$, the existence and the uniqueness of a weak (smooth) solution to (3.4) is guaranteed by the smoothness of the diffusion coefficient $a_\mu(\rho)$ and by $a_\mu(\rho) \geq \mu > 0$ (see [LSU67], Chapter V, Subsection 6, Theorem 6.1). Moreover, it is easily checked that the operator $\mathcal{U} : L^2([0,T];H^1(\Omega)) \to L^2(\Omega_T)$ is well defined.

Then, we denote by $\mathcal{D} : L^2(\Omega) \to H^1(\Omega)$ the solution operator of the elliptic equation

$$-\Delta S + S = \overline{\rho}, \tag{3.5}$$

more precisely we set $\mathcal{D}(\overline{\rho}) = S$. Finally, we set

$$\mathcal{T} = \mathcal{U} \circ \mathcal{D}.$$

In what follows we recover some properties of the maps $\mathcal{U}$ and $\mathcal{D}$ needed to apply Schauder's fixed point theorem.

Proposition 3.3. *The map $\mathcal{U} : L^2([0,T];H^1(\Omega)) \to L^2(\Omega_T)$ is continuous and compact.*

Proof. We denote by $\Delta^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)$ the inverse of the Laplacian on Ω with Dirichlet boundary conditions. Multiplication of equation (3.4) by $\Delta^{-1}\frac{\partial \rho}{\partial t}$ and integration by parts over Ω yield

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \mathcal{A}_{\mu,\nu}(\rho(t))dx + \int_\Omega \left|\nabla\Delta^{-1}\frac{\partial\rho}{\partial t}\right|^2 dx = \int_\Omega b_\nu(\rho)\nabla S \cdot \nabla\Delta^{-1}\frac{\partial\rho}{\partial t}dx \\ &\leq \sup_\rho b_\nu(\rho)\int_\Omega |\nabla S|\left|\nabla\Delta^{-1}\frac{\partial\rho}{\partial t}\right| dx, \end{aligned}$$

where $\mathcal{A}_{\mu,\nu}(z) = \int_0^z\int_0^\xi a_{\mu,\nu}(\zeta)d\zeta d\xi$. Hence, by Young's inequality we obtain

$$\frac{d}{dt}\int_\Omega \mathcal{A}_{\mu,\nu}(\rho(t))dx + \frac{1}{2}\int_\Omega \left|\nabla\Delta^{-1}\frac{\partial\rho}{\partial t}\right|^2 dx \leq \frac{1}{2}\left(\sup_\rho b_\nu(\rho)\right)^2 \int_\Omega |\nabla S|^2 dx, \tag{3.6}$$

and, after integration on $[0,T]$,

$$\int_0^T\int_\Omega \left|\nabla\Delta^{-1}\frac{\partial\rho}{\partial t}\right|^2 dx \leq C\int_0^T\int_\Omega |\nabla S|^2 dx + \int_\Omega \mathcal{A}_{\mu,\nu}(\rho_0)dx. \tag{3.7}$$

Moreover, multiplication of (3.4) by $A_{\mu,\nu}(\rho) := \int_0^\rho a_{\mu,\nu}(\zeta)d\zeta$ and integration by parts imply

$$\frac{d}{dt}\int_\Omega \mathcal{A}_{\mu,\nu}(\rho(t))dx + \int_\Omega |\nabla A_{\mu,\nu}(\rho)|^2 dx = \int_\Omega b_\nu(\rho)\nabla S \cdot \nabla A_{\mu,\nu}(\rho)dx. \tag{3.8}$$

By means of Young's inequality one can manipulate (3.8) in a similar fashion as before and obtain, after double integration on $[0,T]$,

$$\frac{1}{T}\int_0^T\int_\Omega \mathcal{A}_{\mu,\nu}(\rho(t))dx + \int_0^T\int_\Omega |\nabla A_{\mu,\nu}(\rho)|^2\,dxdt$$
$$\leq C\int_\Omega \mathcal{A}_{\mu,\nu}(\rho_0)dx + C\int_0^T\int_\Omega |\nabla S|^2 dxdt. \tag{3.9}$$

The definition of $a_{\mu,\nu}$ implies

$$\rho^2 \leq C(\mu)\mathcal{A}_{\mu,\nu}(\rho), \qquad A'_{\mu,\nu}(\rho) \geq \mu.$$

Therefore, inequalities (3.9) and (3.7) imply the following statement

$$\|\rho\|_{L^2([0,T];H^1(\Omega))} + \|\partial_t\rho\|_{L^2([0,T];H^{-1}(\Omega))} \leq C(\mu,T)\left[\|S\|_{L^2([0,T];H^1(\Omega))} + C(\rho_0)\right], \tag{3.10}$$

where we have used the fact that $\Delta^{-1} : H^{-1}(\Omega) \to H^1_0(\Omega)$ is an isomorphism. Let us now take a sequence $\{S_n\}_n$ uniformly bounded in $L^2([0,T];H^1(\Omega))$ and consider the corresponding $\rho_n = \mathcal{U}(S_n)$. Inequality (3.10) and an embedding result by Lions and Aubin (see for instance [Sho97], Chapter III, Section 1, Proposition 1.3) imply that $\{\rho_n\}$ is compact in $L^2(\Omega_T)$. To prove continuity, take a sequence $S_n \in L^2([0,T];H^1(\Omega))$ such that $S_n \to \widetilde{S}$. Then, by compactness of $\mathcal{U}$, $\mathcal{U}(S_n) = \rho_n$ has a convergent subsequence in $L^2(\Omega_T)$. Therefore, ρ_n has a subsequence ρ_{n_k} converging almost everywhere to some $\widetilde{\rho} \in L^2(\Omega_T)$. By using the weak formulation of problem (3.4) with $S = S_{n_k}$ and $\rho = \rho_{n_k}$, we can easily deduce that $\widetilde{\rho}$ is the unique weak solution to (3.4) where $S = \widetilde{S}$. Therefore, the whole sequence ρ_n converges to $\mathcal{U}(\widetilde{S})$ and the proof is complete. □

For the sake of completeness, we provide the details about the continuity of $\mathcal{D}$, which is of course straightforward.

Proposition 3.4. *The (linear) operator $\mathcal{D} : L^2(\Omega_T) \to H^1(\Omega_T)$ is continuous.*

Proof. It follows from the estimate

$$\|S(t)\|_{H^1(\Omega)} \leq \|\overline{\rho}(t)\|_{L^2(\Omega)}, \tag{3.11}$$

which can be easily proven by multiplying equation (3.5) by S and by integrating over Ω. □

Theorem 3.5. *There exists at least one local–in–time solution of problem (3.3).*

Proof. Relation (3.9) clearly implies that

$$\|\rho\|_{L^2(\Omega_T)} \leq C(\mu)\sqrt{T}\left[C(\rho_0) + \|S\|_{L^2([0,T];H^1(\Omega))}\right]$$

and this, together with (3.11), implies

$$\|\rho\|_{L^2(\Omega_T)} \leq C(\mu)\sqrt{T}\left[C(\rho_0) + \|\overline{\rho}\|_{L^2(\Omega_T)}\right]. \tag{3.12}$$

Hence, for $M > 0$ we can consider the Banach space

$$\mathcal{X}_M = \left\{\rho \in L^2(\Omega_T)\,\Big|\, \int\int_{\Omega_T} |\rho|^2 dxdt \leq M\right\}.$$

Estimate (3.12), then, clearly implies that $\mathcal{T} = \mathcal{U}\circ\mathcal{D}$ is well defined as a map from $\mathcal{X}_M$ into itself provided that T is small enough and M large enough. Moreover, thanks to propositions 3.3 and 3.4, $\mathcal{T} : L^2(\Omega_T) \to L^2(\Omega_T)$ is continuous and compact. Therefore, $\mathcal{T}$ has a fixed point thanks to Schauder's fixed point Theorem (see e.g. [Tay97], Chapter 14, Corollary B.3), which means that (3.3) has a local–in–time solution $\rho \in L^2(\Omega_T)$. □

Remark 3.6. A continuation principle in the spirit of remark 2.3 holds in this case too, where the L^1 and the L^∞ norms are replaced by the L^2 norm. Moreover, by integrating once with respect to time in the identity (3.8), one easily gets an estimate of the local solution ρ in $L^\infty([0,T], L^2(\Omega))$.

Our next aim is to prove that condition (3.2) is preserved by the local solution $\rho(t)$ of problem (3.3) at any $t \in [0,T]$.

Lemma 3.7. *Let $\rho(x,t)$ be the local solution to (3.3) provided by Theorem 3.5, with initial datum ρ_0 satisfying (3.2). Then, for any $0 \le t \le T$, we have*

$$0 \le \rho(x,t) \le 1, \qquad \textit{a. e. in } x.$$

Proof. We claim that $\rho(x,t)$ can be obtained as a uniform limit of the sequence $\rho_n(x,t)$ recursively defined by

$$\rho_n \ \text{ solution of } \begin{cases} \dfrac{\partial \rho}{\partial t} = \nabla \cdot (a_{\mu,\nu}(\rho_{n-1})\nabla\rho - b_\nu(\rho)\nabla S) & \text{as } \ (x,t) \in \Omega_T \\ -\Delta S + S = \rho, & \text{as } \ (x,t) \in \Omega_T, \\ \rho(x,t) = S(x,t) = 0 & \text{as } x \in \partial\Omega, \\ \rho(x,0) = \rho_0(x), \end{cases}$$

where the local existence for the above equation can be proven in the same way as before (the diffusion term is linear!). To see this, one can apply the argument in the proof of proposition 3.3 to the semi–linear equation satisfied by ρ_n and get the estimate (3.10) uniformly in n. Then, in the same way we get compactness in $L^2(\Omega_T)$ of the family ρ_n, and this is enough to get a weak solution to the nonlinear equation (3.3) in the limit as $n \to \infty$. Now, by means of the same comparison argument as in Theorem 2.8, we get the desired estimates for any n (the parabolic part of the operator is linear and nondegenerate), and the same relation holds as $n \to \infty$. □

Remark 3.8. As a consequence of the previous lemma and thanks to the continuation principle 3.6, the local–in–time solution to (3.3) is actually globally defined.

Remark 3.9. Higher regularity of the local solution when the initial datum ρ_0 is smooth enough can be proven by modifying the fixed point argument, involving higher Sobolev norms and exploiting the regularizing properties of the solution operator $\mathcal{S}$ of the elliptic equation $-\Delta S + S = \rho$, the nondegeneracy of the parabolic equation in (3.3) and the result in Lemma 3.7 (we omit the details).

Our next step is the limit as $\nu \to 0$ in (3.3). We have the following theorem.

Theorem 3.10 (Existence for the nondegenerate approximation)**.** *For any small $\mu > 0$, there exists at least one global–in–time weak solution to the problem*

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = \nabla \cdot (a_\mu(\rho)\nabla\rho - [\rho(1-\rho)]_+ \nabla S) & \textit{as } \ (x,t) \in \Omega_T \\ -\Delta S + S = \rho, & \textit{as } \ (x,t) \in \Omega_T, \\ \rho(x,t) = 0 & \textit{as } x \in \partial\Omega, \\ \rho(x,0) = \rho_0(x), \end{cases} \tag{3.13}$$

where $a_\mu = \mu + [\rho(1-\rho)]_+$, *with* ρ_0 *satisfying (3.2). Moreover, the local–in–time solution* ρ *satisfies*

$$0 \le \rho(x,t) \le 1 \tag{3.14}$$

almost everywhere.

Proof. Let $\mu > 0$ be fixed. For any small $\nu > 0$, let us consider the solution provided by Theorem 3.5. We observe that estimate (3.10) is independent on ν. Hence, by the same argument in the proof of Proposition 3.3, we deduce that the family of solutions $\{\rho_{\mu,\nu}\}_{\nu>0}$ is relatively compact in $L^2(\Omega_T)$. By extracting a convergent a. e. subsequence, one can easily prove the consistency of the limit procedure as $\nu \to 0$. Estimate (3.14) is a consequence of Lemma 3.7. □

Remark 3.11. By means of similar arguments as those in Remark 3.9, one can prove that the solution provided by the above theorem enjoys more regularity if so does the initial datum.

Remark 3.12. In view of (3.14), the global solution ρ provided in Theorem 3.10 solves the equation

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left(a_\mu(\rho)\nabla\rho - \rho(1-\rho)\nabla S\right),$$

where we have removed the positive part in the term $\rho(1-\rho)$.

3.1.2. *Existence via approximation.* In this subsection we prove global existence of solutions to the problem (3.1)–(3.2) via approximation by the solution to the regularized problem analyzed in the previous subsection. For a fixed $n \in \mathbb{N}$ we define $\rho_0^n := \rho_0\theta_n$, where θ_n is a smooth cutoff function such that $\theta_n(x) \equiv 1$ if $|x| \le n-1$, $\theta_n(x) = 0$ if $|x| \ge n$ and $0 \le \theta_n \le 1$ otherwise. For a small positive μ, let ρ_n^μ be the solution to (3.13) provided in the previous subsection with $\Omega = \{|x| \le n\}$ and initial datum ρ_0^n. We aim to prove that the family of solutions $\{\rho_n^\mu\}_{n,\mu}$ enjoys the suitable uniform estimates need to be relatively compact in a certain sense. This time, the estimates on the approximating solution of problem (3.13) developed in Proposition 3.3 cannot be used to get the desired compactness, since they blow up as μ tends to zero. We aim to improve them in the following Theorem.

Theorem 3.13 (Global existence in a closed ball)**.** *For any fixed integer* n*, there exists a weak solution* ρ_n *to the problem*

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = \nabla \cdot (\rho(1-\rho)\nabla(\rho - S)) & \text{as } (x,t) \in B(0,n) \times [0,+\infty) \\ -\Delta S + S = \rho, & \text{as } (x,t) \in B(0,n) \times [0,+\infty), \\ \rho(x,t) = 0 & \text{as } x \in \partial B(0,n), \\ \rho(x,0) = \rho_0(x), \end{cases} \tag{3.15}$$

with initial datum ρ_0^n.

Proof. We prove several energy estimates, obtained after suitable manipulations of the approximating equation

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left(a_\mu(\rho)\nabla\rho - \rho(1-\rho)\nabla S\right). \tag{3.16}$$

In the computations below, as usual, we suppose the solution to be smooth enough in order to perform integration by parts, thus requiring the initial datum to be smooth enough. The result for general initial datum follows by approximation. In what follows, C shall denote a generic positive constant independent on μ and on n, and T is a positive time.

STEP 1. Multiplying (3.16) by ρ and integrating by parts we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega \rho^2 dx &= -2\int_\Omega a_\mu(\rho)|\nabla\rho|^2 dx + 2\int_\Omega \rho(1-\rho)\nabla\rho\cdot\nabla S dx\\ &\le -C\int_\Omega a_\mu(\rho)|\nabla\rho|^2 dx + C\int_\Omega |\nabla S|^2,\end{aligned}$$

where we have used the Cauchy–Schwarz inequality and the uniform bound for $a_\mu(\rho)$. Since

$$\|S\|_{H^1(\Omega)} \le \|\rho\|_{L^2(\Omega)}, \tag{3.17}$$

integration over $[0,T]$ and Gronwall inequality yield

$$\sup_{0\le t\le T}\int_\Omega \rho(t)^2 dx + \int_0^T\int_\Omega a_\mu(\rho)|\nabla\rho|^2 dxdt \le e^{CT}. \tag{3.18}$$

STEP 2. Given $A_\mu(\rho) = \int_0^\rho a_\mu(\xi)d\xi$, we compute (as in (3.9))

$$\frac{d}{dt}\int_\Omega \mathcal{A}_\mu(\rho(t))dx + \int_\Omega |\nabla A_\mu(\rho)|^2\, dx = \int_\Omega \rho(1-\rho)\nabla S\cdot\nabla A_\mu(\rho)dx \tag{3.19}$$

and, integrating over $[0,T]$ and using (3.18)

$$\frac{1}{T}\int_0^T\int_\Omega \mathcal{A}_\mu(\rho(t))dx + \int_0^T\int_\Omega |\nabla A_\mu(\rho)|^2\, dx \le C\int_\Omega \mathcal{A}_\mu(\rho_0)dx + Ce^{CT}. \tag{3.20}$$

STEP 3. We have the estimate

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\nabla A_\mu|^2\, dx &= 2\int_\Omega \nabla A_\mu(\rho)\cdot\nabla A_\mu(\rho)_t dx = -2\int_\Omega A_\mu(\rho)_t\Delta A_\mu(\rho)dx\\ &= -2\int_\Omega \rho_t A_\mu(\rho)_t dx - 2\int_\Omega A_\mu(\rho)_t\nabla\cdot(\rho(1-\rho)\nabla S)\, dx\\ &= -2\int_\Omega \frac{1}{a_\mu(\rho)}A_\mu(\rho)_t^2 dx - 2\int_\Omega A_\mu(\rho)_t\rho(1-\rho)\Delta S dx\\ &\quad - 2\int_\Omega A_\mu(\rho)_t(1-2\rho)\nabla\rho\cdot\nabla S dx\\ &\le -C\int_\Omega A_\mu(\rho)_t^2 dx + C\int_\Omega \rho^2 dx + C\int_\Omega \rho(1-\rho)|\nabla\rho|^2 dx,\end{aligned}$$

where we have used $\|\nabla S\|_{L^\infty} \le C\|\rho\|_{L^\infty}$, which can be easily obtained thanks to (2.6) and the Green function method on the ball $B(0,n)$ by means of maximum principle for the operator $-\Delta + \mathbb{I}$, and the uniform bound for ρ. We observe that such an estimate is independent on the diameter of Ω. Multiplying by $t\in[0,T]$ the above computation and integrating over $[0,T]$, using (3.18) and (3.20) we get

$$\begin{aligned}\int_0^T\int_\Omega tA_\mu(\rho)_t^2 dxdt &\le -C\int_0^T t\frac{d}{dt}\int_\Omega |\nabla A_\mu|^2\, dxdt + CTe^{CT}\\ &\le C\int_0^T\int_\Omega |\nabla A_\mu|^2\, dxdt + CTe^{CT} \le C(1+Te^{CT}).\end{aligned} \tag{3.21}$$

Collecting estimates (3.18), (3.20) and (3.21), by Sobolev embedding we recover

$$\sqrt{t}A_\mu(\rho)_t \subset\subset L^2(\Omega_T).$$

Hence, there exists a sequence $\mu_k \to 0$ (as $k \to \infty$) such that $A_{\mu_k}(\rho_n^{\mu_k})$ converges almost everywhere in Ω_T to some function $A(x)$ as $k \to \infty$. Since $A_\mu(\rho) \to \overline{A}(\rho) := \frac{\rho^2}{2} - \frac{\rho^3}{3}$ as $\mu \to 0$, then it is easy to see that $\overline{A}(\rho_n^{\mu_k})$ converges to A almost everywhere, and since $\overline{A}$ has continuous inverse on $[0,1]$, this implies that $\rho_n^{\mu_k}$ has an almost everywhere limit defined on Ω_T. Such limit is a weak solution of problem (3.15). □

As usual, it can be easily seen that the solution provided by the above theorem satisfies (3.14).

The last step in our approximation argument consists in taking the limit of ρ_n as $n \to \infty$. We can skip the details about this procedure, which is rather standard, and it is based on the same estimate and compactness arguments as above. In fact, it can be easily checked that all the estimates above are uniform with respect to the domain Ω. We are ready to state our final existence theorem.

Theorem 3.14 (Global existence of solutions)**.** *There exists at least a global weak solution $\rho(x,t)$ to the Cauchy problem (3.1)–(3.2) in the sense of definition 3.1.*

Proof. We only need to check that the distributional solution $\rho = \lim_{n\to\infty} \rho_n$ satisfies the properties of definition 3.1. Conditions (i) and (ii) can be easily recovered thanks to estimate (3.19), which can be easily obtained in the limiting case $\mu = 0$, $\Omega = \mathbb{R}^d$. Condition (iii) is a trivial consequence of the same condition satisfied by the approximating solutions. Condition (iv) is a consequence of (ii). □

Remark 3.15. The conservation of the total mass can be proven in the same way as in Theorem 2.7.

3.2. **Entropy solutions.** In the following we turn our attention to the problem of uniqueness of suitable solutions. For equations with an interaction of nonlocal fluxes and degenerate diffusions there is no straightforward way to prove the uniqueness of weak solutions (cf. also [BCM03]) and one might even expect non-uniqueness as for nonlocal transport equations (cf. [DGT00]). We therefore turn our attention to a rather natural restriction of weak solutions to so-called *entropy solutions*. Apart from uniqueness, the main motivation for considering entropy solutions is the possibility to obtain correct dissipation of entropy functionals, which will be discussed in the subsections below.

Due to the fact that the convolution $\mathcal{B} * \rho$ is smooth anyway, the behaviour of dissipation functionals on this part seems not important and we shall therefore adapt the definition of entropy solutions for fixed flux $\nabla S(x,t)$ (cf. [BCM03] for a more detailed discussion).

Definition 3.16 (Entropy Solutions)**.** *We shall say that a nonnegative function $\rho \in L^1([0,T]\times\mathbb{R}^d) \cap C(0,T;L^1(\mathbb{R}^d))$ is an* entropy solution *of the Cauchy problem (3.1)–(3.2) on $\mathbb{R}^d \times [0,T]$ if the following conditions are satisfied:*

(i) *For all $c \in \mathbb{R}$ and all non-negative test functions $\varphi \in C_c^\infty([0,T)\times\mathbb{R}^d)$, the following entropy inequality holds*

$$\int_0^T \int_{\mathbb{R}^d} \Big[|\rho - c|\varphi_t + \mathrm{sign}\,(\rho - c)\,(\rho(1-\rho) - c(1-c))\,\nabla S \cdot \nabla\varphi + \varepsilon|A(\rho) - A(c)|\Delta\varphi - \mathrm{sign}\,(\rho - c)c(1-c)\Delta S\varphi \Big]\,dx\,dt \geq 0, \tag{3.22}$$

where $S = S[\rho]$ is the unique H^1 solution to $-\Delta S + S = \rho$.

(ii) $A(\rho) \in L^2(0,T;H^1(\mathbb{R}^d))$
(iii) $0 \le \rho(x,t) \le 1$ *almost everywhere in* $[0,T] \times \mathbb{R}^d$.
(iv) *Essentially, as* $t \downarrow 0$,
$$\int_{\mathbb{R}^d} |\rho(x,t) - \rho_0(x)| \, dx \to 0.$$

We start by verifying the existence of entropy solutions. For this sake we take a closer look at our preceding construction of weak solutions. In the first step, namely the nondegenerate approximation on a bounded domain, we obtain even a unique classical solution of (3.3) and (3.13). The classical solution clearly satisfies the corresponding entropy condition (with smoothed nonlinearities on the bounded domain). Due to a-priori bounds on ρ, $A(\rho)$, and $S = \mathcal{B} * \rho$ we always extract suitably convergent subsequences when passing from (3.13) and (3.15) and from (3.15) and (3.1), so that the entropy inequality (3.22) carries over to the limit. Hence, we obtain the following results

Theorem 3.17. *Let* $\rho_0 \in L^1(\mathbb{R}^d)$ *satisfy* $0 \le \rho_0 \le 1$. *Then there exists an entropy solution of* (3.1), (3.2) *according to Definition* (3.16).

In order to prove uniqueness of the entropy solution, we shall use continuous dependence of entropy solutions on the flux in the L^1-norm:

Lemma 3.18. *Let* $S^1, S^2 \in C(0,T;H^1(\mathbb{R}^d)) \cap L^\infty([0,T] \times \mathbb{R}^d)$, *with*
$$\nabla S^j \in C(0,T;W^{1,1}_{loc}(\mathbb{R}^d)) \cap C([0,T] \times \mathbb{R}^d), \qquad \Delta S^j \in L^\infty([0,T] \times \mathbb{R}^d)$$
be given and let $u^j \in L^\infty(0,T;BV(\mathbb{R}^d))$ *be entropy solutions of*
$$u^j_t = \nabla \cdot \big(u^j(1-u^j)\nabla(\varepsilon u^j - S^j)\big)$$
with initial values $u^j_0 \in BV(\mathbb{R}^d)$ *for* $j = 1,2$.
$$\begin{aligned} \|u^1(t) - u^2(t)\|_{L^1(\mathbb{R}^d)} \quad &\le \quad \|u^1_0 - u^2_0\|_{L^1(\mathbb{R}^d)} + \frac{t}{4}\|\nabla S^1 - \nabla S^2\|_{L^\infty(0,t;BV(\mathbb{R}^d))} \\ &\quad + t\max\{V_1, V_2\}\|\nabla S^1 - \nabla S^2\|_{L^\infty([0,t]\times\mathbb{R}^d)}, \end{aligned} \tag{3.23}$$
where $V_j = \|u^j\|_{L^\infty(0,t;BV(\mathbb{R}^d))}$.

Proof. The proof can be carried out in an analogous way to the proof of Theorem 1.3 in [KR03] by appropriately using the time-dependence of the flux in the estimates and the fact that the function $p \mapsto p(1-p)$ has Lipschitz constant 1 and supremum $\frac{1}{4}$ on the interval $[0,1]$. □

Below we shall prove a uniqueness result in the smaller class of entropy solutions of bounded variation in the case of spatial dimension one. Before proving their uniqueness, we verify the existence of such entropy solutions:

Proposition 3.19 (Regularity of entropy solutions)**.** *Let* $\rho_0 \in BV(\mathbb{R}^1;[0,1])$, *then an entropy solution of* (3.1), (3.2) *satisfies*
$$\rho \in L^\infty(0,T;BV(\mathbb{R}^1)).$$

Proof. We construct the BV-solution by smooth approximation. Let ρ be an L^1 viscosity solution of (3.1), (3.2) and let $\rho^\delta \in C(0,T;C^1(\mathbb{R}^1))$ such that $0 \le \rho^\delta \le 1$ almost everywhere and $\rho^\delta \to \rho$ in $C(0,T;BV(\mathbb{R}^1))$. Then we also have
$$S^\delta = \mathcal{B} * \rho^\delta \in C(0,T;BV(\mathbb{R}^1))$$

and consequently

$$S^\delta_{xx} = S^\delta - \rho^\delta \in C(0,T;BV(\mathbb{R}^d)).$$

The results of [KR01] imply the existence of an entropy solution u^δ of

$$u^\delta_t - \left(u^\delta(1-u^\delta)(\varepsilon u^\delta - S^\delta)_x\right)_x = 0$$

with initial value $u(0) = \rho_0$, u^δ belonging in the space $\in L^\infty(0,T;BV(\mathbb{R}^1))$. Moreover, since $u^\delta_h(x,t) = u^\delta(x+h,t)$ is an entropy solution of the same equation with $S^\delta(x.t)$ replaced by $S^\delta_h(x,t) := S^\delta(x+h,t)$ and initial value $\rho_0(.+h)$, we may apply the continuous dependence estimate to deduce

$$\begin{aligned} \|u^\delta_h(t) - u^\delta(t)\|_{L^1(\mathbb{R}^1)} \quad &\leq \quad \|\rho_0(.+h) - \rho^0\|_{L^1(\mathbb{R}^1)} + \frac{t}{4}\|S^\delta_x - (S^\delta_h)_x\|_{L^\infty(0,t;BV(\mathbb{R}^1))} \\ &\quad + tV^\delta\|S^\delta_x - (S^\delta_h)_x\|_{L^\infty([0,t]\times\mathbb{R}^1)}, \end{aligned}$$

with $V^\delta = \|u^\delta\|_{L^\infty(0,t;BV(\mathbb{R}^1))}$. After division by h we obtain in the limit $h \to 0$

$$V^\delta \quad \leq \quad \|\rho^0\|_{BV(\mathbb{R}^1)} + \frac{t}{4}\|S^\delta_{xx}\|_{L^\infty(0,t;BV(\mathbb{R}^1))} + tV^\delta\|S^\delta_{xx}\|_{L^\infty([0,t]\times\mathbb{R}^1)},$$

From the uniform bounds for ρ^δ in $L^\infty(\mathbb{R}^1) \cap L^1(\mathbb{R}^1)$ one can easily deduce a uniform estimate for $\|S^\delta_{xx}\|_{L^\infty([0,t]\times\mathbb{R}^1)}$. Moreover, there exists a constant $c > 0$ (independent of δ and t) such that

$$\|S^\delta_{xx}\|_{L^\infty(0,t;BV(\mathbb{R}^1))} \leq c\|\rho^\delta\|_{L^\infty(0,t;BV(\mathbb{R}^1))}.$$

Hence, we deduce

$$(1-Ct)\|u^\delta\|_{L^\infty(0,t;BV(\mathbb{R}^1))} \leq \|\rho^0\|_{BV(\mathbb{R}^1)} + \frac{ct}{4}\|\rho^\delta\|_{L^\infty(0,t;BV(\mathbb{R}^1))}$$

As $\delta \to 0$, one can prove in a standard way that $u^\delta \to \rho$. Now let t be such that $(2C + \frac{c}{2})t < 1$, then by lower semicontinuity

$$\begin{aligned} \frac{1}{2}\|\rho\|_{L^\infty(0,t;BV(\mathbb{R}^1))} \quad &\leq \quad \liminf\left((1-Ct)\|u^\delta\|_{L^\infty(0,t;BV(\mathbb{R}^1))} - \frac{ct}{4}\|\rho^\delta\|_{L^\infty(0,t;BV(\mathbb{R}^1))}\right) \\ &\leq \quad \|\rho^0\|_{BV(\mathbb{R}^1)}, \end{aligned}$$

and hence, $\rho \in L^\infty(0,t;BV(\mathbb{R}^1))$.

By applying the same argument consecutively to time intervals of length smaller than $\frac{2}{4C+c}$ we finally obtain that $\rho \in L^\infty(0,T;BV(\mathbb{R}^1))$. □

Finally, from the continuous dependence it is a small step to prove the main uniqueness result:

Theorem 3.20 (Uniqueness)**.** *The entropy solution ρ of* (3.1), (3.2) *is unique in the space* $L^\infty(0,T;BV(\mathbb{R}^1))$

Proof. Assume that ρ^1 and ρ^2 be two different entropy solutions belonging in the space $L^\infty(0,T;BV(\mathbb{R}^1))$. Then we can assume without restriction of generality that $\|\rho^1(t) - \rho^2(t)\| \neq 0$ for $t > 0$ arbitrarily small (otherwise we can take τ as the maximal time before which the solutions are equal and rescale time to $t - \tau$).

One can verify in a straight-forward way that S^1 and S^2 satisfy the assumptions of Lemma 3.18 and that there exists a constant $C > 0$ such that

$$\|S^1_x - S^2_x\|_{L^\infty(0,t;BV(\mathbb{R}^1))} \leq C\|\rho^1 - \rho^2\|_{L^\infty(0,t;L^1(\mathbb{R}^1))}$$

and

$$\|S^1_x - S^2_x\|_{L^\infty([0,t]\times\mathbb{R}^1)} \leq C\|\rho^1 - \rho^2\|_{L^\infty(0,t;L^1(\mathbb{R}^1))}.$$

Hence, by Lemma 3.18 the estimate

$$\|\rho^1(t) - \rho^2(t)\|_{L^1(\mathbb{R}^1)} \leq \tilde{C}t\|\rho^1(t) - \rho^2(t)\|_{L^1(\mathbb{R}^1)}$$

holds for some constant $\tilde{C}$. Since we can choose t small enough such that $Ct < 1$ this yields a contradiction. □

3.3. **Finite speed of propagation in one space dimension.** In this subsection we focus our attention on the Cauchy problem in one space dimension

$$\begin{cases} \partial_t \rho = \partial_x \left(\rho(1-\rho)\partial_x \left(\varepsilon\rho - S\right)\right) \\ \rho(x,0) = \rho_0(x), \end{cases} \tag{3.24}$$

where ρ_0 is compactly supported and satisfying the usual condition $0 \leq \rho_0 \leq 1$. Our aim is to prove that the solution $\rho(t)$ at any time $t > 0$ is still compactly supported. This feature is usually referred to as the *the finite rate of propagation property*, and it is typically satisfied by nonlinear diffusion equations of the form

$$\rho_t = A(\rho)_{xx},$$

when $A : \mathbb{R}_+ \to \mathbb{R}_+$ is a smooth nondecreasing function such that $A'(0) = 0$ (see for instance [Kal87, CT05]). Our approach in proving such a property is based upon certain estimates of the moments of a solution. This is closely related with the estimates of the Wasserstein distances between any two solutions to a nonlinear diffusion equation developed in [CGT04].

For a positive integer n, we define the $2n$–th moment of a nonnegative, integrable function ρ as

$$M_{2n}(\rho) := \int_{-\infty}^{+\infty} x^{2n}\rho(x)dx.$$

The distribution function of ρ is given by

$$F(x) = \int_{-\infty}^{x} \rho(y)dy.$$

The *pseudo inverse* function of F, defined on the interval $[0, m]$, $m = \int_{-\infty}^{+\infty} \rho(x)dx$, is given by

$$F^{-1}(\xi) = \inf\left\{x \in \mathbb{R} \,\middle|\, F(x) > \xi\right\}.$$

If $\rho(x) > 0$ almost everywhere, then F^{-1} is a real inverse, and the change of variables

$$x \in \mathrm{supp}(\rho) \mapsto \xi = F(x) \in [0, m]$$

is a bijection. Therefore, one can change variable into the definition of the moments above and get

$$M_{2n}(\rho) = \int_{-\infty}^{+\infty} x^{2n}\rho(x)dx = \int_0^m F^{-1}(\xi)^{2n}d\xi.$$

Obviously we have

$$\left[\frac{1}{m}\int_0^m F^{-1}(\xi)^{2n}d\xi\right]^{1/2n} \to \|F^{-1}\|_{L^\infty([0,m])},$$

as $n \to +\infty$. Moreover, it is clear that

$$\mathrm{meas}(\mathrm{supp}(\rho)) \leq 2\|F^{-1}\|_{L^\infty([0,m])}. \tag{3.25}$$

Thus, a uniform estimate with respect to n of $M_{2n}^{1/2n}(\rho(t))$ where $\rho(t)$ is the solution to (3.24) at time t automatically provides the finiteness of the size of the support of $\rho(t)$. We shall prove that such an estimate is true. In our proof we shall suppose that ρ is strictly positive on its support. The general result follows by the same approximation argument developed in [CGT04]. For further reference we briefly recall the following result in [Kne77, CDFG05] for a nonlinear friction equation.

Theorem 3.21. *Let $A(\rho) = \frac{\rho^2}{2} - \frac{\rho^3}{3}$. Let $\overline{\rho}(t)$ be a nonnegative solution to the equation*

$$\overline{\rho}_t = A(\overline{\rho})_{xx}. \tag{3.26}$$

Let $\overline{F}^{-1}(t)$ be the pseudo–inverse of the distribution function of $\overline{\rho}(t)$. Then, there exists a continuous (increasing) function of time $t \mapsto C(t)$ such that

$$\|\overline{F}^{-1}(t)\|_{L^{2n}} \leq C(t), \tag{3.27}$$

where $C(t)$ does not depend on n.

We now state our result for equation (3.24).

Theorem 3.22 (Finite speed of propagation)**.** *Let $\rho(x,t)$ be the unique entropy solution to (3.24) with compactly supported initial datum ρ_0. Then, the profile $\rho(t)$ has compact support at any positive time t.*

Proof. As pointed out before, to pursue our aim we only need to control $M_{2n}^{1/2n}$ uniformly in n in any finite time interval $[0,T]$. In the sequel we shall suppose that ρ enjoys enough regularity in order to justify integration by parts in the estimates of the moments. The rigorous result in the general case follows as usual by approximation. Moreover, in order to deal with some boundary term we shall encounter in our computations, we shall need to work with an approximation of the Cauchy problem (3.24) in a bounded domain $[-b,b]$ for b positive integer. The compactness estimates proven in section 3.1 then allow us to take the limit as $b \to \infty$, since our estimate does not depend on b. The same technique is used in [CGT04].

We denote by ρ_b the solution of (3.24) on $[-b,b]$ with Dirichlet boundary conditions and with an initial datum $\rho_{0,b}$ having mass $m = \int \rho_0$ and converging uniformly to ρ_0 as $b \to \infty$. Let $u_b(t) : [0,m] \to \mathbb{R}$ be the pseudo–inverse of the distribution function of ρ_b. Let us also consider the solution $\overline{\rho}(t)$ to the nonlinear diffusion equation (3.26) on $[-b,b]$ having $\rho_{0,b}$ as initial datum and satisfying Dirichlet boundary conditions. Let $\overline{u}_b(t)$ the pseudo inverse of its distribution function. We shall omit the subscript b in the computations below for simplicity. A standard computation with pseudo–inverses (see e.g. [CT05]) shows that u and $\overline{u}$ satisfies the following equations

$$\partial_t u = -\partial_\xi A\left((\partial_\xi u)^{-1}\right) + \left(1 - (\partial_\xi u)^{-1}\right)\int_0^m B'(u(\xi) - u(\eta))d\eta$$
$$\partial_t \overline{u} = -\partial_\xi A\left((\partial_\xi \overline{u})^{-1}\right)$$

For fixed n we have, after integration by parts

$$\frac{d}{dt}\int_0^m [u(t)-\overline{u}(t)]^{2n}d\xi = -2n(u-\overline{u})^{2n-1}\left[A((\partial_\xi u)^{-1})-A((\partial_\xi \overline{u})^{-1})\right]\Big|_{\xi=0}^{\xi=m}$$
$$+2n(2n-1)\int_0^m (u-\overline{u})^{2n-2}\left[\partial_\xi u-\partial_\xi \overline{u}\right]\left[A((\partial_\xi u)^{-1})-A((\partial_\xi \overline{u})^{-1})\right]d\xi$$
$$+2n\int_0^m (u-\overline{u})^{2n-1}\left(1-(\partial_\xi u)^{-1}\right)\int_0^m B'(u(\xi)-u(\eta))d\eta d\xi.$$

Hence, since u and $\overline{u}$ take values in a bounded interval and since $A((\partial_\xi u)^{-1})$ and $A((\partial_\xi \overline{u})^{-1})$ are zero when $\xi = 0, 1$ (we recall that $(\partial_\xi u)^{-1} = \rho$, see [CGT04]), the boundary term above disappears. Moreover, since the function $t \to A(t^{-1})$ is non increasing we can get rid of the second addend in the right hand side above. Therefore, due to the uniform bound of ρ in L^∞ and to the definition of $\mathcal{B}$, there exists a fixed constant $C > 0$ such that

$$\frac{d}{dt}\int_0^m [u(t)-\overline{u}(t)]^{2n}d\xi \leq Cn\left[\int_0^m [u(t)-\overline{u}(t)]^{2n}d\xi + 1\right],$$

and, by Gronwall inequality,

$$\|u(t)-\overline{u}(t)\|_{L^{2n}[0,m]} \leq e^{Ct}$$

for some positive constant C. Now, (3.27) and a similar estimate for $\overline{u}_b$ as in [CGT04] imply

$$\|\overline{u}_b(t)\|_{L^{2n}[0,m]} \leq C(t)$$

and $C(t)$ is independent of b and n. This fact yields

$$\|u(t)\|_{L^{2n}[0,m]} \leq C(t)$$

for some function $C(t)$ of time independent of b and n. We now want to send $b \to \infty$ in order to extend the result to the solution ρ of the original Cauchy problem. By using the same technique as in section 3.1 we can recover some compactness estimates for the family ρ_b in such a way that, up to subsequences, $\rho_b \to \rho$ almost everywhere in $\mathbb{R}\times[0,+\infty)$. Recalling that

$$\|u_b(t)\|_{L^{2n}[0,m]} = d_{2n}(\rho_b, \delta_0),$$

where $d_p(\cdot,\cdot)$ is the p–Wasserstein distance between two measures with finite p–th moment (see e. g. [Vil03]), and where δ_0 is the Dirac delta measure centered at zero, we use the lower semi continuity of such a distance with respect to the weak convergence in the sense of measures (see [Vil03]) and we obtain the desired estimate for the original solution ρ and the corresponding pseudo–inverse u

$$(M_{2n}(\rho(t)))^{1/2n} = \|u(t)\|_{L^{2n}[0,m]} \leq C(t).$$

We can now send $n \to \infty$ to get

$$\|u(t)\|_{L^\infty[0,m]} \leq C(t)$$

and the proof is complete. □

3.4. **Asymptotic behaviour.** In the following we investigate the asymptotic behaviour of weak solutions to (3.1), (3.2) for large time. The main idea in this case is the analysis of the behaviour of the associated energy functional

$$\tilde{E}(\rho) := \int_{\mathbb{R}^d} [\rho(\varepsilon\rho - S(\rho)] \, dx, \tag{3.28}$$

where $S(\rho)$ is the unique solution to $-\Delta S + S = \rho$ decaying at infinity. This energy functional is to be considered on the admissible set

$$\mathcal{K} := \{\rho \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \mid 0 \le \rho \le 1 \text{ a.e. }\}. \tag{3.29}$$

For the nonlinear diffusion case, the change from $\varepsilon > 1$ to $\varepsilon < 1$ is of particular interest, since $\tilde{E}(\rho)$ changes from a strictly convex (for $\varepsilon > 1$) to a nonconvex functional, which we verify in the following:

Lemma 3.23. *The functional $\tilde{E} : \mathcal{K} \to \mathbb{R}$ is bounded below by $-\int_{\mathbb{R}^d} \rho \, dx$ for $\varepsilon > 0$. Moreover, $\tilde{E}$ is positive and strictly convex for $\varepsilon > 1$.*

Proof. First of all, since $\rho \le 1$ and $S(\rho) \ge 0$, we have

$$\tilde{E}(\rho) := \int_{\mathbb{R}^d} [\rho(\varepsilon\rho - S(\rho)] \, dx \ge -\int_{\mathbb{R}^d} S(\rho) \, dx.$$

The property $\int_{\mathbb{R}^d} S(\rho) \, dx = \int_{\mathbb{R}^d} \rho \, dx$ can be deduced immediately from the elliptic equation satisfied by $S(\rho)$ and hence,

$$\tilde{E}(\rho) \ge -\int_{\mathbb{R}^d} \rho \, dx.$$

Since $\tilde{E}$ is a quadratic functional, convexity is equivalent to strict positivity. From the Cauchy-Schwarz inequality we have

$$\tilde{E}(\rho) = \int_{\mathbb{R}^d} \rho(\varepsilon\rho - S(\rho)) \, dx \ge (\varepsilon - 1) \int_{\mathbb{R}^d} \rho^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^d} \left(\rho^2 - S(\rho)^2\right) \, dx.$$

Finally, a standard energy estimate for the elliptic equation satisfied by $S(\rho)$ shows that the second term is nonnegative, and hence,

$$\tilde{E}(\rho) \ge (\varepsilon - 1) \int_{\mathbb{R}^d} \rho^2 \, dx > 0$$

for $\rho \ne 0$ and $\varepsilon > 1$. $\square$

A fundamental property of the model is the dissipation of the energy $\tilde{E}$. Formally, if we compute the time derivative of $E(\rho(t))$, insert the equations and apply Gauss' Theorem, then we obtain

$$\begin{aligned} \frac{d}{dt}\tilde{E}(\rho(t)) &= \int_{\mathbb{R}^d} \rho_t(2\varepsilon\rho - \mathcal{B} * \rho) \, dx - \int_{\mathbb{R}^d} \rho\mathcal{B} * \rho_t \, dx \\ &= 2\int_{\mathbb{R}^d} \rho_t(\varepsilon\rho - \mathcal{B} * \rho) \, dx \\ &= 2\int_{\mathbb{R}^d} \nabla \cdot [\rho(1-\rho)\nabla(\varepsilon\rho - \mathcal{B} * \rho)] \, (\varepsilon\rho - \mathcal{B} * \rho) \, dx \\ &= -2\int_{\mathbb{R}^d} \rho(1-\rho) \left|\nabla(\varepsilon\rho - \mathcal{B} * \rho)\right|^2 \, dx := -2I(\rho, S) < 0. \end{aligned} \tag{3.30}$$

This estimate can be made rigorous by standard smooth approximation techniques, and thus, we have derived the following result:

Proposition 3.24 (Energy Dissipation)**.** *Let ρ be a weak solution of* (3.1), (3.2)*. Then the functional*

$$e : \mathbb{R}^+ \to \mathbb{R}, t \mapsto e(t) := \tilde{E}(\rho(t))$$

is nonincreasing. Moreover $e(s) = e(t)$ for $s > t$ if and only if ρ is stationary in the interval $[s,t]$ and

$$\rho(1-\rho)\nabla(\varepsilon\rho - S(\rho)) = 0 \qquad a.e. \text{ in } \mathbb{R}^d \times [s,t].$$

3.4.1. *Non decaying solutions in 1–d for moderate diffusivity.* In one space dimension and for $\varepsilon < 1$, we can verify that the functional $\tilde{E}$ is not positive by an explicit construction of appropriate densities:

Proposition 3.25. *Let $d = 1$ and $\varepsilon < 1$, then for each $m > 0$ there exists $\rho \in \mathcal{K}$ satisfying*

$$\tilde{E}(\rho) < 0, \qquad \text{and} \qquad \int_{\mathbb{R}^d} \rho \; dx = m. \tag{3.31}$$

Proof. Let $0 < \alpha^2 < \frac{1}{\varepsilon} - 1$ and let ψ^α be a continuous function satisfying

$$\begin{array}{rl} -\frac{d^2\psi^\alpha}{dx^2} + \psi^\alpha = \frac{1}{c} & 0 \le x < a \\ -\frac{d^2\psi^\alpha}{dx^2} - \alpha^2\psi^\alpha = 0 & a \le x \le b \\ -\frac{d^2\psi^\alpha}{dx^2} + \psi^\alpha = 0 & x > b \end{array}$$

with boundary conditions $\psi^\alpha(0) = 1$, $\frac{d\psi^\alpha}{dx}(0) = 0$ and $\psi^\alpha(x) \to 0$ as $x \to \infty$, and some constant c satisfying

$$0 < \varepsilon < c < \frac{1}{\alpha^2 + 1} < 1. \tag{3.32}$$

We find that if $a < \ln \frac{c}{1-c}$, a continuously differentiable, nonnegative solution exists and is given by

$$\psi^\alpha(x) = \begin{cases} \frac{1}{c}(1 - (c-1)\cosh x) & 0 \le x < a \\ c_1 \sin(\alpha(x-a)) + c_2 \cos(\alpha(x-a)) & a \le x \le b \\ c_3 e^{b-x} & b < x, \end{cases}$$

where the constants satisfy

$$\begin{array}{ll} c_1 & = \frac{c-1}{\alpha c} \sinh a < 0 \\ c_2 & = (1 + (c-1)\cosh a)/c \\ c_3 & = c_1 \sin(\alpha(b-a)) + c_2 \cos(\alpha(b-a)), \end{array}$$

and the length of the second interval is fixed by the relation

$$b = a + \frac{1}{\alpha} \arctan\Big(-\frac{c_2 + \alpha c_1}{c_1 - \alpha c_2}\Big).$$

If we choose S as the symmetric extension of $c\psi^\alpha$ to $\mathbb{R}$ and

$$\rho(x) = \begin{cases} 1 & -a < x < a \\ c(\alpha^2+1)\psi^\alpha(x) & a \le x \le b \\ c(\alpha^2+1)\psi^\alpha(-x) & -b \ge x \ge -a \\ 0 & |x| > b, \end{cases}$$

then S satisfies $-\frac{d^2 S}{dx^2} + S = \rho$ and $\rho \in \mathcal{K}$. Moreover, we have

$$\tilde{E}(\rho) = \int_{\mathbb{R}^d} \rho(\varepsilon\rho - S) \; dx = 2\Big[\int_0^a (\varepsilon - S) \; dx + (\varepsilon(\alpha^2+1) - 1)(\alpha^2+1)\int_a^b S^2 \; dx\Big].$$

This expression is negative if $a < \operatorname{arccosh}\Big(\frac{1-\varepsilon}{1-c}\Big)$. The mass m is given by

$$\frac{1}{2}m = a+(\alpha^2+1)\int_a^b S\,dx = a+c(\alpha^2+1)\int_a^b c_1\sin(\alpha(x-a))+c_2\cos(\alpha(x-a))\,dx.$$

Because of (3.32), we have that $\frac{1}{2}m < a+\int_a^b c_1\sin(\alpha(x-a))+c_2\cos(\alpha(x-a))\,dx$. If α is chosen small enough, we can choose the constant c to be so close to 1 that the restraints on a, and hence on m become arbitrarily large. Thus, for any given mass m, we can always construct a solution $\rho\in\mathcal{K}$ such that (3.31) holds. □

If we now consider an initial value ρ_0 for (3.1) such that $\tilde{E}(\rho_0)<0$, then due to the energy dissipation we have $\tilde{E}(\rho(t))\le\tilde{E}(\rho)<0$ and hence, such solutions cannot decay for $t\to\infty$ (since otherwise one would have $\liminf\tilde{E}(\rho(t))\ge\tilde{E}(0)=0$). Together with Proposition 3.25 this implies that there exist non-decaying solutions for every $\varepsilon<1$ and $m>0$.

3.4.2. *Attractors of the semigroup in* 1*–*d. As another interesting consequence of the energy estimate (3.30), for any positive ε we can characterize the set of attractors of the semigroup (3.1), (3.2) as the set of its stationary entropy solutions, i. e. the set of all $\rho\in L^1$, $0\le\rho\le 1$ such that

$$\rho(1-\rho)\nabla(\varepsilon\rho-S(\rho))=0 \qquad \text{a.e. in } \mathbb{R}\times[0,+\infty), \tag{3.33}$$

where $S(\rho)$ is the unique solution to $-\Delta S+S=\rho$ decaying at infinity. To this aim, we set

$$B(\rho)=\int_0^\rho\sqrt{r(1-r)}dr$$

and we observe that (3.30) implies that the quantity

$$\int_0^{+\infty}I(\rho(t),S(t))dt=\int_0^{+\infty}\int_{-\infty}^{+\infty}\left[\varepsilon^2B(\rho)_x^2-2\varepsilon A(\rho)_xS_x+\rho(1-\rho)S_x^2\right]dxdt$$

is uniformly bounded. Therefore, any sequence of times tending to infinity has a subsequence $t_k\to+\infty$ such that $I(\rho(t_k),S(t_k))\to 0$ as $k\to\infty$. Since

$$\int_{-\infty}^{+\infty}\rho(1-\rho)S_x^2dx\le\frac{1}{4}\int_{\mathbb{R}^d}\rho^2dx,$$

and $\rho(t)$ is uniformly bounded in $L^1\cap L^\infty$, and because of

$$-2\varepsilon\int_{\mathbb{R}^d}A(\rho)_xS_xdx=2\varepsilon\int_{\mathbb{R}^d}A(\rho)(\rho-S)dx,$$

we easily get a uniform (with respect to time) bound for

$$\int_{-\infty}^{+\infty}B(\rho(t_k))_x^2dx.$$

By means of the uniform bound in $L^1\cap L^\infty$ for the solution $\rho(t)$, by Sobolev embedding we can extract a new subsequence of times (still denoted by t_k for simplicity) such that

$$\begin{cases}\rho(t_k)\to\rho^\infty \ \text{ a. e. in } \mathbb{R}\times[0+\infty)\\ \rho(t_k)\to\rho^\infty \ \text{ in } \ L^2_{loc}(\mathbb{R})\\ B(\rho(t_k))_x\to v^\infty \ \text{ weakly in } L^2(\mathbb{R})\\ I(\rho(t_k),S(t_k))\to 0,\end{cases}$$

as $k \to \infty$. Now, for any test function $\phi \in C_0^\infty(\mathbb{R})$ we have

$$\int_{-\infty}^{+\infty} v^\infty \phi \, dx = \lim_{k\to\infty} \int_{-\infty}^{+\infty} B(\rho(t_k))_x \phi \, dx$$
$$= -\lim_{k\to\infty} \int_{-\infty}^{+\infty} B(\rho(t_k))\phi_x dx = -\int_{-\infty}^{+\infty} B(\rho^\infty)\phi_x dx$$

and therefore $v^\infty = B(\rho^\infty)_x$ almost everywhere. Hence, denoting by S^∞ the solution to $-S_{xx} + S = \rho^\infty$, by weak lower semi–continuity of the L^2 norm and by the strong compactness of $S_x = \mathcal{B}' * \rho$ in L^2, by extracting another subsequence we have

$$I(\rho^\infty, S^\infty) = \varepsilon^2 \int_{-\infty}^{+\infty} \varepsilon B(\rho^\infty)_x^2 dx - 2\varepsilon \int_{-\infty}^{+\infty} A(\rho^\infty)_x S_x^\infty dx$$
$$+ \int_{-\infty}^{+\infty} \rho^\infty(1-\rho^\infty)(S_x^\infty)^2 dx \le \liminf_{k\to\infty} \left\{ \int_{-\infty}^{+\infty} B(\rho(t_k))_x dx \right.$$
$$\left. -2\varepsilon \int_{-\infty}^{+\infty} A(\rho(t_k))_x S(t_k)_x dx + \int_{-\infty}^{+\infty} \rho(t_k)(1-\rho(t_k))S(t_k)_x^2 dx \right\}$$
$$= \lim_{k\to\infty} I(\rho(t_k), S(t_k)) = 0.$$

Recalling that $I(\rho, S) = \int_{-\infty}^{+\infty} \rho(1-\rho)(\varepsilon\rho - S)_x^2 dx$, we have thus proven the following

Theorem 3.26 (Attractors of the semigroup)**.** *Let $\rho(t)$ be the solution to* (3.1)*,* (3.2) *in one space dimension. Then, any sequence of times admits a subsequence t_k such that $\rho(t_k) \to \rho^\infty$ almost everywhere. Moreover, ρ^∞ is a solution to (3.33).*

Remark 3.27. In the arguments above we have implicitly supposed existence of stationary solutions. Actually, the convergence of subsequences in the above theorem is strong enough to prove existence of stationary, weak, entropy solutions of (3.1), (3.2) in one space dimension. Thanks to the non decay result in subsection 3.4.1, we deduce existence of stationary solutions which are not identically zero in case of $\varepsilon < 1$.

3.4.3. *Characterization of the attractors for large diffusivity in* 1*–d.* In this subsection we prove that the only attractor of the semigroup, and hence the only solution to the stationary problem (3.33), in case of large diffusivity $\varepsilon > 1$ is the constant solution $\rho^\infty \equiv 0$. To perform this task, we need an additional energy estimate, namely, we compute the evolution of the logarithmic functional

$$L(\rho) = \int_{-\infty}^{+\infty} [\rho \log \rho + (1-\rho)\log(1-\rho)] dx.$$

As for the energy estimate in the previous subsections, we compute the evolution of $L(\rho)$ by means of a formal computation which can be made rigorous by approximation. Integration by parts and conservation of the total mass yield

$$L(\rho(t)) - L(\rho(0)) = \int_0^t \int (\log \rho - \log(1-\rho))\rho_t dx d\tau$$
$$= -\int_0^t \int \left(\frac{\rho_x}{\rho} + \frac{\rho_x}{1-\rho} \right) (\varepsilon\rho(1-\rho)\rho_x - \rho(1-\rho)S_x) \, dx d\tau$$
$$= -\varepsilon \int_0^t \int \rho_x^2 dx d\tau + \int_0^t \int \rho_x S_x dx d\tau \le -(\varepsilon - 1) \int_0^t \int \rho_x^2 dx d\tau. \tag{3.34}$$

The logarithmic functional L cannot be used directly to achieve an asymptotic behaviour of the solution ρ. However, the estimate performed above can be used to characterize the stationary solutions in case $\varepsilon > 1$. We have the following theorem.

Theorem 3.28 (Attractors for large diffusivity in 1–d)**.** *Let ρ, S be a solution to (3.1) with $\varepsilon > 1$ such that ρ has finite support at any time. Then, the support of ρ is not uniformly bounded with respect to t. As a consequence of that, there exist no nonzero compactly stationary solutions ρ, S to (3.1) if $\varepsilon > 1$.*

Proof. Suppose that $\rho(t)$ is a solution with uniformly bounded support. Since the function

$$[0,1] \ni \rho \mapsto \rho \log \rho + (1-\rho)\log(1-\rho)$$

is bounded, then $L(\rho(t))$ is uniformly bounded in time. Therefore, in a similar way as in the proof of Theorem 3.26, we can handle the right hand side of estimate (3.34) in a clever way in order to get some strong compactness. More precisely, there exists a divergent sequence of times t_k such that $\rho(t_k)$ converge to some ρ^∞ almost everywhere and strongly in L^2_{loc}, and such that $\rho(t_k)_x$ converges to zero strongly in L^2. As in Theorem 3.26, we can easily prove that $\rho^\infty_x = 0$ and, by Fatou's lemma we conclude that $\rho^\infty = 0$. By Sobolev interpolation lemma we get $\rho(t_k) \to 0$ uniformly, and this is in contradiction with $\rho(t)$ having uniformly bounded support because of the conservation of the mass. This proves the first assertion of the theorem. In particular, we have also proven that any compactly supported stationary solution must equal zero. □

As a consequence of the previous theorem, we have the following asymptotic decay result in case of large diffusivity.

Corollary 3.29 (Decay of solutions in 1–d for large diffusivity)**.** *Let ρ be the solution to (3.1) with compactly supported initial datum ρ_0 satisfying (3.2). Then,*

$$\lim_{t\to\infty} \|\rho(t)\|_{L^\infty(\mathbb{R})} = 0.$$

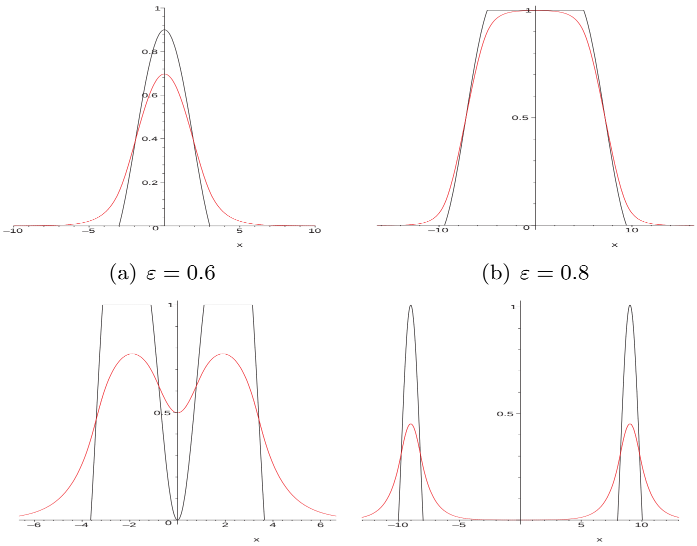
Proof. From Theorem 3.26, any divergent sequence of times admits a subsequence t_k such that $\rho(t_k)$ converges almost everywhere to a stationary solutions satisfying (3.33). Thanks to the results in theorems 3.22 and 3.28, such a solution must be the constant solution $\rho \equiv 0$. Moreover, the convergence to zero holds in L^∞ in view of $B(\rho(t_k) \to 0$ in L^2 and by Sobolev interpolation lemma. □

3.5. **Stationary solutions.** As stated in proposition 3.25, stationary solutions of (3.1), (3.2) have to satisfy $\rho = 0$, $\rho = 1$ or $\varepsilon\nabla\rho = \nabla S$. In one space dimension, this means that we can construct stationary solutions by arranging subintervals on $\mathbb{R}$ such that ρ is in the admissible set $\mathcal{K}$ and satisfies one of these conditions in every interval.

Proposition 3.30. *Let $d = 1$ and $\varepsilon < 1$, then for each $m > 0$ small enough there exists a stationary solution of* (3.1) *satisfying $\rho \in \mathcal{K}$.*

Proof. Let ρ and S be the symmetric extension to $\mathbb{R}$ of

$$\bar{S}(x) = \begin{cases} \frac{\varepsilon}{\varepsilon-1}c_1 + c_2\cos(\sqrt{\frac{1-\varepsilon}{e}}x) & 0 \le x \le a \\ c_3 e^{a-x} & a \le x, \end{cases} \qquad \bar{\rho}(x) = \begin{cases} \frac{1}{\varepsilon}S + c_1 & 0 \le x \le a \\ 0 & a \le x, \end{cases}$$



(a) $\varepsilon = 0.6$ (b) $\varepsilon = 0.8$

(c) $\varepsilon = 0.2$ (d) $\varepsilon = 0.2$

FIGURE 1. Stationary solutions ρ (dark lines) and S (light lines) of (3.1), (3.2).

and let the constants satisfy

$$c_1 = c\frac{\sqrt{\varepsilon}-1}{\sqrt{\varepsilon}} < 0, \quad c_2 = \frac{c(\varepsilon\sqrt{\varepsilon}-\varepsilon)}{\varepsilon-1} > 0, \quad c_3 = -\varepsilon c_1,$$

where c is the maximal value of ρ and a is given by

$$a = \sqrt{\frac{\varepsilon}{1-\varepsilon}}\arccos(-\sqrt{\varepsilon}).$$

Then, a simple calculation shows that for any given values of $\varepsilon < 1$ and $0 \le c \le 1$, a non-negative solution $\rho \in C(\mathbb{R})$ and $S \in C^1(\mathbb{R})$ exists. Moreover, S and ρ are decreasing functions on $[0, a]$, implying the assertion. $\square$

An example of this type of solution is shown in fig. 1(a), where we set $c = 0.9$ and $\varepsilon = 0.6$.

In general, also more complicated stationary solutions can be constructed, for instance solutions with

$$\bar{\rho}(x) = \begin{cases} 1 & 0 \le x \le a \\ \frac{1}{\varepsilon}S + c_1 & a \le x \le b \\ 0 & b \le x, \end{cases} \tag{3.35}$$

or solutions with several peaks (see fig. 1(b)-1(d)). It is no longer straightforward to show that these solutions exist for any choice of $\varepsilon < 1$, but there is strong numerical evidence. As an example, we chose a stationary solution of type (3.35): it seems that for any mass m large enough and $\varepsilon < 1$, a solution can be uniquely determined. Fig. 3.5 shows the interval a as a function of the mass for different values of ε (left to right: $\varepsilon = 10^{-3}$, $\varepsilon = 0.2$, $\varepsilon = 0.6$ and $\varepsilon = 0.9$). Depending on ε, there exists a

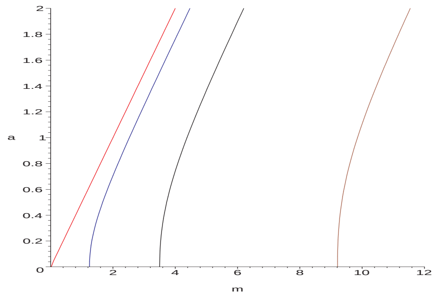


FIGURE 2. Dependence on the parameter a on m for different values of ε.

minimal value for m, which is due to the fact that the slope of ρ, and hence also the minimal distance between intervals where $\rho = 1$ and $\rho = 0$ is proportional to ε.

4. Numerical Simulation

In the following we discuss the numerical simulation of the linear and nonlinear model. In order to obtain a unified presentation, we start from the general model (1.1) specifying the particular form of the potential if needed.

4.1. **Numerical Schemes.** In the following we briefly discuss two different schemes for the simulation of (1.1), both being based on forward Euler time and upwind finite difference space discretizations. The difference in the methods is the perspective used for their construction, in the first case we use an Eulerian approach, i.e., we directly discretize (1.1), while in the second case we take a Lagrangian perspective, i.e., we discretize the equation for the pseudo-inverse of the distribution function. The main advantage of the second method is that it is posed on the interval $(0, 1)$ and therefore does not require any artificial cut of the computational domain. On the other hand, the distribution function and its pseudo-inverse do not exist in multiple dimensions so that the Lagrangian method cannot be used anymore. We will therefore carry out most computations using the Eulerian method, but use the Lagrangian approach as a reference to check the error caused by cutting the domain in the Eulerian approach.

4.1.1. *Eulerian Approach.* A straightforward approach to solve (2.1) and (3.1) numerically is to apply a finite difference method to the original equation: for each time step, S is computed using an implicit discretization of the elliptic equation, then the cell density ρ is calculated using the updated value for S. This is done by standard operator splitting of the equation for ρ. First the advection term is calculated with an upwind scheme (cf. [LeV90]). The solution obtained from this step is subsequently used as an initial value for a time step in the diffusion problem $\rho_t = \varepsilon\Delta\rho$ and $\rho_t = \varepsilon\Delta A(\rho)$ for the linear and the nonlinear case respectively. If ε is relatively large and the parabolic CFL condition $\Delta t \leq C\frac{\Delta x^2}{\varepsilon}$ becomes too expensive compared to the hyperbolic CFL condition for the advection term, we use a Crank-Nicholson scheme (an implicit, weighted average method, cf. [Qua03]) for the time integration linear diffusion term.

4.1.2. *Lagrangian Approach.* In spatial dimension one, we can use the equation satisfied by the pseudo-inverse of the distribution function (as used in the proof of finite speed above) to construct a numerical scheme that avoids approximation by a finite domain. Let $F : \mathbb{R} \times [0,T] \to [0,m]$ be the distribution function satisfying

$$F_x(x,t) = \rho(x,t) \quad \text{a.e,} \qquad \lim_{x\to-\infty} F(x,t) = 0 \quad \forall\ t.$$

The pseudo-inverse $u : [0,m] \times [0,T] \to \mathbb{R}$ is defined via

$$u(\xi,t) = \sup\{\ x \in \mathbb{R} \mid F(x,t) \leq \xi\ \}.$$

By analogous reasoning as in [GT05, LT04] we can derive the equation

$$\partial_t u = -\partial_\xi A\left((\partial_\xi u)^{-1}\right) + \left(1 - (\partial_\xi u)^{-1}\right) \int_0^m \mathcal{B}'(u - u(\eta,.))d\eta \tag{4.1}$$

to be satisfied by the pseudo-inverse u. Note that this equation corresponds to the Eulerian description of the system, roughly speaking the pseudo-inverse describes the location of particles (which would be exact for piecewise constant u). In the construction of a finite-difference method we follow the approach [GT05], which can be carried over in a one-to-one fashion and we only have to take care of the additional term $1 - (\partial_\xi u)^{-1}$ not appearing in [GT05].

If we use a grid $0 = \xi_0 < \xi_1 < \ldots < \xi_N = m$ and denote $u_k(t) = u(x_k,t)$, then a step of an explicit upwind finite difference method can be written as

$$u_k(t+\tau) = u(t) - \tau D_+ A\left(\frac{1}{D_- u_k(t)}\right) + \tau\left(1 - \frac{1}{D u_k(t)}\right) \sum_j w_j \mathcal{B}'(u_k(t) - u_j(t)),$$

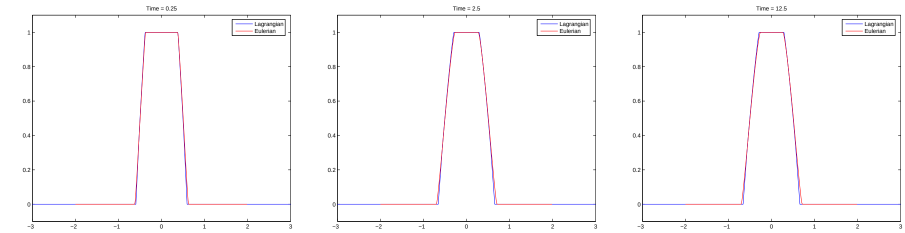
where D_+ and D_- denote the standard forward and backward difference quotients.

4.2. **Numerical Tests.** In the following we present the results of some numerical examples carried out in spatial dimension one and two, respectively.

4.2.1. *Comparison of Eulerian and Lagrangian Approaches.* As mentioned above, the Eulerian method for the numerical solution of (1.1) requires the approximation of the solution on an unbounded domain by (artificial) boundary conditions on a bounded domain. For the numerical simulations presented below, we take Dirichlet boundary conditions for S and ρ and perform the computations on large domains.

In order to check whether this method is reliable, we compare solutions of the nonlinear model (3.1) obtained by the Eulerian and the Lagrangian approach, where the latter is used as a reference due to the fact that it does not need any domain approximation. Figure 3 illustrates the resulting solutions ρ at different time steps. Here we used the parameter value $\varepsilon = 0.1$, the time step $\Delta t = 5 \times 10^{-4}$, and the grid sizes $\Delta x = 10^{-2}$ (Eulerian) and $\Delta\xi = 10^{-2}$ (Lagrangian).

Figure 4 provides a plot of the 2-Wasserstein distance between the numerical solutions as a function of time. One can observe that the error grows only moderately in the beginning, and then stops when the solution gets close to the stationary state. The maximal value of the error is approximately Δx and thus of the same order as the as the error caused by the discretizations anyway. Hence, the error due to the cutting of the domain in the Eulerian method seems to introduce only a negligible error, and we will apply this approach in the following numerical experiments. As an additional test, the numerical results in one and two space dimensions were checked by successively doubling the size of the domain and comparing the



(a) $t = 0.25$ (b) $t = 2.5$ (c) $t = 12.5$

FIGURE 3. Comparison of the Eulerian and the Lagrangian Method

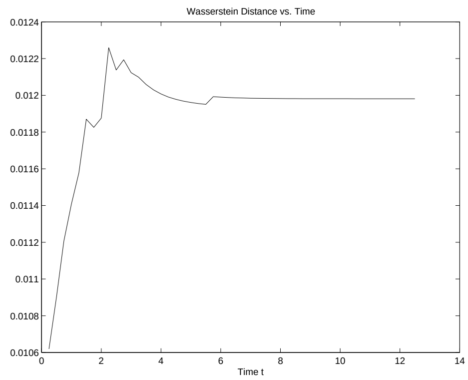
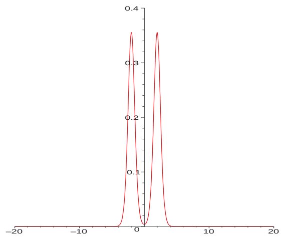


FIGURE 4. Wasserstein-distance between the solutions obtained with the Eulerian and the Lagrangian method as a function of time

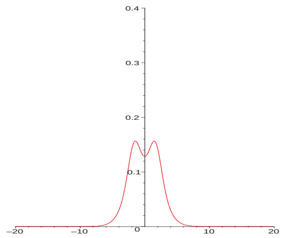
corresponding numerical solutions, which also produced negligible variations for reasonably large domain size.

4.2.2. *One-dimensional Simulations.* The evolution of the cell density in (1.1) is illustrated in figures 5 and 6 for the linear and the nonlinear case, respectively: starting with a symmetrical initial condition for ρ consisting of two peaks both with mass $\frac{1}{2}$, we compute the solutions with the Eulerian scheme described above, using $\Delta x = 10^{-2}$, $\Delta t = 5 \times 10^{-3}$ in the linear and $\Delta x = 10^{-2}$, $\Delta t = 5 \times 10^{-4}$ in the nonlinear case, respectively. In order to reduce the computation time, we consider symmetrical initial data with compact support on $0 \leq x \leq a$ and prescribe Neumann boundary conditions at $x = 0$, so that we only have to compute on half of the domain.

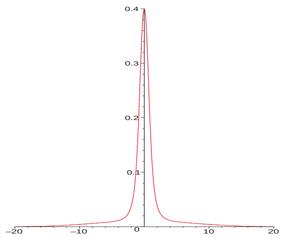
¿From Figure 5 one observes that the two initial peaks merge into a single peak that eventually decays with time. In Figure 6 the time evolution of the nonlinear diffusion problem (3.1) is illustrated. In order to have more fair comparison with the linear case we set $\varepsilon = 0.5$, i.e., ten times the value we took for the linear case (but simulations with smaller ε, e.g. $\varepsilon = 0.2$ showed a very similar behavior). Starting with the same initial conditions, the solution first behaves as in the linear case: there is attraction between the two peaks and they merge to a single one. In



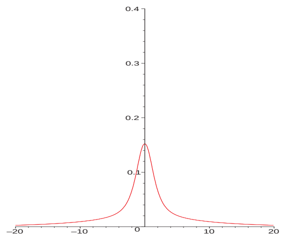
(a) $t = 4$



(b) $t = 40$

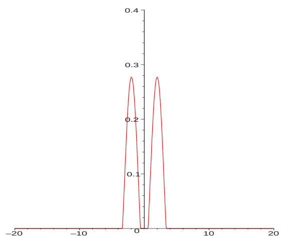


(c) $t = 400$

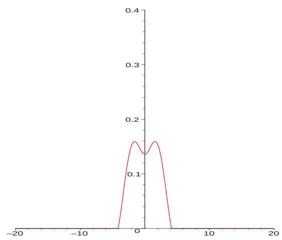


(d) $t = 2000$

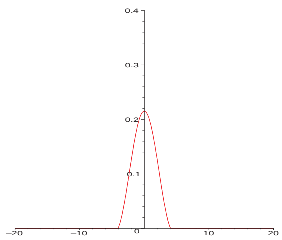
FIGURE 5. Linear problem, $\varepsilon = 0.05$



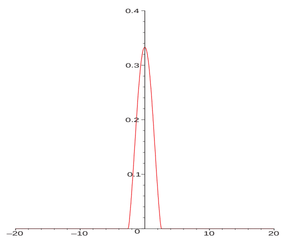
(a) $t = 4$



(b) $t = 40$



(c) $t = 80$

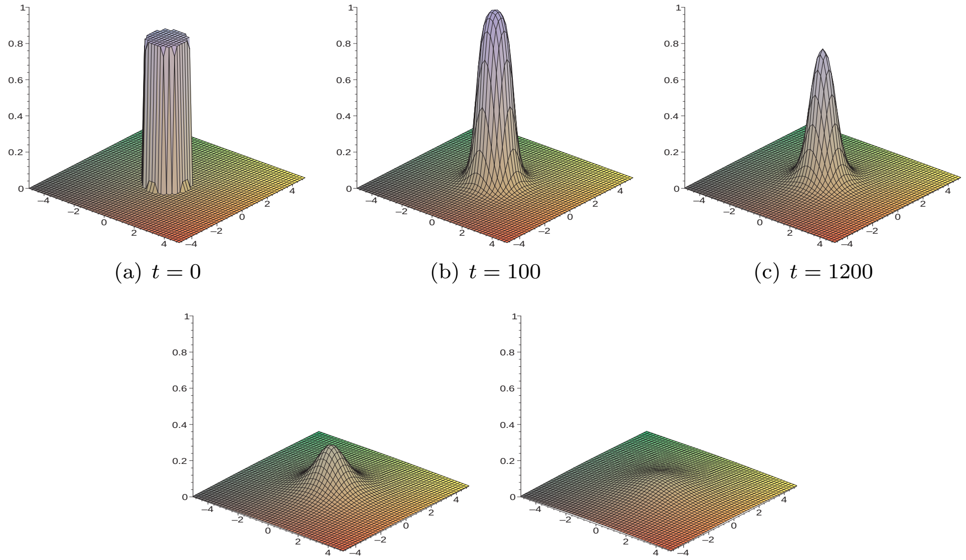


(d) $t = 400$

FIGURE 6. Nonlinear problem, $\varepsilon = 0.5$

contrast to to the linear case, however, this peak does not decay, but approaches a nontrivial stationary solution as characterized in Proposition 3.30.

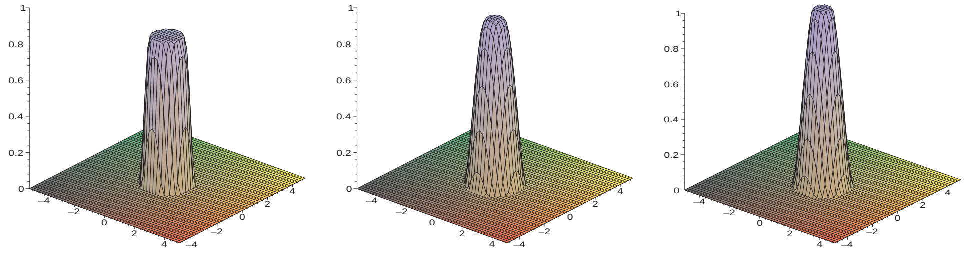
4.2.3. *Two-dimensional Simulations.* In two space dimensions, we perform numerical experiments on a rectangular grid with $\Delta x = \Delta t = 5 \times 10^{-2}$ for the linear,



(a) $t = 0$ (b) $t = 100$ (c) $t = 1200$

(d) $t = 1400$ (e) $t = 1500$

FIGURE 7. Linear problem, $\varepsilon = 0.04$



(a) $t = 0.25$ (b) $t = 2$ (c) $t = 35$

FIGURE 8. Nonlinear problem, $\varepsilon = 0.2$

and $\Delta x = 5 \times 10^{-2}$, $\Delta t = 4 \times 10^{-4}$ for the nonlinear model, respectively. The diffusivities are $\varepsilon = 0.04$ for the linear and $\varepsilon = 0.2$ for the nonlinear model. Figure 7 illustrates the temporal evolution of the solution ρ of the linear problem, starting from the initial condition shown in the first picture. Similar to the $1-d$ case, we see that the maximal value of the density remains one for a long time interval. However, for even larger time diffusion dominates aggregation and the density starts to decay. In Figure 8 the behaviour of the nonlinear model starting with the same initial conditions is shown. One observes that the solution is not decaying and a stationary state with finite support is obtained in the limit.

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Martin Burger – Industrial Mathematics Institute, Johannes Kepler University, Altenbergerstr. 69, A-4040 Linz, Austria
E-mail address: martin.burger@jku.at

Marco Di Francesco – Johann Radon Institute for Computational and Applied Mathematics (RICAM) Austrian Academy of Sciences (ÖAW), Altenbergerstr. 69, A-4040 Linz, Austria
E-mail address: marco.difrancesco@oeaw.ac.at

Yasmin Dolak-Struss– Johann Radon Institute for Computational and Applied Mathematics (RICAM) Austrian Academy of Sciences (ÖAW), Altenbergerstr. 69, A-4040 Linz, Austria
E-mail address: yasmin.dolak@oeaw.ac.at