UNIVERSITY OF CAMBRIDGE

CENTRE FOR MATHEMATICAL SCIENCES

DEPARTMENT OF APPLIED MATHEMATICS
& THEORETICAL PHYSICS

# On a Lie–Poisson system and its Lie algebra

Tony Bloch & Arieh Iserles

# Part I:
# A matrix ODE system

Let

$$X' = [N, X^2] = NX^2 - X^2N, \qquad t \geq 0,$$

where $X(0) = X_0 \in \mathrm{Sym}(n)$ and $N \in \mathfrak{so}(n)$. Here

$\mathrm{Sym}(n)$ : $n \times n$ real symmetric matrices,
$\mathfrak{so}(n)$ : $n \times n$ real skew-symmetric matrices.

Why is this system interesting?

**Reason 1:** It is *isospectral:* defining a skew-symmetric matrix function

$$B(X) = NX + XN,$$

we can rewrite it at once in the form

$$X' = [B(X), X], \qquad X(0) = X_0 \in \mathrm{Sym}(n).$$

The system above is isospectral for any

$$B : \mathrm{Sym}(n) \to \mathfrak{so}(n)$$

– its invariants are the eigenvalues of $X_0$.

Other isospectral systems, more well-known:

- The Toda lattice equations (Flaschka; Lax; Moser);
- The QR flow (Symes; Deift, Nanda & Tomei; Watkins);
- The double-bracket flow (Brockett; Chu & Driessel; Bloch, Brockett & Crouch; Bloch & Iserles);
- The Toeplitz annihilator flow (Chu & Driessel).

Why are isospectral ODEs isospectral?

Because they are an outcome of orthogonal group action. Thus, it is easy to verify that

$$X(t) = Q(t)X_0Q^\top(t), \qquad t \geq 0,$$

where

$$Q' = B(QX_0Q^\top)Q, \qquad Q(0) = I.$$

Since

$$A(Q) = B(QX_0Q^\top) : \mathrm{SO}(n) \to \mathfrak{so}(n)$$

and $\mathfrak{so}(n)$ is the Lie algebra of the special orthogonal group $\mathrm{SO}(n)$, it follows that $Q$ evolves in $\mathrm{SO}(n)$ and $X(t)$ is similar to $X_0$.

**Reason 2:** The ODE is acted by congruence. Given

$$A : \mathrm{Sym}(n) \to \mathrm{M}(n),$$

where $\mathrm{M}(n)$ is the set of real $n \times n$ matrices, it is easy to verify that the solution of

$$X' = A(X)X + XA^\top(X), \quad t \geq 0,$$

where $X(0) = X_0 \in \mathrm{Sym}(n)$, is *congruent* to $X_0$:

$$X(t) = V(t)X_0V^\top(t), \qquad t \geq 0,$$

where

$$V' = A(VX_0V^\top)V, \qquad V(0) = I,$$

is a flow in the general linear group $\mathrm{GL}(n)$.

Congruent flows preserve the angular field of values

$$F'(X) = \{\boldsymbol{y}^* X \boldsymbol{y} \,:\, \boldsymbol{y} \in \mathbb{C}^n,\ \boldsymbol{y} \neq \mathbf{0}\}$$

and the signature of $X_0$. Also, if $X_0 = LL^\top$ is the Cholesky factorization of $X_0 \in \mathrm{Sym}_+(n)$ then a factorization of $X(t)$ is

$$X(t) = [V(t)L][V(t)L]^\top.$$

Setting $A(X) = [N, X]$, it is easy to verify that our ODE system is acted by congruence.

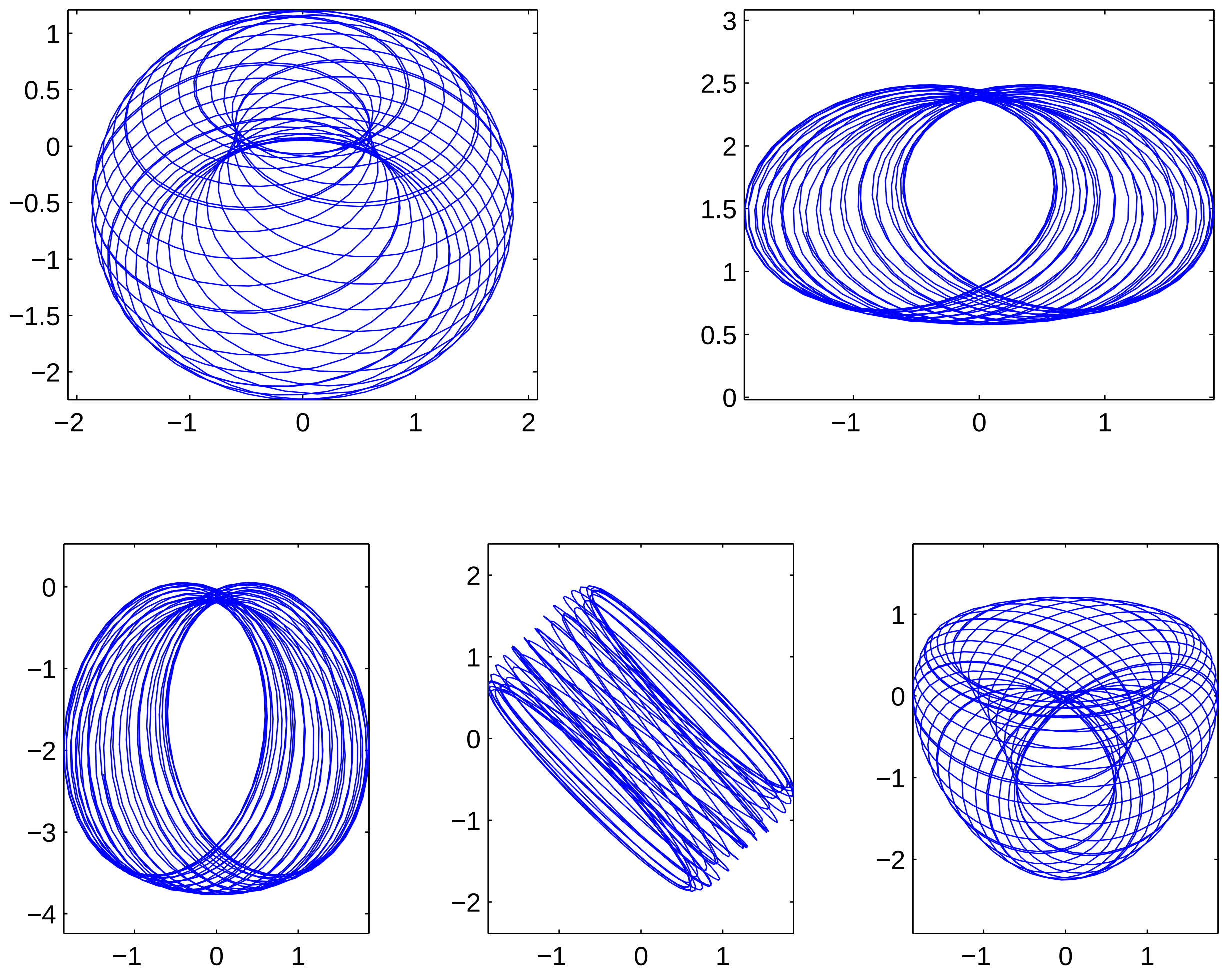
Although action by congruence is much “weaker” than action by similarity, the really interesting freature of our system is that it is acted by two different groups. This is particularly interesting in the context of *Lie-group methods* since we are faced with the choice which action to retain under discretization.

**Reason 3:** The system is a "dual" of *generalized rigid body equations*

$$M' = [\Omega, M],$$

where $\Omega \in \mathfrak{so}(n)$ and $M = J\Omega + \Omega J$, where $J$ lives in $\mathrm{Sym}(n)$ (Arnold).

**Reason 4:**



These are phase portraits (specifically, we display planes $(x_{1,2}, x_{k,l})$) for $n = 3$,

$$N = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

and random initial conditions.

A persuasive observation: The solution evolves on invariant tori in $\mathbb{R}^{\frac{1}{2}n(n+1)}$.

Such behaviour is hardly ever accidental and it is reasonable to suspect that there is a deeper structure hiding within the equation $X' = [N, X^2]$.
This suspicion is well founded...

# Part II:
# Poisson systems

Given

**1** A smooth function $H : \mathbb{R}^m \to \mathbb{R}$ (a Hamiltonian); and

**2** A linear, homogeneous function $S : \mathbb{R}^m \to \mathfrak{so}(m)$

the ODE system

$$\boldsymbol{x}' = S(\boldsymbol{x})\boldsymbol{\nabla} H(\boldsymbol{x}), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0 \in \mathbb{R}^m,$$

is said to be *almost Poisson.*

By "linear, homogeneous" we mean that there exist structure constants $c_{i,j}^k$ such that

$$S_{i,j}(\boldsymbol{x}) = \sum_{k=1}^{m} c_{i.j}^k x_k, \quad i,j = 1,\ldots,m.$$

Note that skew-symmetry of $S$ implies $c_{i,j}^k + c_{j,i}^k = 0$.

We say that the structure constants obey the Jacobi condition if

$$\sum_{k=1}^{m} (c_{p,q}^{k} c_{k,r}^{l} + c_{q,r}^{k} c_{k,p}^{l} + c_{r,p}^{k} c_{k,q}^{l}) = 0$$

for all $p, q, r, l = 1, \ldots, m$. In that case the ODE is a ***Poisson system***, a.k.a. Kostant–Kirillov–Lie–Soriou system.

Why are Poisson systems interesting?

- They represent a generalization of Hamiltonian systems. In particular, the Hamiltonian energy $H(\boldsymbol{y})$ is conserved by the flow.

- Define a Poisson bracket of two functions as

  $$\{f, g\} = [\boldsymbol{\nabla} f(\boldsymbol{y})]^{\top} S(\boldsymbol{y}) \boldsymbol{\nabla} g(\boldsymbol{y}).$$

  A *Casimir* is a function $f$ which is in involution with the Hamiltonian:

  $$\{f, H\} = 0.$$

  Each Casimir is a first integral of a Poisson system. (Note that $H$ itself is a Casimir.) Casimirs are typical of Poisson systems.

- Each Poisson system can be represented as a ***Lie–Poisson system:*** Suppose that we have square matrices $E_1, E_2, \ldots, E_m$ such that

$$[E_i, E_j] = \sum_{k=1}^{n} c_{i,j}^k E_k, \qquad i, j = 1, \ldots, m.$$

We generate the free Lie algebra

$$\mathcal{E} = \mathrm{FLA}(E_1, E_2, \ldots, E_m)$$

with the basis $\boldsymbol{E} = \{E_1, \ldots, E_m\}$. Thus, $\mathcal{E}$ is the closure of the basis elements (the generators) with respect to

1. Linear operations; and
2. Commutation.

Now, let $\mathcal{E}^*$ be the dual Lie algebra of $\mathcal{E}$: the Lie algebra of all linear functionals acting on $\mathcal{E}$, with the natural definition of a commutator. We let $\langle F, E \rangle$ for $F \in \mathcal{E}^*$ and $E \in \mathcal{E}$ be a positive bilinear form: it is enough to consider in our setting

$$\langle F, E \rangle = \mathrm{tr} F^\top E$$

(i.e., Frobenius norm or the Killing form.)

It is possible to reformulate the Lie–Poisson flow so that it evolves in $\mathcal{E}^*$. Let $\boldsymbol{F} = \{F_1, F_2, \ldots, F_m\}$ be a dual basis of $\mathcal{E}^*$: a basis such that

$$\langle F_k, E_l \rangle = \begin{cases} 1, & k = l, \\ 0, & k \neq l. \end{cases}$$

We set

$$Y(t) = \sum_{k=1}^{m} y_k(t) F_k \in \mathcal{E}^*$$

and (abusing notation) let $H(Y) = H(\boldsymbol{y})$. Then the Lie–Poisson system can be formulated as

$$Y' = -\mathrm{ad}^*_{\mathrm{d}H(Y)} Y.$$

Here

$$\mathrm{d}H(Y) = \left( \frac{\partial H(Y)}{\partial Y_{i,j}} \right)_{i,j=1}^{m}$$

and $\mathrm{ad}^*$ is the dual adjoint operator which, within our context, can be taken as

$$\mathrm{ad}^*_A B = [A^\top, B].$$

Therefore a Lie–Poisson system possesses a crucial geometric feature: it evolves in a Lie algebra.

Remark: There exist several numerical methods for Lie–Poisson systems that keep the solution within $\mathcal{E}^*$ (Lewis & Simo; Faltinsen).

BACK TO $X' = [N, X^2]$.

We set

$$H(X) = \tfrac{1}{2}\|X\|^2_{\mathrm{Frob}} = \tfrac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n} x_{k,l}^2.$$

Our equations are

$$x'_{k,l} = \sum_{i=1}^{n}\sum_{j=1}^{n}(n_{k,i}x_{i,j}x_{l,j} - n_{i,j}x_{k,i}x_{l,j} - n_{i,j}x_{k,j}x_{l,i} + n_{l,i}x_{k,j}x_{i,j})$$

and, with much algebra, we have the structure matrix

$$S_{(p,q),(r,s)} = \tfrac{1}{2}(n_{p,r}x_{q,s} + n_{p,s}x_{q,r} + n_{q,r}x_{p,s} + n_{q,s}x_{p,r})$$

for all $1 \le p \le q \le n$, $1 \le r \le s \le n$ (note that $\dim \mathcal{E} = m = \frac{1}{2}n(n+1)$).

## The Jacobi condition:

Remember, we need to check whether

$$\sum_{i=1}^{n}\sum_{j=i}^{n}\Big[c^{(i,j)}_{(p,q),(r,s)}c^{(u,v)}_{(i,j),(k,l)} + c^{(i,j)}_{(r,s),(k,l)}c^{(u,v)}_{(i,j),(p,q)} + c^{(i,j)}_{(k,l),(p,q)}c^{(u,v)}_{(i,j),(r,s)}\Big]$$

vanishes for all $(p,q),(r,s),(k,l),(u,v)$. Here

$$c^{(i,j)}_{(p,q),(r,s)} = \tfrac{1}{2}\left[\delta^{(q,s)}_{(i,j)}n_{p,r} + \delta^{(q,r)}_{(i,j)}n_{p,s} + \delta^{(p,s)}_{(i,j)}n_{q,r} + \delta^{(p,r)}_{(i,j)}n_{q,s}\right],$$

with

$$\delta^{(p,q)}_{(i,j)} = \begin{cases} 1, & (i,j) = (p,q), \\ 0, & (i,j) \neq (p,q). \end{cases}$$

This requires a very great deal of nasty algebra. Employing symmetry, it is enough to check for a single choice of parameters and we end up with terms of the form

$$n_{p,s}n_{q,s} + n_{q,s}n_{p,s} + n_{s,p}n_{q,s} + n_{s,q}n_{p,s},$$

which are zero by the skew-symmetry of $N$.

***THEOREM*** *The system $X' = [N, X^2]$ is Poisson.*

# Part III:
# Lie–Poisson systems

Can every Poisson system be converted into a Lie–Poisson system?

This is true iff, given a set of structure constants $c_{k,l}^{i}$ that obey skew-symmetry and the Jacobi condition, we can identify matrices that form a basis of the underlying free Lie algebra. This is precisely the statement of the...

***ADO THEOREM*** *Every finite-dimensional Lie algebra possesses a finite-dimensional faithful representation.*

Here, an algebra representation is a homomorphism

$$\rho : \mathfrak{g} \to \mathrm{End}\, V,$$

where $\mathfrak{g}$ is a (formal) Lie algebra, while $\mathrm{End}\, V$ is a matrix Lie algebra over the linear space $V$.
A Lie-algebra homomorphism a linear map s.t.

$$\rho([a,b]) = [\rho(a), \rho(b)], \qquad a, b \in \mathfrak{g}.$$

The representation is faithful if $\rho$ is injective.

## Can such a representation be derived explicitly?

The proof of Ado's theorem has been converted by Willem de Graaf into a (very complicated!) symbolic algorithm. Yet, his algorithm falls short of our needs:

- It produces matrices whose size increases exponentially with the dimension of $g$, while we (bearing in mind eventual application to geometric integration) wish to find a small (ideally, the smallest!) representation $\mathcal{E}$;

- We should bear it in mind that ultimately we wish to work in the dual algebra $\mathcal{E}^*$. In particular, we seek

  $$\mathcal{E} = \mathsf{FLA}(E_1, \ldots, E_m), \qquad \mathcal{E}^* = \mathsf{FLA}(F_1, \ldots, F_m),$$

  where $m = \frac{1}{2}n(n+1)$, s.t. $\mathrm{tr}\, E_k^\top F_l = \delta_{k,l}$.

  An ideal situation is if $\mathrm{tr}\, E_k^\top E_l = \pi_k \delta_{k,l}$ for some $\pi_1, \ldots, \pi_m > 0$ (an orthogonal basis), since then we may identify $\mathcal{E}^*$ with $\mathcal{E}$, choosing $F_k = \pi_k^{-1} E_k$.

THE GOAL Find matrices $E_{p,q}$ s.t.

$$[E_{p,q}, E_{r,s}] = \frac{1}{2}(n_{q,r}E_{p,s} + n_{p,r}E_{q,s} + n_{q,s}E_{p,r} + n_{p,s}E_{q,r}).$$

THE ALGORITHM

**Step 1:** We assume without loss of generality that $\|N\|_2 = 1$: otherwise, later multiply elements of the basis by $\|N\|_2$. Consider the matrix

$$I + \mathrm{i}N.$$

This is a Hermitian matrix, since $N \in \mathfrak{so}(n)$.
Moreover, it is positive semi-definite and singular. The reason is that

$$\lambda \in \sigma(I + \mathrm{i}N) \quad \Leftrightarrow \quad \lambda = 1 - \mu,\ \mathrm{i}\mu \in \sigma(N).$$

However, $|\mu| \leq \|N\|_2$ and $\max \mu = \|N\|_2 = 1$.

We deduce that there exists

$$(I + \mathrm{i}N)^{1/2}$$

which is also Hermitian, positive semi-definite and singular.

For reasons that will be made apparent soon, we wish to evaluate the QR factorization of $(I + \mathrm{i}N)^{1/2}$.

**Step 2:** We evaluate the standard form of the QR factorization of $(I + \mathrm{i}N)^{1/2}$. In other words, we find $Q \in \mathrm{U}(n)$ and an upper-tridiagonal $R$ such that

$$QR = (I + \mathrm{i}N)^{1/2}$$

and such that the first nonzero element along each row of $R$ is real and positive. Actually, we require just the matrix $R$.

Although this can be done the long way (find first the square root, then its (complex) QR factorization – the latter is less trivial than it looks), we can do it more neatly in one fell swoop. Essentially, the algorithm is, for $k = 1, 2, \ldots, n-1$

$$r_{k,k} = \left(1 - \sum_{j=1}^{k-1} |r_{j,k}|^2\right)^{1/2},$$

$$r_{k,l} = \frac{1}{r_{k,k}} \left(\mathrm{i}n_{k,l} - \sum_{j=1}^{k-1} \bar{r}_{j,k} r_{j,l}\right), \quad l = k+1, \ldots, n.$$

We need to cater for the case $r_{k,k} = 0$ and do so by column changes, but that's a minor technical point.

**Step 3:** Note that singularity of $(I+\mathrm{i}N)^{1/2}$ implies that the bottom row of $R$ is zero. We remove that row, hence $R$ is now an $(n-1)\times n$ complex matrix. We let

$$A=\begin{bmatrix}\mathrm{Re}\,R\\ \mathrm{Im}\,R\end{bmatrix}=[\boldsymbol{a}_1,\boldsymbol{a}_1,\ldots,\boldsymbol{a}_n].$$

Note that $\boldsymbol{a}_k\in\mathbb{R}^{2n-2}$, $k=1,\ldots,n$. We set

$$E_{p,q}=\tfrac{1}{2}(\boldsymbol{a}_p\boldsymbol{a}_q^\top+\boldsymbol{a}_q\boldsymbol{a}_p^\top)J,\qquad 1\le p\le q\le n,$$

where

$$J=\begin{bmatrix}O & I\\ -I & O\end{bmatrix}.$$

But why does it make sense? Letting $B=\mathrm{Re}\,R$ and $C=\mathrm{Im}\,R$, we note from

$$(B+\mathrm{i}C)^*(B+\mathrm{i}C)=R^*Q^*QR=I+\mathrm{i}N$$

that

$$B^\top B+C^\top C=I,$$
$$B^\top C-C^\top B=N.$$

Let

$$B = [\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n], \qquad C = [\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n].$$

Then

$$\begin{aligned}\boldsymbol{a}_p^\top J \boldsymbol{a}_q &= [\boldsymbol{b}_p^\top \, \boldsymbol{c}_p^\top] \begin{bmatrix} O & I \\ -I & O \end{bmatrix} \begin{bmatrix} \boldsymbol{b}_q \\ \boldsymbol{c}_q \end{bmatrix} = \boldsymbol{b}_p^\top \boldsymbol{c}_q - \boldsymbol{c}_p^\top \boldsymbol{b}_q \\ &= (B^\top C - C^\top B)_{p,q} = n_{p,q}.\end{aligned}$$

Therefore

$$\begin{aligned}[E_{p,q}, E_{r,s}] &= \tfrac{1}{4}[(\boldsymbol{a}_p\boldsymbol{a}_q^\top + \boldsymbol{a}_q\boldsymbol{a}_p^\top)J, (\boldsymbol{a}_r\boldsymbol{a}_s^\top + \boldsymbol{a}_s\boldsymbol{a}_r^\top)J] \\ &= \tfrac{1}{4}[(\boldsymbol{a}_q J \boldsymbol{a}_r)\boldsymbol{a}_p\boldsymbol{a}_s^\top + (\boldsymbol{a}_p^\top J \boldsymbol{a}_r)\boldsymbol{a}_q\boldsymbol{a}_s^\top + (\boldsymbol{a}_q^\top J \boldsymbol{a}_s)\boldsymbol{a}_p\boldsymbol{a}_r^\top \\ &\quad + (\boldsymbol{a}_p^\top J \boldsymbol{a}_s)\boldsymbol{a}_q\boldsymbol{a}_r^\top - (\boldsymbol{a}_s^\top J \boldsymbol{a}_p)\boldsymbol{a}_r\boldsymbol{a}_q^\top - (\boldsymbol{a}_r^\top J \boldsymbol{a}_p)\boldsymbol{a}_s\boldsymbol{a}_q^\top \\ &\quad - (\boldsymbol{a}_s^\top J \boldsymbol{a}_q)\boldsymbol{a}_r\boldsymbol{a}_p^\top - (\boldsymbol{a}_r^\top J \boldsymbol{a}_q)\boldsymbol{a}_s\boldsymbol{a}_p^\top]J \\ &= \tfrac{1}{4}[n_{q,r}(\boldsymbol{a}_p\boldsymbol{a}_s^\top + \boldsymbol{a}_s\boldsymbol{a}_p^\top)J + n_{p,r}(\boldsymbol{a}_q\boldsymbol{a}_s^\top + \boldsymbol{a}_s\boldsymbol{a}_q^\top)J \\ &\quad + n_{q,s}(\boldsymbol{a}_p\boldsymbol{a}_r^\top + \boldsymbol{a}_r\boldsymbol{a}_p^\top)J + n_{p,s}(\boldsymbol{a}_q\boldsymbol{a}_r^\top + \boldsymbol{a}_r\boldsymbol{a}_q^\top)J] \\ &= \tfrac{1}{2}(n_{q,r}E_{p,s} + n_{p,r}E_{q,s} + n_{q,s}E_{p,r} + n_{p,s}E_{q,r}).\end{aligned}$$

We deduce that we have a representation of our Lie algebra in $\mathbb{R}^{2n-2}$.

But is it faithful? Orthogonal?

Both questions can be answered in a single calculation: since $JJ^\top = -J^2 = I$,

$$\begin{aligned}\langle E_{p,q}, E_{r,s}\rangle &= \tfrac{1}{4}\mathrm{tr}(\boldsymbol{a}_p\boldsymbol{a}_q^\top + \boldsymbol{a}_q\boldsymbol{a}_p^\top)JJ^\top(\boldsymbol{a}_s\boldsymbol{a}_r^\top + \boldsymbol{a}_r\boldsymbol{a}_s^\top) \\ &= \tfrac{1}{4}\mathrm{tr}[(\boldsymbol{a}_q^\top\boldsymbol{a}_s)\boldsymbol{a}_p\boldsymbol{a}_r^\top + (\boldsymbol{a}_q^\top\boldsymbol{a}_r)\boldsymbol{a}_p\boldsymbol{a}_r^\top + (\boldsymbol{a}_p^\top\boldsymbol{a}_s)\boldsymbol{a}_q\boldsymbol{a}_r^\top \\ &\quad + (\boldsymbol{a}_p^\top\boldsymbol{a}_r)\boldsymbol{a}_q\boldsymbol{a}_s^\top] \\ &= \tfrac{1}{2}[(\boldsymbol{a}_p^\top\boldsymbol{a}_r)(\boldsymbol{a}_q^\top\boldsymbol{a}_s) + (\boldsymbol{a}_p^\top\boldsymbol{a}_s)(\boldsymbol{a}_q^\top\boldsymbol{a}_r)] \\ &= \tfrac{1}{2}(\delta_{p,r}\delta_{q,s} + \delta_{p,s}\delta_{q,r}) \\ &= \begin{cases} 1, & p=q=r=s, \\ \frac{1}{2}, & p\neq q,\ p=r,\ q=s, \\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

Therefore the representation is of full dimension $\frac{1}{2}n(n+1)$, hence faithful, and it is orthogonal.

***THEOREM*** *The above algorithm results in a faithful and orthogonal representation of the underlying Lie algebra in* $\mathbb{R}^{2n-2}$.

An example: Assume $a^2+b^2+c^2=1$ and set

$$N = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}, \qquad \|N\|_2 = 1.$$

We compute

$$R = \begin{bmatrix} 1 & \mathrm{i}a & \mathrm{i}b \\ 0 & \sqrt{b^2+c^2} & \frac{-ab+\mathrm{i}c}{\sqrt{b^2+c^2}} \\ 0 & 0 & 0 \end{bmatrix}$$

(verify that $R^*R = I + \mathrm{i}N$), hence (removing the bottom row)

$$A = \begin{bmatrix} \operatorname{Re} R \\ \operatorname{Im} R \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{b^2+c^2} & -\frac{ab}{\sqrt{b^2+c^2}} \\ 0 & a & b \\ 0 & 0 & \frac{c}{\sqrt{b^2+c^2}} \end{bmatrix}$$

and

$$\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{a}_2 = \begin{bmatrix} 0 \\ \sqrt{b^2+c^2} \\ a \\ 0 \end{bmatrix}, \quad \boldsymbol{a}_3 = \begin{bmatrix} 0 \\ -\frac{ab}{\sqrt{b^2+c^2}} \\ b \\ \frac{c}{\sqrt{b^2+c^2}} \end{bmatrix}.$$

Hence,

$$E_{1,1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$E_{1,2} = \begin{bmatrix} -\frac{1}{2}a & 0 & 0 & \frac{1}{2}\sqrt{b^2+c^2} \\ 0 & 0 & \frac{1}{2}\sqrt{b^2+c^2} & 0 \\ 0 & 0 & \frac{1}{2}a & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$E_{1,3} = \begin{bmatrix} -\frac{1}{2}b & -\frac{c}{2\sqrt{b^2+c^2}} & 0 & -\frac{ab}{2\sqrt{b^2+c^2}} \\ 0 & 0 & -\frac{ab}{2\sqrt{b^2+c^2}} & 0 \\ 0 & 0 & \frac{1}{2}b & 0 \\ 0 & 0 & \frac{c}{2\sqrt{b^2+c^2}} & 0 \end{bmatrix},$$

$$E_{2,2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -a\sqrt{b^2+c^2} & 0 & 0 & b^2+c^2 \\ -a^2 & 0 & 0 & a\sqrt{b^2+c^2} \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$E_{2,3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{b(a^2-b^2-c^2)}{2\sqrt{b^2+c^2}} & -\frac{1}{2}c & 0 & -ab \\ -ab & -\frac{ac}{2\sqrt{b^2+c^2}} & 0 & -\frac{b(a^2-b^2-c^2)}{2\sqrt{b^2+c^2}} \\ -\frac{ac}{2\sqrt{b^2+c^2}} & 0 & 0 & \frac{1}{2}c \end{bmatrix},$$

$$E_{3,3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{ab^2}{\sqrt{b^2+c^2}} & \frac{abc}{b^2+c^2} & 0 & \frac{a^2b^2}{b^2+c^2} \\ -b^2 & -\frac{bc}{\sqrt{b^2+c^2}} & 0 & -\frac{ab^2}{\sqrt{b^2+c^2}} \\ -\frac{bc}{\sqrt{b^2+c^2}} & -\frac{c^2}{b^2+c^2} & 0 & -\frac{abc}{b^2+c^2} \end{bmatrix}.$$

# Conclusion

Our point of departure was the ODE matrix system $X' = [N, X^2]$, which is subject to two distinct group actions. We have proved that it is a Poisson system and constructed an algorithm for the construction of its orthogonal faithful representation in the underlying Lie algebra.

Themes for ongoing and future research:

- Is the system integrable? What are its Casimirs? First integrals? What are features of the flow in the dual algebra?
- Are there bi-Hamiltonians? If so, are they nondegenerate? Can integrability follow by this route?
- Are there other isospectral flows, in addition to this one and the Toda lattice, which are Lie–Poisson? What are their Lie algebras?
- Is it possible to consider the "Ado problem" (find a faithful representation, given structure constants) in its full generality, in a linear-algebraic, numerical setting?

ACKNOWLEDGMENTS: Brad BAXTER, Matania BEN ARTZI, John BUTCHER, Willem de GRAAF, Adi LEVIN, David LEVIN, Hans MUNTHE-KAAS, Peter OLVER, Jeremy SCHIFF and Tel Aviv University School of Mathematics.